Black Hole Entropy Related to Measures of Entanglement

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Plan of the talk

1. Motivation: the Black Hole-Qubit Correspondence (BHQC)

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4. Hitchin functionals related to measures of entanglement
5. Hitchin functionals and black hole entropy
6. Hitchin functionals and Freudenthal systems
7. Speculations on the physical basis of the BHQC.

P. L. and G Sárosi: arXiv:1206.5066, P.L, Sz. Nagy, J. Pipek and G. Sárosi, P.L and F. Holweck, arXiv:1502.04537

The main correspondence is between the structure of the Bekenstein-Hawking entropy formulas in extremal BPS or non BPS black hole solutions in supergravity and certain multipartite entanglement measures of composite quantum systems with either distinguishable or indistinguishable constituents.

As an other aspect of the correspondence it has also been realized that the classification problem of entanglement types of special entangled systems and special types of black hole solutions can be mapped to each other .

Apart from structural correspondences the BHQC also addressed issues of dynamics. The attractor mechanism as a "distillation" procedure.

1. Similar symmetry structures

On the string theory side there are the U-duality group leaving invariant the black hole entropy formulas, on the other hand on the quantum information theoretic side there are the groups of admissible transformations used to represent local manipulations on the entangled subsystems leaving invariant the corresponding entanglement measures.

Motivation: What is the reason for the BHQC?

1. Similar symmetry structures
2. **Stability**

On the string theory side there are the U-duality group leaving invariant the black hole entropy formulas, on the other hand on the quantum information theoretic side there are the groups of admissible transformations used to represent local manipulations on the entangled subsystems leaving invariant the corresponding entanglement measures.

Measures of entanglement for special entangled systems.

Fermions on an M dimensional single particle Hilbert space $V = \mathbb{C}^M$. The Hilbert space is spanned by the basis

$$\left(f_1^\dagger\right)^{n_1}\left(f_2^\dagger\right)^{n_2}\dots\left(f_M^\dagger\right)^{n_M}|0\rangle$$

For the N particle subspace $\sum_j n_j = N$.
E.g. a three fermion state ($N = 3$) with six single particle states or *modes* ($M = 6$) is

$$|P\rangle = \sum_{1\leq i_1<i_2<i_3\leq 6} P_{i_1 i_2 i_3} f_{i_1}^\dagger f_{i_2}^\dagger f_{i_3}^\dagger |0\rangle$$

$$P = \sum_{1\leq i_1<i_2<i_3\leq 6} P_{i_1 i_2 i_3} e^{i_1} \wedge e^{i_2} \wedge e^{i_3} \in \wedge^3 V^*$$

$\{e^j\}$ basis of V^* , $\{e_j\}$ basis of V.

SLOCC transformations

$$|P\rangle \mapsto (S \otimes S \otimes S)|P\rangle$$

$$P_{i_1 i_2 i_3} \mapsto P_{j_1 j_2 j_3} S^{j_1}{}_{i_1} S^{j_2}{}_{i_2} S^{j_3}{}_{i_3}, \qquad S^j{}_i \in GL(6, \mathbb{C})$$

The SLOCC entanglement classes are the orbits under this action.

Measures of entanglement for special entangled systems.

Let

$$(1,2,3,4,5,6) \leftrightarrow (1,2,3,\overline{1},\overline{2},\overline{3})$$

$$\eta \equiv P_{123}, \qquad \xi \equiv P_{\overline{123}}$$

$$X = \begin{pmatrix} X_{11} & X_{12} & X_{13} \\ X_{21} & X_{22} & X_{23} \\ X_{31} & X_{32} & X_{33} \end{pmatrix} \equiv \begin{pmatrix} P_{1\overline{23}} & P_{1\overline{31}} & P_{1\overline{12}} \\ P_{2\overline{23}} & P_{2\overline{31}} & P_{2\overline{12}} \\ P_{3\overline{23}} & P_{3\overline{31}} & P_{3\overline{12}} \end{pmatrix}$$

$$Y = \begin{pmatrix} Y_{11} & Y_{12} & Y_{13} \\ Y_{21} & Y_{22} & Y_{23} \\ Y_{31} & Y_{32} & Y_{33} \end{pmatrix} \equiv \begin{pmatrix} P_{\overline{1}23} & P_{\overline{1}31} & P_{\overline{1}12} \\ P_{\overline{2}23} & P_{\overline{2}31} & P_{\overline{2}12} \\ P_{\overline{3}23} & P_{\overline{3}31} & P_{\overline{3}12} \end{pmatrix}.$$

$$\mathcal{D}(P) = [\eta\xi - \mathrm{Tr}(XY)]^2 - 4\mathrm{Tr}(X^\sharp Y^\sharp) + 4\eta\mathrm{Det}(X) + 4\xi\mathrm{Det}(Y)$$

$\mathcal{D}(P)$ defines an entanglement measure.

$$0 \leq \mathcal{T}_{123} = 4|\mathcal{D}(P)|$$

where $\mathcal{T}_{123} \leq 1$ for normalized states.

