Monopoles, instantons and non-Abelian black holes

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Seminar given on June 15th, 2015 at the Workshop on Theoretical Aspects of BHs and Cosmology, IIP, Natal, Brazil

Based on 0802.1799, 0803.0684 (by *P. Meessen*) 0806.1477, 1412.5547, 1501.02078 and 1503.01044

Work done in collaboration with *P. Bueno, M. Hübscher, P.F. Ramírez and S. Vaulà* (IFT UAM/CSIC, Madrid) and *P. Meessen* (U. Oviedo)

Plan of the Talk:

1 – Introduction

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Now it is natural to ask what happens in the gauged theories. There are several possible gaugings in $N = 2, d = 4$ theories. Let's review the theory.

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We are not going to consider hypermultiplets in this seminar.

All vector fields are collectively denoted by $A^{\Lambda}{}_{\mu} = (A^{0}{}_{\mu}, A^{i}{}_{\mu})$. They are combined with the dual (magnetic) vector fields $A_{\Lambda\,\mu}$ into a symplectic vector

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$\mathcal{V}^{M}(Z, Z^{*})$ defines completly this sector of the theory (it defines a Special Kähler geometry). Alternatively, one can use a prepotential.

The action of the bosonic fields of the ungauged theory is

$$
\begin{aligned}
S &= \int d^4x \sqrt{|g|}\, \Big[R + 2\mathcal{G}_{ij^*}\partial_\mu Z^i \partial^\mu Z^{*\, j^*} + 2\Im\mathrm{m}\mathcal{N}_{\Lambda\Sigma} F^{\Lambda\,\mu\nu} F^{\Sigma}{}_{\mu\nu} \\
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These theories have supersymmetric, extreme, charged black holes which are very well known by all of you. To study solutions with non-Abelian vector fields we must gauge these theories. In absence of hypermultiplets there are just three possibilities:

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1. We gauge an $U(1)\subset SU(2)_R\subset U(2)_R$ using Fayet-Iliopoulos terms.
2. We gauge a subgroup G of the isometry group of $\mathcal{G}_{ij^*}$ in combination with $U(1)_R\in U(2)_R$ (Kähler trans.).
3. If G contains an $SU(2)$ factor we can combine this gauging with that of $SU(2)_R$ using $SU(2)$ Fayet-Iliopoulos terms.

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- The preservation of the Hermitean structure implies the holomorphicity of the $k_\Lambda{}^i$ components of the Killing vectors: $k_\Lambda{}^i = k_\Lambda{}^i(Z)$.

These conditions can be formally expressed as follows:

➢ The global transformations to consider are

$$\delta_\alpha Z^i = \alpha^\Lambda k_\Lambda{}^i(Z)\,, \qquad [K_\Lambda, K_\Sigma] = -f_{\Lambda\Sigma}{}^\Omega K_\Omega\,,$$

where $K_\Lambda = k_\Lambda{}^i\partial_i + \text{c.c.}$.

➢ The vector fields and period matrix must transform as

$$\delta_\alpha A^\Lambda{}_\mu = \alpha^\Sigma f_{\Sigma\Omega}{}^\Lambda A^\Omega{}_\mu\,, \qquad \delta_\alpha \mathcal{N}_{\Lambda\Sigma} = -2\alpha^\Omega f_{\Omega(\Lambda}{}^\Gamma \mathcal{N}_{\Sigma)\Gamma}\,.$$

➢ The preservation of the metric implies that the K_Λ are Killing vectors of $\mathcal{G}_{ij^*}$.

➢ The preservation of the Hermitean structure implies the holomorphicity of the $k_\Lambda{}^i$ components of the Killing vectors: $k_\Lambda{}^i = k_\Lambda{}^i(Z)$.

➢ The Kähler structure will be preserved if

1. The Kähler potential is preserved (up to Kähler transformations)

$$\pounds_\Lambda \mathcal{K} \equiv k_\Lambda{}^i\partial_i\mathcal{K} + k^*_\Lambda{}^{i^*}\partial_{i^*}\mathcal{K} = \lambda_\Lambda(Z) + \lambda^*_\Lambda(Z^*)\,.$$

2. The Kähler 2-form $\mathcal{J} = i\mathcal{G}_{ij^*}dZ^i \wedge dZ^{*j^*}$ is also preserved:

$$\pounds_\Lambda \mathcal{J} = 0\,.$$

Then,

$$\left.\begin{array}{l} d\mathcal{J} = 0 \ \ \Rightarrow \pounds_{\Lambda}\mathcal{J} = d(i_{k_{\Lambda}}\mathcal{J})\,, \\ \\ \pounds_{\Lambda}\mathcal{J} = 0\,, \end{array}\right\} \Rightarrow d(i_{k_{\Lambda}}\mathcal{J}) = 0\,, \ \Rightarrow i_{k_{\Lambda}}\mathcal{J} = d\mathcal{P}_{\Lambda}\,, \Leftrightarrow k_{\Lambda\, i^{*}} = i\partial_{i^{*}}\mathcal{P}_{\Lambda}\,.$$

for some real 0-forms $\mathcal{P}_{\Lambda}$: the *momentum maps* or *Killing prepotentials*.