Measures of entanglement for special entangled systems.

SLOCC classes over $\mathbb{C}$

$$P = e^1\wedge e^2\wedge e^3+e^1\wedge e^{\overline{2}}\wedge e^{\overline{3}}+e^2\wedge e^{\overline{3}}\wedge e^{\overline{1}}+e^3\wedge e^{\overline{1}}\wedge e^{\overline{2}}, \quad \mathcal{D}(P)\neq 0$$

$$P = e^1\wedge e^2\wedge e^3+e^1\wedge e^{\overline{2}}\wedge e^{\overline{3}}+e^2\wedge e^{\overline{3}}\wedge e^{\overline{1}}, \qquad \mathcal{D}(P)=0, \quad \tilde{P}\neq 0$$

$$P = e^1\wedge (e^2\wedge e^3+e^{\overline{2}}\wedge e^{\overline{3}}, \qquad \mathcal{D}(P)=0, \quad \tilde{P}=0$$

$$P = e^1\wedge e^2\wedge e^3, \qquad \mathcal{D}(P)=0, \quad \tilde{P}=0.$$

SLOCC classes over $\mathbb{R}$

$$\varrho_+ = e^1\wedge e^2\wedge e^3+e^1\wedge e^{\overline{2}}\wedge e^{\overline{3}}+e^2\wedge e^{\overline{3}}\wedge e^{\overline{1}}+e^3\wedge e^{\overline{1}}\wedge e^{\overline{2}}, \qquad \mathcal{D}(\varrho_+)>0$$

$$\varrho_- = e^1\wedge e^2\wedge e^3-e^1\wedge e^{\overline{2}}\wedge e^{\overline{3}}-e^2\wedge e^{\overline{3}}\wedge e^{\overline{1}}-e^3\wedge e^{\overline{1}}\wedge e^{\overline{2}}, \qquad \mathcal{D}(\varrho_-)<0$$

Measures of entanglement for special entangled systems

$$F^{1,2,3} \equiv e^{1,2,3} + e^{\overline{1},\overline{2},\overline{3}}, \qquad F^{\overline{1},\overline{2},\overline{3}} \equiv e^{1,2,3} - e^{\overline{1},\overline{2},\overline{3}}$$

$$E^{1,2,3} \equiv e^{1,2,3} + ie^{\overline{1},\overline{2},\overline{3}}, \qquad E^{\overline{1},\overline{2},\overline{3}} \equiv e^{1,2,3} - ie^{\overline{1},\overline{2},\overline{3}}$$

$$\varrho_+ = \frac{1}{2}\left(F^1 \wedge F^2 \wedge F^3 + F^{\overline{1}} \wedge F^{\overline{2}} \wedge F^{\overline{3}}\right), \qquad \mathcal{D}(\varrho_+)$$

$$\varrho_- = \frac{1}{2}\left(E^1 \wedge E^2 \wedge E^3 + E^{\overline{1}} \wedge E^{\overline{2}} \wedge E^{\overline{3}}\right), \qquad \mathcal{D}(\varrho_-)$$

Notice that these are states similar to the GHZ states known for three-qubits. This is not a coincidence. Ordinary three qubits and three *bosonic qubits* can be described in this formalism. The invariants $D(P)$ and $d(P)$ arising from $\mathcal{D}(P)$ are Cayley's hyperdeterminant and the discriminant function for cubic curves. Other special entangled tripartite systems containing bosons and fermions can also be described by **embedding** them into **three fermion systems** with **six modes**.

Stability

A prehomogeneous vector space (PV) is a triple $(G, R, \mathcal{V})$ where $\mathcal{V}$ is a finite dimensional vector space over $\mathbb{C}$, G is a group and R is a representation $R : G \to GL(V)$ such that for a generic element $v \in \mathcal{V}$ G **has an open dense orbit** $R(G)v$ in $\mathcal{V}$. An element $v \in \mathcal{V}$ is called *stable* if it lies in such an open orbit of G.

For a PV one should have $\mathrm{dim}G - \mathrm{dim}G_v = \mathrm{dim}\mathcal{V}$ where G_v is the stabilizer of a $v \in \mathcal{V}$.

Stability means that states in a neighborhood of a particular one are equivalent with respect to the group G of local manipulations.

Now $G = GL(6, \mathbb{C})$, $\mathcal{V} = \wedge^3 V^*$, R is just the SLOCC action. One can show that $G_v = SL(3, \mathbb{C}) \times SL(3, \mathbb{C})$ for the GHZ class with $\mathcal{D} \neq 0$.

$$36 - 16 = 20 \leftrightarrow \mathrm{dim}G - \mathrm{dim}G_v = \mathrm{dim}\mathcal{V}$$

Hitchin's functional

Let us consider the *real vector space* $W = \mathbb{R}^6$ and a three-form $\varrho \in \wedge^3 W^*$. Define

$$(K_\varrho)^a{}_b = \frac{1}{2!3!}\varepsilon^{ac_2c_3c_4c_5c_6}\varrho_{bc_2c_3}\varrho_{c_4c_5c_6}$$

Hitchin's invariant is

$$\lambda(\varrho) = \frac{1}{6}\mathrm{Tr}K_\varrho^2.$$

Notice that for $\varrho \equiv P$ then

$$\mathcal{D}(\varrho) = \lambda(\varrho)$$

Hitchin's invariant

Define

$$I_\varrho \equiv K_\varrho / \sqrt{|\mathcal{D}(\varrho)|}$$

Now one can show that

$$I_\varrho^2 = -id, \qquad \mathcal{D}(\varrho) < 0$$

This means that I_ϱ defines a ϱ dependent complex structure on W.

On the other hand define

$$\tilde{\varrho}_{abc} = \varrho_{dbc} (K_\varrho)^d{}_a$$

then

$$\hat{\varrho}(\varrho) = \frac{\tilde{\varrho}}{\sqrt{|\mathcal{D}(\varrho)|}}, \qquad \textbf{Freudenthal dual}$$

which satisfies

$$2\mathrm{sgn}(\mathcal{D})\sqrt{|\mathcal{D}(\varrho)|}\epsilon = \varrho \wedge \hat{\varrho}(\varrho) \qquad \epsilon = e^1 \wedge e^2 \wedge e^3 \wedge e^4 \wedge e^5 \wedge e^6$$

Hitchin's invariant

If $\mathcal{D}(\varrho) \neq 0$ i.e. it belongs to one of the real **stable** orbits then

$$\alpha = \varrho + \hat{\varrho}(\varrho), \quad \beta = \varrho - \hat{\varrho}(\varrho), \qquad \mathcal{D}(\varrho) > 0$$

$$\Omega = \varrho + i\hat{\varrho}(\varrho), \quad \overline{\Omega} = \varrho - i\hat{\varrho}(\varrho), \qquad \mathcal{D}(\varrho) < 0$$

are belonging to the fully separable entanglement class, hence

$$\varrho = \frac{1}{2}(\alpha + \beta), \qquad \mathcal{D}(\varrho) > 0$$

$$\varrho = \frac{1}{2}(\Omega + \overline{\Omega}), \qquad \mathcal{D}(\varrho) < 0$$

are of the GHZ forms.

With respect to the complex structure I_ϱ the separable state (complex Slater determinant) Ω is of type $(3,0)$. This metod of finding the GHZ form of any stable state works also over $\mathbb{C}$.

Hitchin's functional

Now M is a **real** closed oriented 6-manifold and ϱ is a three-form.

$$\varrho = \frac{1}{3!}\varrho_{abc}(x)dx^a \wedge dx^b \wedge dx^c \in \wedge^3 T^* M$$

Hitchin's functional is defined as

$$V_H(\varrho) = \int_M \sqrt{|\mathcal{D}(\varrho)|}d^6x = \frac{1}{2}\mathrm{sgn}(\mathcal{D}(\varrho)) \int_M \varrho \wedge \hat{\varrho}(\varrho).$$

Let $\mathcal{D} < 0$ and $[\varrho] \in H^3(M, \mathbb{R})$ i.e. $\varrho = \varrho_0 + d\sigma$ and $d\varrho = 0$ then

$$\delta_\sigma V_H = 0 \implies d\hat{\varrho}(\varrho) = 0.$$

Hence the separable state $\Omega = \varrho + i\hat{\varrho}(\varrho)$ of type $(3, 0)$ is closed and one can show that the almost complex structure I_ϱ is integrable. Hence a critical point or a classical solution of $V_H(\varrho)$ defines a complex structure on M with a non-vanishing holomorphic three-form Ω. **Calabi-Yau structures are coming from a functional related to an entanglement measure $\mathcal{D}$.**

Hichin functionals and black hole entropy

Basic idea: The microstates of **extremal** black holes are coming from wrapping configurations of branes and strings around nontrivial homology cycles of extra dimensions. Our manifold M of extra dimensions is a real 6 dimensional manifold. The duals of the cycles on M give rise to **cohomology classes of forms**. These are the objects interpreted as **entangled states**.