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➛ The preservation of the Hodge structure requires that we accompany the transformations δ_α with $U(1)_R$ transformations. In particular, the spinors must transform as

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$$\begin{aligned} \delta_\alpha \mathcal{L}^{\Lambda} &= \alpha^{\Lambda}\pounds_{\Lambda}\mathcal{L}^{\Lambda} = -\tfrac{1}{2}\alpha^{\Sigma}(\lambda_{\Sigma}-\lambda^*_{\Sigma})\mathcal{L}^{\Lambda} + \alpha^{\Sigma} f_{\Sigma\Omega}{}^{\Lambda}\mathcal{L}^{\Omega}\,, \\ \delta_\alpha \mathcal{M}_{\Lambda} &= \alpha^{\Lambda}\pounds_{\Lambda}\mathcal{M}_{\Lambda} = -\tfrac{1}{2}\alpha^{\Sigma}(\lambda_{\Sigma}-\lambda^*_{\Sigma})\mathcal{M}_{\Lambda} - \alpha^{\Sigma} f_{\Sigma\Lambda}{}^{\Omega}\mathcal{M}_{\Omega}\,, \end{aligned}$$

➢ This last requirement leads to this expression of the Killing vectors:

$$k_{\Lambda\, i^*} = i f_{\Lambda\Sigma}{}^{\Gamma} \left(\mathcal{D}_{i^*} \mathcal{L}^{*\Sigma} \mathcal{M}_{\Gamma} + \mathcal{L}^{\Sigma} \mathcal{D}_{i^*} \mathcal{M}^{*}_{\Gamma} \right) \,,$$

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1. (Always) A $U(1) \subset SU(2)_R$ via FI terms. The timelike supersymmetric solutions of these theories have been classified in Caldarelli & Klemm, `hep-th/0307022`, Cacciatori, Caldarelli, Klemm & Mansi, `hep-th/0406238`, Cacciatori, Caldarelli, Klemm, Mansi & Roest, `arXiv:0704.0247` and Cacciatori, Klemm, Mansi & Zorzan, `arXiv:0804.0009`. There are very few regular black holes among them (Toldo & Vandoren `arXiv:1207.3014`).

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$3 - N = 2, d = 4$ SEYM

To gauge the theory we replace the standard by gauge-covariant derivatives

$$\begin{aligned}
\partial_\mu Z^i &\longrightarrow \mathfrak{D}_\mu Z^i \equiv \partial_\mu Z^i + g A^\Lambda{}_\mu k_\Lambda{}^i\,,\\
\mathcal{D}_\mu \psi_{I\nu} &\longrightarrow \mathfrak{D}_\mu \psi_{I\nu} \equiv \{\nabla_\mu + \tfrac{i}{2}(\mathcal{Q}_\mu + g A^\Lambda{}_\mu \mathcal{P}_\Lambda)\}\psi_{I\nu}\,,\\
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$$\mathfrak{D}_\mu \epsilon_I \quad = \quad \{\nabla_\mu + \tfrac{i}{2}(\mathcal{Q}_\mu + g A^\Lambda{}_\mu \mathcal{P}_\Lambda)\}\epsilon_I \,.$$

The supersymmetry transformations of the bosons stay unchanged, but those of the fermions get shifted by terms proportional to g which will enter quadratically in the scalar potential:

$$\delta_\epsilon \psi_{I\,\mu} \quad = \quad \mathfrak{D}_\mu \epsilon_I + \varepsilon_{IJ} T^+{}_{\mu\nu}\gamma^\nu \epsilon^J \,,$$

$$\delta_\epsilon \lambda^{Ii} \quad = \quad i \not{\mathfrak{D}} Z^i \epsilon^I + \varepsilon^{IJ}(\not{G}^{i\,+} + \tfrac{1}{2} g \mathcal{L}^{*\Lambda} k_\Lambda{}^i)\epsilon_J \,,$$

The action of the bosonic fields takes the form

$$\begin{aligned} S &= \int d^4x\sqrt{|g|}\,\left[R + 2\mathcal{G}_{ij^*}\mathfrak{D}_\mu Z^i \mathfrak{D}^\mu Z^{*\,j^*} + 2\Im\mathrm{m}\mathcal{N}_{\Lambda\Sigma}F^{\Lambda\,\mu\nu}F^{\Sigma}{}_{\mu\nu} \right. \\ & \left. \qquad -2\Re\mathrm{e}\mathcal{N}_{\Lambda\Sigma}F^{\Lambda\,\mu\nu\star}F^{\Sigma}{}_{\mu\nu} - V(Z,Z^*)\right]\,, \end{aligned}$$

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We will be interested in asymptotically-flat solutions.