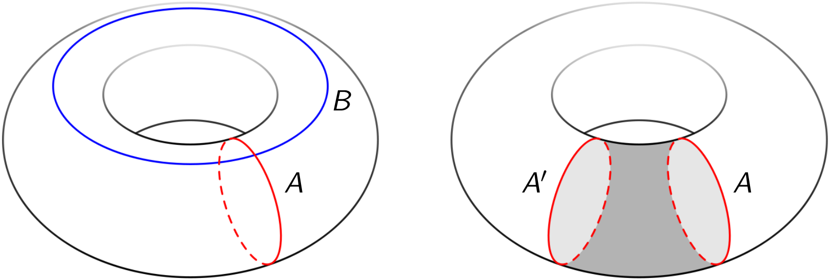
$$\gamma \in H_3(M, \mathbb{Z})$$

$$\Gamma \in H^3(M, \mathbb{Z})$$

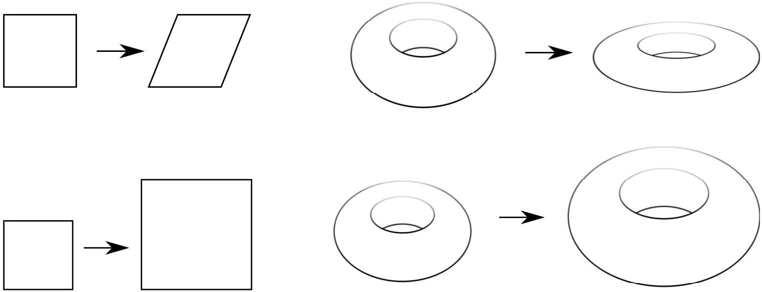
We make the identification

$$[\Gamma] \equiv \varrho \in H^3(M, \mathbb{R})$$

Hichin functionals and black hole entropy



Complex structure and Kähler structure deformations



Define a partition function (Dijkgraaf et. al. 2005) as

$$Z_H(\gamma) = \int_{[\varrho]=\Gamma} e^{V_H(\varrho)} \mathcal{D}\varrho$$

Using the method of steepest descent it is easy to demonstrate that

$$S_{BH} = \pi V_H(\varrho_{\mathrm{crit}}), \qquad [\varrho] = \Gamma.$$

This establishes a link between the value of the extremized action $V_H(\varrho)$ based on an entanglement measure $\mathcal{D}(\varrho)$ and the semiclassical (Bekenstein-Hawking) black hole entropy.
However one can even be more ambitious and conjecture that this is the correst formula accounting for also the quantum corrections. If this is true then we would be able to use the BHQC in a more general context.

Example: T^6

Real coordinates $u^i, v^i, i = 1, 2, 3$.

$$\alpha_0 = du^1 \wedge du^2 \wedge du^3, \qquad \alpha_{ij} = \frac{1}{2}\varepsilon_{ii'j'} du^{i'} \wedge du^{j'} \wedge dv^j$$

$$\beta^0 = -dv^1 \wedge dv^2 \wedge dv^3, \qquad \beta^{ij} = \frac{1}{2}\varepsilon_{ji'j'} du^i \wedge dv^{i'} \wedge dv^{j'} \tag{1}$$

as

$$\Gamma = p^0\alpha_0 + P^{ij}\alpha_{ij} - Q_{ij}\beta^{ij} - q_0\beta^0$$

The real three-form ϱ belonging to the class with $\mathcal{D}(\varrho) < 0$

$$\varrho = \sum_{1 \le a < b < c \le 6} \varrho_{abc} f^a \wedge f^b \wedge f^c$$

where

$$(f^1, f^2, f^3, f^4, f^5, f^6) \equiv (du^1, du^2, du^3, dv^1, dv^2, dv^3)$$

Example: T^6

$$\Gamma = [\varrho]$$

$$p^0 = \varrho_{123}, \qquad \begin{pmatrix} P^{11} & P^{12} & P^{13} \\ P^{21} & P^{22} & P^{23} \\ P^{31} & P^{32} & P^{33} \end{pmatrix} = \begin{pmatrix} \varrho_{23\bar{1}} & \varrho_{23\bar{2}} & \varrho_{23\bar{3}} \\ \varrho_{31\bar{1}} & \varrho_{31\bar{2}} & \varrho_{31\bar{3}} \\ \varrho_{12\bar{1}} & \varrho_{12\bar{2}} & \varrho_{12\bar{3}} \end{pmatrix}$$

$$q_0 = \varrho_{\overline{123}}, \qquad \begin{pmatrix} Q_{11} & Q_{12} & Q_{13} \\ Q_{21} & Q_{22} & Q_{23} \\ Q_{31} & Q_{32} & Q_{33} \end{pmatrix} = \begin{pmatrix} \varrho_{1\overline{23}} & \varrho_{1\overline{31}} & \varrho_{1\overline{12}} \\ \varrho_{2\overline{23}} & \varrho_{2\overline{31}} & \varrho_{2\overline{12}} \\ \varrho_{3\overline{23}} & \varrho_{3\overline{31}} & \varrho_{3\overline{12}} \end{pmatrix}$$