4 – The supersymmetric solutions of $N = 2, d = 4$ SEYM theories

The supersymmetric (or BPS) solutions of all these theories have been classified in Hübscher, Meessen, O., Vaulà arXiv:0806.1477 using the method pioneered by Gauntlett and collaborators (Class. Quant. Grav. **20** (2003) 4587 [hep-th/0209114])

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The timelike class contains very interesting non-Abelian generalizations of the Abelian black-hole solutions.

We are going to focus on this case.

Our results for the timelike case can be summarized in the following

RECIPE:

☞ Find a set of Yang-Mills fields $\tilde{A}^{\Lambda}{}_{m}$ and functions $\mathcal{I}^{\Lambda}$ in $\mathbb{R}^3$ satisfying

$$\tilde{F}^{\Lambda}{}_{mn} = -\frac{1}{\sqrt{2}}\epsilon_{mnp}\tilde{\mathfrak{D}}_{p}\mathcal{I}^{\Lambda} ,$$

which is the Bogomol'nyi equation satisfied by known magnetic monopole solutions (more on this, later).

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☞ Use the above solution to solve the equation

$$\tilde{\mathfrak{D}}_{m}\tilde{\mathfrak{D}}_{m}\mathcal{I}_{\Lambda} = \tfrac{1}{2}g^{2} \left[f_{\Lambda(\Sigma}{}^{\Gamma}f_{\Delta)\Gamma}{}^{\Omega}\,\mathcal{I}^{\Sigma}\mathcal{I}^{\Delta}\right]\mathcal{I}_{\Omega} ,$$

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$$\tilde{\mathfrak{D}}_{m}\tilde{\mathfrak{D}}_{m}\mathcal{I}_{\Lambda} = \tfrac{1}{2}g^2 \left[f_{\Lambda(\Sigma}{}^{\Gamma}f_{\Delta)\Gamma}{}^{\Omega}\,\mathcal{I}^{\Sigma}\mathcal{I}^{\Delta}\right]\mathcal{I}_{\Omega} ,$$

for the $\mathcal{I}_{\Lambda}$s. For compact gauge groups

$$\mathcal{I}_{\Lambda} \propto \mathcal{I}^{\Lambda} ,$$

always provides a solution.

The real symplectic vector $(\mathcal{I}^M) = \begin{pmatrix} \mathcal{I}^\Lambda \\ \mathcal{I}_\Lambda \end{pmatrix}$ determines completely the solution.

The physical fields $g_{\mu\nu}, A^\Lambda{}_\mu, Z^i$ are derived from them as follows:

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- ☞ First we must solve the stabilization (or Freudenthal duality) equations to find $\mathcal{R}^M(\mathcal{I})$ identifying

$$\mathcal{I}^M \equiv \Im\mathrm{m}(\mathcal{V}^M/X)\,, \qquad \mathcal{R}^M \equiv \Re\mathrm{e}(\mathcal{V}^M/X)\,,$$

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☞ The scalars are, then, given by

$$Z^i = \frac{\mathcal{L}^i}{\mathcal{L}^0} = \frac{\mathcal{L}^i/X}{\mathcal{L}^0/X} = \frac{\mathcal{R}^i + i\mathcal{I}^i}{\mathcal{R}^0 + i\mathcal{I}^0}\,.$$

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- The 1-form $\omega = \omega_m dx^m$ on $\mathbb{R}^3$ is found by solving the equation

$$(d\omega)_{mn} = 2\epsilon_{mnp}\mathcal{I}_M \, \tilde{\mathfrak{D}}_p \mathcal{I}^M = 2\epsilon_{mnp}\left[\mathcal{I}_\Lambda \tilde{\mathfrak{D}}_p \mathcal{I}^\Lambda - \mathcal{I}^\Lambda \tilde{\mathfrak{D}}_p \mathcal{I}_\Lambda\right] \, ,$$

(if $\mathcal{I}_\Lambda \propto \mathcal{I}^\Lambda$ then $\omega = 0$).

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5 – A simple example with gauge group $SU(2)$

This is the simplest case.

Acoording to the general discussion we must consider a model of $N = 2, d = 4$ supergavity must have at least 3 vector multiplets transforming in the adjoint of $SU(2)$ (in practice, $SO(3)$).

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Just follow the RECIPE!

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If we identify the Higgs field Φ^{i}

$$\Phi^{i} \equiv -\tfrac{1}{\sqrt{2}}\mathcal{I}^{i},$$

then the non-Abelian equation is the Bogomol'nyi equation.

6 – The $SU(2)$ Bogomol'nyi equation

Let us consider the Georgi–Glashow model: an $SU(2)$ gauge field A^i coupled to a Higgs fields Φ^i with a potential $V(\Phi) = \frac{1}{2}\lambda[\mathrm{Tr}(\Phi^2) - 1]^2$

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In the Bogomol'nyi–Prasad–Sommerfield (BPS) $\lambda = 0$ and for time-independent, magnetic (${A^i}_t = 0$) configurations, the above action can be rewritten, up to a total derivative, in the form

$$S = -\tfrac{1}{2} \int d^4x \, \mathrm{Tr}(F_{mn} \pm \epsilon_{mnp}\mathfrak{D}_p\Phi)^2 \ ,$$

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Configurations that satisfy this first-order equation satisfy the second-order Yang–Mills–Higgs equations automatically.