Now a critical point of $V_H(\varrho)$ gives rise to a fully separable state of the form $\Omega = \varrho + i\hat{\varrho}(\varrho)$ where $\hat{\varrho}$ is the Freudenthal dual of ϱ expressed in terms of the charges.

Example: T^6

$$\hat{p}^0 = \frac{\tilde{p}^0}{\sqrt{-\mathcal{D}}}, \quad \hat{P} = \frac{\tilde{P}}{\sqrt{-\mathcal{D}}}$$

$$\hat{q}^0 = \frac{\tilde{q}^0}{\sqrt{-\mathcal{D}}}, \quad \hat{Q} = \frac{\tilde{Q}}{\sqrt{-\mathcal{D}}}$$

$$\tilde{p}^0 = -2N(P) - p^0(p^0 q_0 - (P, Q)),$$

$$\tilde{P} = 2(p^0 Q^\sharp - Q \times P^\sharp) - (p^0 q_0 - (P, Q))P$$

$$\tilde{q}^0 = 2N(Q) + q^0(p^0 q_0 - (P, Q))$$

$$\tilde{Q} = -2(q^0 P^\sharp - P \times Q^\sharp) + (p^0 q_0 - (P, Q))Q.$$

$$(A, B) = \mathrm{Tr}(AB), \qquad N(A) = \mathrm{Det}(A)$$

$$A \times B = (A + B)^\sharp - A^\sharp - B^\sharp$$

$$\mathcal{D} = [p^0 q_0 - (P, Q)]^2 - 4(P^\sharp, Q^\sharp) + 4p^0 N(Q) + 4q_0 N(P)$$

Example: T^6

Now this particular Ω arising from the critical point of $V_H(\varrho)$ can be expanded as

$$\Omega = C\Omega_0 = C\left(\alpha_0 + \tau^{jk}\alpha_{jk} + \tau^{\sharp}{}_{jk}\beta^{kj} - (\mathrm{Det}\tau)\beta^0\right)$$

One can then introduce complex coordinates

$$z^i = u^i + \tau^{ij}v^j$$

such that the separable form is manifest

$$\Omega = C\Omega_0 = Cdz^1 \wedge dz^2 \wedge dz^3 = \varrho + i\hat{\varrho}(\varrho)$$

Here for the expansion coefficients τ^{ij} fixing the complex structure of T^6 we chose the convention

$$\tau^{ij} = x^{ij} - iy^{ij}, \qquad y^{ij} > 0$$

The complex structure obtained from the extremization of Hitchin's functional is

$$\tau = \frac{P + i\hat{P}}{p^0 + i\hat{p^0}}$$

Finally

$$\tau = \frac{1}{2}\left[-(2PQ + [p^0 q_0 - (P, Q)]) + i\sqrt{-\mathcal{D}}\right](P^\sharp - p^0 Q)^{-1}.$$

Using this we obtain the final result

$$S_{BH} = \pi V_H(\varrho_{crit}) = \pi\sqrt{-\mathcal{D}}$$

This result shows that the semiclassical black hole entropy is given by the entanglement measure $\mathcal{D}$ for the three-fermion state.

Example: T^6

It is instructive to express $[\varrho] = \Gamma$ from in the form

$$\Gamma = \frac{1}{2}(C\Omega_0 + \overline{C\Omega_0})$$

Let us introduce the Hermitian inner product for three-forms as

$$\langle\varphi|\psi\rangle = \int_{T^6} \varphi \wedge *\overline{\psi}$$

One can then regard $H^3(T^6, \mathbb{C})$ equipped with $\langle\cdot|\cdot\rangle$ as a 20 dimensional Hilbert space. One can then see that

$$|\Gamma\rangle = (-\mathcal{D})^{1/4}\left(e^{i\alpha}|123\rangle - e^{-i\alpha}|\overline{123}\rangle\right), \qquad \tan\alpha = \frac{p^0}{\hat{p}^0}.$$

Notice that this "state" is of the GHZ-like form.

Quantum corrections

Use Hitchin's functional also to recover the quantum corrections that has already been calculated via topological string techniques. It turned (Pestun and Witten) that after appropriate gauge fixing at the one loop level there is a discrepancy between the result based on Hitchin's functional and the result of topological string theory. In order to resolve this discrepancy Pestun and Witten suggested to use a partition function based on the *generalized Hitchin functional* instead. Hitchin's functional is connected to Calabi-Yau structures on the other hand the generalized Hitchin functional (GHF) is connected to generalized Calabi-Yau structure. For the resolution they have chosen manifolds with $b_1(M) = 0$ where the critical points and classical values of both functionals coincide, however the quantum fluctuating degrees of the two functionals are different. The upshot of these consideration was that after a convenient interpretation it turns out that the conjecture of Dijkgraaf et.al. remains true even at the one loop level.

Three fermions with seven modes

Let $V = \mathbb{C}^7$ and

$$\mathcal{P} = \frac{1}{3!}\mathcal{P}_{I_1 I_2 I_3} e^{I_1} \wedge e^{I_2} \wedge e^{I_3} \in \wedge^3 V^*$$

Now $I, J, A, B, C = 1, \dots 7$ and $i, j, a, b, c = 1, \dots 6$. The SLOCC group is $GL(V) = GL(7, \mathbb{C})$ with the usual diagonal action.
We define

$$(M^A)^B{}_C = \frac{1}{12}\varepsilon^{ABI_1 I_2 I_3 I_4 I_5}\mathcal{P}_{CI_1 I_2}\mathcal{P}_{I_3 I_4 I_5}$$

$$N_{AB} = \frac{1}{24}\varepsilon^{I_1 I_2 I_3 I_4 I_5 i_6 I_7}\mathcal{P}_{AI_1 I_2}\mathcal{P}_{BI_3 I_4}\mathcal{P}_{I_5 I_6 I_7}$$

$$L^{AB} \equiv (M^A)^C{}_D (M^B)^D{}_C.$$

They are covariants with transformation properties

$$(M^A)^B{}_C \mapsto (\mathrm{Det} g')g^A{}_D g^B{}_E g'{}_C{}^F (M^D)^E{}_F$$

$$N_{AB} \mapsto (\mathrm{Det} g')g'{}_A{}^C g'{}_B{}^D N_{CD}$$

$$L^{AB} \mapsto (\mathrm{Det} g')^2 g^A{}_C g^B{}_D L^{CD}$$

Invariants

It is worth relating the seven mode case to the six mode one via a $35 = 20 + 15$ split.

$$\mathcal{P} = P + \omega \wedge e^7$$
$$\omega = \frac{1}{2}\omega_{ij} e^i \wedge e^j.$$

We form the following relative invariant

$$\mathcal{J}(\mathcal{P}) = \frac{1}{2^4 3^2 7} L^{AB} N_{AB}$$

with transformation property

$$\mathcal{J}(\mathcal{P}) \mapsto (\mathrm{Det} g')^3 \mathcal{J}(\mathcal{P}).$$

The invariant is very complicated, but if we employ the constraint

$$\omega \wedge P = 0$$

then we get

$$\mathcal{J}(\mathcal{P}) = \frac{1}{4} \mathrm{Pf}(\omega) \mathcal{D}(P)$$

One might think that a new relative invariant is $\mathrm{Det}(\boldsymbol{N})$ or a $\mathrm{Det}(\boldsymbol{L})$. However,

$$\mathrm{Det}(\boldsymbol{N}) = -6 \cdot (9\mathrm{Pf}(\omega)\mathcal{D}(P))^3$$

Note that when $V = \mathbb{R}^7$ the quantity

$$\mathcal{B}_{IJ} = -\frac{1}{6}N_{IJ}.$$

is used in string theory. Then we have the nice formula

$$\mathrm{Det}\mathcal{B} = (\mathcal{J}(\mathcal{P}))^3$$

Note that originally it was Engel who showed in 1900 that the polynomial $\mathcal{J}$ exists and it must be related to a symmetric bilinear form $(\mathcal{B}_{IJ})$ in this way.

SLOCC classification

This classification problem has been solved by Reichel in 1907. However, his classification was not complete. The number of nontrivial classes is nine and not seven as claimed by him. It was Schouten in 1931 who used much simpler methods to obtain a full classification. In this scheme one of the SLOCC classes plays a similar role than the famous GHZ class in the six mode case. Denote by e^A the basis vectors of the seven dimensional vector space V^* and by e^a the basis vectors of its six dimensional subspace. Define

$$E^{1,2,3} = e^{1,2,3} + ie^{4,5,6}, \qquad E^{\overline{1},\overline{2},\overline{3}} = e^{1,2,3} - ie^{4,5,6}, \qquad E^7 = ie^7$$

Then we take *GHZ*-like state of the six mode case

$$E^{123} + E^{\overline{123}} = 2(e^{123} - e^{156} + e^{246} - e^{345})$$

and we add to this the one: $(E^{1\overline{1}} + E^{2\overline{2}} + E^{3\overline{3}}) \wedge E^7$.