A well-known Ansatz to solve the Bogomol'nyi equations in the $SU(2)$ case is the "hedgehog" Ansatz, which mixes space and Lie-algebra indices:

$$\Phi^i = \delta^i{}_m f(r) x^m, \qquad A^i{}_m = -\epsilon^i{}_{mn} x^n h(r),$$

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The Bogomol'nyi equations become an system of ODFs for $f(r)$ and $h(r)$

$$\begin{cases} r\partial_r h + 2h - f(1 + gr^2 h) &= 0, \\ r\partial_r (h + f) - gr^2 h(h + f) &= 0. \end{cases}$$

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$$f_s = \frac{1}{gr^2} [1 - \mu r \coth(\mu r + s)], \qquad h_s = -\frac{1}{gr^2} \left[1 - \frac{\mu r}{\sinh(\mu r + s)}\right],$$

$$f_* = \frac{1}{gr^2} \left[\frac{1}{1 + \lambda^2 r}\right], \qquad h_* = -f_*.$$

Let us study a bit these solutions, which are going to use as seeds of $N = 2$, $d = 4$ SEYM solutions.

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☞ The solutions are potentially singular at $r = 0$ only. The only globally regular solution is the one corresponding to the value $s = 0$:

$$A^i{}_m = \frac{\mu}{g}\,\delta^{ip}\epsilon_{pmn}\,\frac{x^n}{r}\mathsf{G}_0(\mu r)\,, \quad \mathsf{G}_0(r) = \frac{1}{r} - \frac{1}{\sinh r}\,,$$

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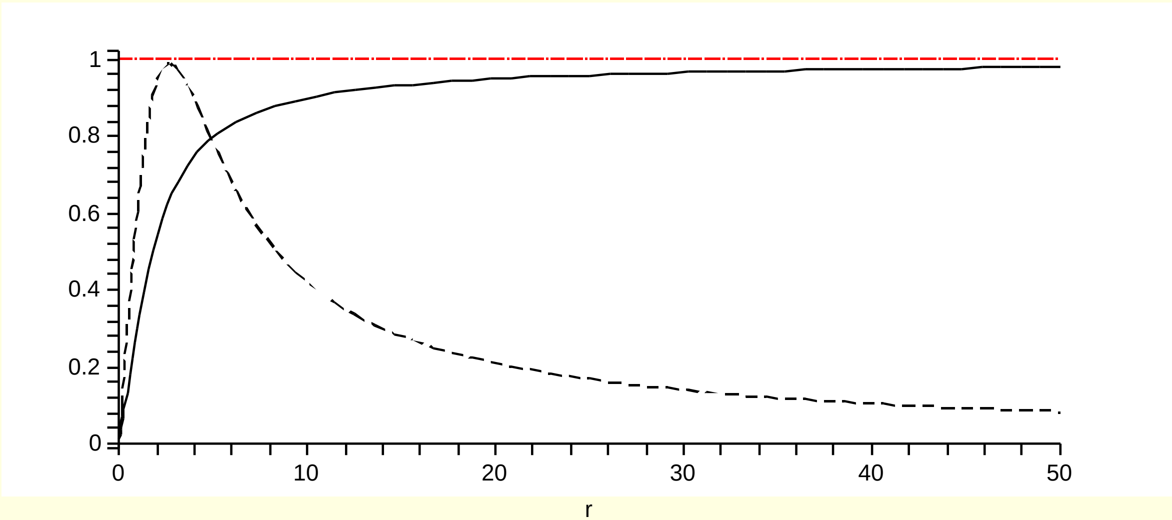
The profiles of the functions G_0 and H_0 are



$\mathcal{I}^{i}$ is regular at $r=0$ for $s=0$, and describes the 't Hooft-Polyakov monopole in the BPS limit.

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All the solutions of the 1-parameter family have magnetic charge $1/g$.

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...which is the construction of the physical fields out of $\mathcal{R}^\Lambda, \mathcal{I}^\Lambda, \mathcal{R}_\Lambda, \mathcal{I}_\Lambda$

This construction will impose constraints on the integration constants $\mu, s, A^0, A_0, p^0, q_0, \lambda$.

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$$ds^2 \;=\; e^{2U}(dt + \omega)^2 - e^{-2U} dx^m dx^m \,,$$

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  Regularity requires either $H^0 \neq 0$ or $H_0 \neq 0$ (some times $p^0 \neq 0$ or $q_0 \neq 0$).

Finally...

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$$Z^i = \frac{\mathcal{L}^i}{\mathcal{L}^0} = \frac{\mathcal{L}^i/X}{\mathcal{L}^0/X} = \frac{\mathcal{R}^i + i\mathcal{I}^i}{\mathcal{R}^0 + i\mathcal{I}^0}\,.$$

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$$Z^i = \frac{-\sqrt{2} r f}{H^0 + 2iH_0} \delta^i{}_m \frac{x^m}{r} .$$

Asymptotically ($r \to \infty$)

$$Z^i \longrightarrow \begin{cases} \dfrac{\sqrt{2}\mu/g}{A^0 + 2iA_0} \delta^i{}_m \dfrac{x^m}{r} , & \text{for } f_s \\ 0 & \text{for } f_* \end{cases}$$

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Now, study the solutions case by case

7 – Global 't Hooft-Polyakov Monopoles

Let us consider first the $p^0 = q_0 = 0$ case ($H^0 = A^0$, $H_0 = A_0$).

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Asymptotic flatness of the global monopole requires

$$\tfrac{1}{2}(A^0)^2 + 2(A_0)^2 = 1 + (\mu/g)^2\,.$$

Asymptotically, the scalars are covariantly constant:

$$Z^i \sim Z_\infty \delta^i{}_m \frac{x^m}{r}, \qquad Z_\infty \equiv \frac{-\mu/g}{1+(\mu/g)^2}\left(\tfrac{1}{\sqrt{2}}A^0 - \sqrt{2}iA_0\right)\,.$$

$|Z_\infty|^2$ is gauge-invariant and we get an expression for μ in terms of g and moduli:

$$\mu^2 = \frac{|Z_\infty|^2}{1 - |Z_\infty|^2} g^2 \,,$$

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Using all this, we get for the mass of the global monopole solution

$$M_{\text{monopole}} = \sqrt{\frac{|Z_\infty|^2}{1-|Z_\infty|^2}} \frac{1}{g} > 0 \, .$$

It saturates a moduli-dependent BPS bound.

8 – Coloured supersymmetric black holes

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We solve the constraint $q_0 A^0 - p^0 A_0 = 0$ by introducing a non-vanishing constant β

$$\frac{A^0}{p^0/\sqrt{2}} = \frac{A_0}{q_0/\sqrt{2}} \equiv 1/\beta, \quad \Rightarrow \quad \left\{ \begin{array}{rcl} H^0 & = & H p^0/(\sqrt{2}\beta), \\ H_0 & = & H q_0/(\sqrt{2}\beta), \end{array} \right. \qquad \text{where} \quad H \equiv 1 + \frac{\beta}{r}.$$

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The normalization of $e^{-2U} = 1$ at infinity implies that

$$\beta^2 = \frac{W_{\rm RN}(\mathcal{Q})/2}{1 + (\mu/g)^2}, \qquad W_{\rm RN}(\mathcal{Q})/2 \equiv \tfrac{1}{2}(p^0)^2 + 2(q_0)^2\,.$$

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(For the isolated solution f_* we have $\mu = 0$.)

The asymptotic behavior of the scalars is the same as in the previous case with Z_∞ given by

$$Z_\infty \equiv \frac{\beta\mu/g}{W_{\rm RN}(\mathcal{Q})/\sqrt{2}} \left(\tfrac{1}{\sqrt{2}} p^0 - \sqrt{2} i q_0 \right), \qquad |Z_\infty|^2 \equiv \frac{\beta^2 (\mu/g)^2}{W_{\rm RN}(\mathcal{Q})/2} \,,$$

Then we can identify

$$\mu^2 = \frac{|Z_\infty|^2}{1-|Z_\infty|^2} g^2\,, \qquad \beta^2 = (1-|Z_\infty|^2) W_{\mathrm{RN}}(\mathcal{Q})/2\,.$$

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Now we can write the full solution in terms of physical parameters (plus s, the Protogenov hair and λ, which is another kind of non-Abelian hair.

In particular, the mass and entropy are given by

$$M = \sqrt{\frac{W_{RN}(\mathcal{Q})/2}{1-|Z_\infty|^2}} + M_{\rm monopole} \,, \qquad M_{\rm monopole} = \sqrt{\frac{|Z_\infty|^2}{1-|Z_\infty|^2}} \frac{1}{g} \,,$$

$$S/\pi = \tfrac{1}{2}\left[W_{\rm RN}(\mathcal{Q}) - \frac{1}{g^2}\right] , \quad \text{for} \quad s \neq 0 \ \text{ and } \ |Z_\infty| = 0 \,,$$

$$S/\pi = \tfrac{1}{2} W_{\rm RN}(\mathcal{Q}) , \quad \text{for} \quad s = 0 \,.$$

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- in the family $s \neq 0$, the charges p^0, q_0 must be such that

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  When the equality is possible (only for certain values of g because p^0 and q_0 must be quantized), the solutions are global monopoles.