The GHZ-like state for seven modes

In this way we obtain the state $\mathcal{P}_0$ of the form

$$\mathcal{P}_0 = e^{123} - e^{156} + e^{246} - e^{345} + e^{147} + e^{257} + e^{367}$$

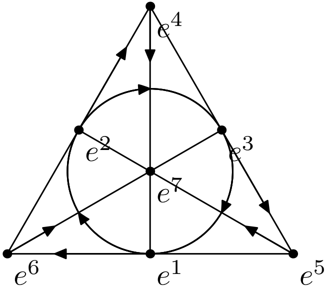


Figure: The oriented Fano plane. The points of the plane correspond to the basis vectors of the seven dimensional single particle space. The lines of the plane represent three fermion basis vectors with the arrows indicating the order of single particle states in them to get a plus sign.

The SLOCC classes and their relation to the Fano plane

Type	Kanonical form
NULL	0
SEP	e^{367}
BISEP	$e^{367} + e^{257}$
W	$e^{246} - e^{345} - e^{156}$
GHZ	$e^{123} - e^{156} + e^{246} - e^{345}$
SYMPL/NULL	$e^{147} + e^{257} + e^{367}$
SYMPL/SEP	$e^{123} + e^{147} + e^{257} + e^{367}$
SYMPL/BISEP	$e^{123} - e^{156} + e^{246} - e^{345} + e^{147}$
SYMPL/W	$e^{123} - e^{156} + e^{246} - e^{345} + e^{147} + e^{257}$
SYMPL/GHZ	$e^{123} - e^{156} + e^{246} - e^{345} + e^{147} + e^{257} + e^{367}$

Table: SLOCC classes for three fermions with seven modes

Metrics with G_2 holonomy

Now the three-form $\mathcal{P}$ belonging to the stable (dense) SLOCC orbit playing the role of an associative 3-form of a G_2 holonomy metric. The entanglement measure $\mathcal{J}(\mathcal{P})$ gives rise to a Hitchin functional defined for a real seven-manifold. The rank of the basic covariant $\mathcal{B}_{IJ}$ is characterizing the entanglement classes. Interestingly this quantity is a defining a metric tensor on the seven manifold.

$$g_{AB} = \mathrm{Det}(\mathcal{B})^{-1/9}\mathcal{B}_{AB}$$

This gives rise to an interesting link between the structure of g_{AB} and patterns of entanglement.

Rank of the metric as related to patterns of entanglement

Name	Type	Rank $N_{IJ}(\mathcal{P})$	$\mathcal{J}(\mathcal{P})$
I	NULL	0	0
II	SEP	0	0
III	BISEP	0	0
IV	W	0	0
V	GHZ	0	0
VI	SYMPL/NULL	1	0
VII	SYMPL/SEP	1	0
VIII	SYMPL/BISEP	2	0
IX	SYMPL/W	4	0
X	SYMPL/GHZ	7	$\neq 0$

Table: Entanglement classes of three fermions with seven single particle states.

An example

A. Brandhuber, J. Gomis, S. S. Gubser, S. Gukov, Nucl.Phys. B611 (2001) 179-204

$$\begin{aligned}\mathcal{P} &= \frac{9}{16} r_0^3 \varepsilon_{abc} \left(\sigma_a \wedge \sigma_b \wedge \sigma_c - \Sigma_a \wedge \Sigma_b \wedge \Sigma_c\right) \\ &+ d\left[\frac{r}{18}(r^2 - \frac{27}{4} r_0^2)(\sigma_1 \wedge \Sigma_1 + \sigma_2 \wedge \Sigma_2) + \frac{r_0}{3}(r^2 - \frac{81}{8})\sigma_3 \wedge \Sigma_3\right]\end{aligned}$$

$$\begin{aligned}\mathcal{P}_{r^*} &= \frac{54}{16} r_0^3 (E_1 - E_{\overline{1}}) \wedge (E_2 - E_{\overline{2}}) \wedge (E_3 - E_{\overline{3}}) \\ &+ A(r) dr \wedge (\sigma_1 \wedge \Sigma_1 + \sigma_2 \wedge \Sigma_2) + B(r) dr \wedge \sigma_3 \wedge \Sigma_3\end{aligned}$$

$$r^* \equiv \frac{9}{2} r_0, \qquad E_{1\overline{23}} \equiv \sigma_1 \wedge \Sigma_2 \wedge \Sigma_3.$$

At $r = r^*$ $\mathcal{P}$ belongs to the degenerate class "SYMPL/SEP".