- The near-horizon limit of the scalars is in all cases (except $s = 0$ in which $Z^i_{\rm h} = 0$)

$$Z^i_{\rm h} = \frac{-1/g}{(\frac{1}{2}p^0 + iq_0)}\delta^i{}_m\frac{x^m}{r}\,.$$

  Since the magnetic charge is $1/g$ in all cases except in the isolated one, we can say that the attractor mechanism also works here (in a covariant way) except in

the isolated case in which $1/g$ is not a charge.

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9 – Black Hedgehogs

In the $s \to \infty$ limit (Wu–Yang $SU(2)$ monopole, rf_∞ harmonic) the scalars are covariantly constant everywhere

$$Z^i = Z\delta^i{}_m \frac{x^m}{r}, \qquad Z = \frac{-\sqrt{2}/g}{p^0/\sqrt{2} + i\sqrt{2}q_0} = Z_\infty .$$

and their energy-momentum tensor vanishes. The solutions are also solutions of the pure Einstein–Yang–Mills theory.

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These solutions have been called *black merons* (Canfora & Giacomini, 2012) and *black hedgehogs* (Hübscher, Meessen, O., Vaula 2007) but were also previously obtained by Perry (1977), Wang (1975), Bais & Russell (1975), Cho & Freund (1975), Yasskin (1975).

10 – Two-center non-Abelian solutions

Using two-center solutions of the Bogomol'nyi equations one can construct two-center $N = 2, d = 4$ supergavity solutions (`arXiv:1412.5547`).

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Define the coordinates relative to each of those centers and the relative position by

$$r^m \equiv x^m - x_0^m \,, \qquad u^m \equiv x^m - x_1^m \,, \qquad d^m \equiv u^m - r^m = x_0^m - x_1^m \,,$$

andd their norms by respectively, r, u and d.

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andd their norms by respectively, r, u and d.

The, the Higgs and gauge fields are given by...

$$\pm\Phi^i = \frac{1}{g}\delta^i{}_m\left\{\left[\frac{1}{r}-\left(\mu+\frac{1}{u}\right)\frac{K}{L}\right]\frac{r^m}{r}+\frac{2r}{uL}\left(\delta^{mn}-\frac{r^m r^n}{r^2}\right)d^n\right\},$$

$$A^i = -\frac{1}{g}\left[\frac{1}{r}-\frac{\mu\mathrm{D}+2d+2u}{\mathrm{L}}\right]\frac{\varepsilon^i{}_{mn}r^m dx^n}{r} + 2\frac{\mathrm{K}}{\mathrm{L}}\frac{\varepsilon_{npq}d^n u^p dx^q}{u\mathrm{D}}\,\delta^i{}_m\frac{r^m}{r}$$

$$-\frac{2r}{u\mathrm{L}}\,\delta^i{}_m\left(\delta^{mn}-\frac{r^m r^n}{r^2}\right)\varepsilon_{npq}u^p dx^q\,,$$

where the functions K, L, D of u and r are defined by

$$K \equiv \left[(u+d)^2+r^2\right]\cosh\mu r+2r(u+d)\sinh\mu r\,,$$

$$L \equiv \left[(u+d)^2+r^2\right]\sinh\mu r+2r(u+d)\cosh\mu r\,,$$

$$\mathrm{D} = 2\,(ud+u^m d^m) = (d+u)^2-r^2\,.$$

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A^i &= -\frac{1}{g}\left[\frac{1}{r}-\frac{\mu\mathrm{D}+2d+2u}{\mathrm{L}}\right]\frac{\varepsilon^i{}_{mn}r^m dx^n}{r} + 2\frac{\mathrm{K}}{\mathrm{L}}\frac{\varepsilon_{npq}d^n u^p dx^q}{u\mathrm{D}}\,\delta^i{}_m\frac{r^m}{r} \\
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K &\equiv \left[(u+d)^2+r^2\right]\cosh\mu r + 2r(u+d)\sinh\mu r\,, \\
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This solution is completely regular (Blair & Cherkis, 2010) and we can just use it as the main ingredient in our recipe for the $\overline{\mathbb{CP}}^3$ model.

The two-center solution of $N = 2, d = 4$ supergavity is completely defined by

$$
\begin{aligned}
\mathcal{I}^0 &= A^0 + \frac{p_r^0/\sqrt{2}}{r} + \frac{p_u^0/\sqrt{2}}{u}\,, \\
\mathcal{I}_0 &= A_0 + \frac{q_{r,0}/\sqrt{2}}{r} + \frac{q_{u,0}/\sqrt{2}}{u}\,, \\
\mathcal{I}^i &= \mp\sqrt{2}\Phi^i(r,u)\,, \\
\mathcal{I}_i &= 0\,.
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$$

The two-center solution of $N = 2, d = 4$ supergavity is completely defined by

$$\begin{aligned} \mathcal{I}^0 &= A^0 + \frac{p^0_r/\sqrt{2}}{r} + \frac{p^0_u/\sqrt{2}}{u}\,, \\ \mathcal{I}_0 &= A_0 + \frac{q_{r,0}/\sqrt{2}}{r} + \frac{q_{u,0}/\sqrt{2}}{u}\,, \\ \mathcal{I}^i &= \mp\sqrt{2}\Phi^i(r,u)\,, \\ \mathcal{I}_i &= 0\,. \end{aligned}$$

The metric and scalar fields are given by

$$e^{-2U} = \tfrac{1}{2}(\mathcal{I}^0)^2 + 2(\mathcal{I}_0)^2 - \Phi^i\Phi^i\,, \qquad Z^i = \frac{\mp\sqrt{2}\Phi^i}{\mathcal{I}^0 + 2i\mathcal{I}_0}\,.$$

and we just have to tune the integration constants for these fields to be regular and the metric static and normalized at infinity.

In the general case, with all the charges $p_r^0, p_u^0, q_{r\,0}, q_{u\,0}$ switched on the system describes two black holes in equilibrium with entropies

$$S_u/\pi = \tfrac{1}{2}W_{\rm RN}(\mathcal{Q}_u)/2 - \frac{1}{g^2}\,, \qquad S_r/\pi = \tfrac{1}{2}W_{\rm RN}(\mathcal{Q}_r)/2\,,$$

and *masses*

$$\begin{aligned} M &= M_r + M_u\,, \\ M_r &= -M_{\rm monopole}\,, \\ M_u &= \sqrt{\tfrac{1}{2}\frac{W_{RN}(\mathcal{Q}_u)}{1-|Z_\infty|^2}} + M_{\rm monopole}\,, \\ M_{\rm monopole} &= \sqrt{\frac{|Z_\infty|^2}{1-|Z_\infty|^2}\frac{1}{g}}\,. \end{aligned}$$

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We would like to find non-Abelian black holes in higher dimensions. The simplest class of theories to be considered are $N = 2, d = 5$ non-Abelian gauged supergravities.

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- ☞ First we want to know how the monopoles become instantons by that mechanism.

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$$d\hat{s}^{\,2} = H^{-1}(dz + \omega)^2 + H dx^m dx^m \qquad (m = 1, 2, 3)\,,$$

where (unhatted $\Rightarrow \mathbb{E}^3$)

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Then, the 3-dimensional gauge and Higgs fields A and Φ defined by

$$\Phi \equiv -H\hat{A}_z\,,$$

$$A_m \equiv \hat{A}_m - \omega_m \hat{A}_z\,,$$

satisfy the Bogomol'nyi equation in $\mathbb{E}^3$ $\mathcal{D}_m\Phi = \frac{1}{2}\epsilon_{mnp}F_{np}$.

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We may obtain black holes, but beware of the singularities!!.

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Let's see what we can get

from the coloured monopole

13 – From d=4 to d=5 N=2 gauged supergravity

(Just the basic facts)

The dimensional reduction of any $N = 2, d = 5$ ungauged supergravity gives a $N = 2, d = 4$ ungauged supergravity of the *cubic* type.

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A $d = 4$ model admitting a $SO(3)$ gauging which can be uplifted to $d = 5$ is the $ST[2, 4]$ (a consistent truncation of the Heterotic string on T^6)

$$\mathcal{F}(\mathcal{X}) = -\tfrac{1}{3!}\frac{d_{ijk}\mathcal{X}^i\mathcal{X}^j\mathcal{X}^k}{\mathcal{X}^0}\,, \qquad (d_{1\alpha\beta}) = (\eta_{\alpha\beta}) = \mathrm{diag}(+---)\,, \qquad \alpha, \beta = 2, 3, 4, 5\,.$$

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$SO(3)$ acts on $\alpha = 3, 4, 5$. The $d = 5$ model admits exactly the same gauging.

Instead of giving the relation between all the fields of both theories we can just give the relation between the *H-variables* which are harmonic functions on $\mathbb{R}^3$.

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An alternative definition of the theory is in terms of the Hesse potential $\mathsf{W}(\mathcal{I})$ which gives the metric function of black-hole solutions:

$$e^{-2U} = 2\sqrt{\left(\eta^{\alpha\beta}\mathcal{I}_\alpha\mathcal{I}_\beta - 2\mathcal{I}^1\mathcal{I}_0\right)\left(\eta_{\alpha\beta}\mathcal{I}^\alpha\mathcal{I}^\beta + 2\mathcal{I}^0\mathcal{I}_1\right) - \left(\mathcal{I}^0\mathcal{I}_0 - \mathcal{I}^1\mathcal{I}_1 + \mathcal{I}^\alpha\mathcal{I}_\alpha\right)^2}\,.$$

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The metric of static supersymmetric 5-dimensional solutions is of the form

$$d\hat{s}^2 = f^2 dt^2 - f^{-1} h_{mn} dx^m dx^n\,,$$

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In particular

$$f^{-1} = \frac{1}{H}\left[\tfrac{1}{4}\left(6L_0H + \eta_{\alpha\beta}K^\alpha K^\beta\right)\left(9H^2\eta^{\alpha\beta}L_\alpha L_\beta + 6HK^0L_\alpha K^\alpha + (K^0)^2\eta_{\alpha\beta}K^\alpha K^\beta\right)\right]^{1/3}\,.$$

The relation between the 4- and 5-dimensional harmonic functions is

$$H = -2\mathcal{I}^0\,, \quad M = -\mathcal{I}_0\,, \quad L_\alpha = -\tfrac{2}{3}\mathcal{I}_\alpha\,, \quad L_0 = -\tfrac{2}{3}\mathcal{I}_1\,, \quad K^0 = -2\mathcal{I}^1\,, \quad K^\alpha = -2\mathcal{I}^\alpha\,,$$

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Thus, in order to use Kronheimer's inverse mechanism to produce black holes we need 4-dimensional solutions with $\mathcal{I}^0 = -\frac{1}{2r}$ and $\mathcal{I}^\alpha = -\sqrt{2}\delta^\alpha{}_i\Phi^i$ for the Higgs field of the "coloured monopole". Adding $U(1)$ fields to have a regular horizon

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The integration constants can be adjusted to have a regular BH as in the $\overline{\mathbb{CP}}^3$ model, but regular in 4d means in there singular in 5d and, therefore, it is convenient to choose them only after uplifting. Remember we must change the radial coordinate $r = \rho^2/4$!!

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We have obtained the first non-Abelian, supersymmetric, statisc and asymptotically flat black hole in $d = 5$, which I have the pleasure to introduce to you $\longrightarrow$

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The black hole has only one non-trivial scalar, ϕ^1.

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All the fields are determined by the harmonic functions

$$\mathcal{I}^0 = \frac{-2}{\rho^2}\,, \quad \mathcal{I}_1 = -2^{-4/3}(\phi^1_\infty)^{2/3} + \frac{4q_1}{\rho^2}\,, \quad \mathcal{I}_2 = -(2\phi^1_\infty)^{-1/3} + \frac{4q_2}{\rho^2}\,,$$

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The metric of the solution is

$$d\hat{s}^2 = f^2 dt^2 - f^{-1}\left(d\rho^2 + \rho^2 d\Omega^2_{(3)}\right)\,, \qquad f = -\left[2(\mathcal{I}_2)^2\left(2\mathcal{I}_1 - \frac{(\mathcal{I}^\alpha)^2}{\mathcal{I}^0}\right)\right]^{-1/3}\,,$$

and describes a regular static black hole under the conditions

$$\text{sign}(q_1) = -1\,, \qquad \text{sign}(q_2) \neq \text{sign}(\phi^1_\infty)\,.$$

The rest of the non-vanishing physical fields are

$$\phi^1 = \frac{-(\mathcal{I}^\alpha)^2 + 2\mathcal{I}^0\mathcal{I}_1}{\mathcal{I}_2\mathcal{I}^0}\,,$$

and the vectors

$$\begin{cases} \hat{A}^0 = -\dfrac{4\sqrt{3}\mathcal{I}^0(\mathcal{I}_2)^2}{e^{-4U}}dt\,, \\ \hat{A}^1 = -\dfrac{\sqrt{3}}{\mathcal{I}_2}dt\,, \\ \hat{A}^\alpha = -\dfrac{2\sqrt{6}}{g(1+\lambda^2\rho^2/4)}\delta^\alpha{}_i v^i\,, \end{cases}$$

where v^i are the $SU(2)$ left-invariant Maurer-Cartan 1-forms.

The rest of the non-vanishing physical fields are

$$\phi^1 = \frac{-(\mathcal{I}^\alpha)^2 + 2\mathcal{I}^0\mathcal{I}_1}{\mathcal{I}_2\mathcal{I}^0}\,,$$

and the vectors

$$\begin{cases} \hat{A}^0 = -\dfrac{4\sqrt{3}\mathcal{I}^0(\mathcal{I}_2)^2}{e^{-4U}}dt\,, \\[2ex] \hat{A}^1 = -\dfrac{\sqrt{3}}{\mathcal{I}_2}dt\,, \\[2ex] \hat{A}^\alpha = -\dfrac{2\sqrt{6}}{g(1+\lambda^2\rho^2/4)}\delta^\alpha{}_i v^i\,, \end{cases}$$

where v^i are the $SU(2)$ left-invariant Maurer-Cartan 1-forms.

The mass and entropy of the black hole are given by

$$M = 2^{4/3}\pi\left[\frac{1}{(\phi^1_\infty)^{2/3}}|q_1| + (\phi^1_\infty)^{1/3}q_2\right]\,, \qquad S = 8\pi^2\left[\left(-2\frac{1}{g^2} + |q_1|\right)q_2^2\right]^{1/2}\,.$$

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- There many new, potentially interesting, black-hole solutions than can be obtained in this way **whose entropies need to be explained**. Also string- and black-ring solutions (work in progress).

THANKS!