The $4D - 5D$ lift and the coupled cluster method

Let us split the six single particle states to ones that are occupied and not occupied.

$$i, j, k = 1, 2, 3, \qquad a, b, c = \overline{1}, \overline{2}, \overline{3}.$$

Define

$$|\psi_0\rangle \equiv \hat{p}^1 \hat{p}^2 \hat{p}^3 |0\rangle$$

Now the Coupled Cluster (CC) and full CI expansions are respectively

$$|\psi\rangle = e^{\hat{T}_1 + \hat{T}_2 + \hat{T}_3} |\psi_0\rangle$$

and

$$|\psi\rangle = (\hat{1} + \hat{C}_1 + \hat{C}_2 + \hat{C}_3)|\psi_0\rangle.$$

Here

$$\hat{T}_1 = T_a{}^i \hat{p}^a \hat{n}_i, \qquad \hat{T}_2 = \frac{1}{4} T_{ab}{}^{ij} \hat{p}^a \hat{n}_i \hat{p}^b \hat{n}_j \qquad \hat{T}_3 = T_{\overline{123}}{}^{123} \hat{p}^{\overline{1}} \hat{n}_1 \hat{p}^{\overline{2}} \hat{n}_2 \hat{p}^{\overline{3}} \hat{n}_3$$

and similar expressions for $\hat{C}_{1,2,3}$. Notice that we have two $1 + 9 + 9 + 1$ splits of the 20 amplitudes.

The $4D - 5D$ lift and the coupled cluster method

Hence we have

$$(\alpha, A, B, \beta) \leftrightarrow (\hat{1}, \hat{C}_2, \hat{C}_1, \hat{C}_3) \qquad (\eta, X, Y, \xi) \leftrightarrow (\hat{1}, \hat{T}_2, \hat{T}_1, \hat{T}_3)$$

$$\alpha = 1, \qquad A^a{}_i = \frac{1}{4}\varepsilon^{abc}\varepsilon_{ijk}C_{bc}{}^{jk}, \qquad B^i{}_a = C_a{}^i, \qquad \beta = C_{\overline{123}}{}^{123}$$

$$\eta = 1, \qquad X^a{}_i = \frac{1}{4}\varepsilon^{abc}\varepsilon_{ijk}T_{bc}{}^{jk}, \qquad Y^i{}_a = T_a{}^i, \qquad \xi = T_{\overline{123}}{}^{123}$$

$$1 = \psi_{123}, \qquad C_a{}^i = \frac{1}{2}\varepsilon^{ijk}\psi_{jka}, \qquad C_{ab}{}^{ij} = \varepsilon^{ijk}\psi_{abk}, \qquad C_{\overline{123}}{}^{123} = \psi_{\overline{123}}$$

We obtain the following dictionary between the CC and CI pictures

$$\alpha = \eta = 1, \qquad B = Y, \qquad A = Y^\sharp + X, \qquad \beta = \mathrm{Det}\,Y + (X, Y) + \xi$$

where

$$(X, Y) \equiv \mathrm{Tr}(XY), \qquad XX^\sharp = (\mathrm{Det}X)I$$

The $4D - 5D$ lift and the coupled cluster method

The inverse relations are

$$\eta = \alpha = 1, \quad Y = B, \quad X = A - B^\sharp, \quad \xi = \beta + 2\mathrm{Det}B - (A, B)$$

Now

$$\mathcal{D}(\psi) = 4[\kappa^2 - (A^\sharp, B^\sharp) + \alpha \mathrm{Det}A + \beta \mathrm{Det}B], \qquad 2\kappa = \alpha\beta - (A, B).$$

This expression displays the parameters (α, A, B, β) i.e. the ones of the full CI expansion of $|\psi\rangle$. Its new expression in terms of the CC expansion parameters (η, Y, X, ξ) is

$$\mathcal{D}(\psi) = \xi^2 + 4\mathrm{Det}X$$

which is much simpler and not featuring the matrix Y at all! The reason for this is the fact that $e^{\hat{T}_1} \in SL(6, \mathbb{C})$ is a SLOCC transformation...

The $4D - 5D$ lift and the coupled cluster method

There is a simple correspondence between between the entropy of $4D$ BPS black holes in type IIA compactified on a Calabi-Yau M and 5D BPS black holes in M-theory on $M \times TN_{\alpha}$. Using this correspondence the electric black hole Q_e charge and spin J_{β} or the magnetic black string charge Q_m and spin J_{α} maybe identified with the dyonic charges of the $4D$ black hole.
See D. Gaiotto, A. Strominger and X. Yin, JHEP 02 (2006) 024, L. Borsten, D. Dahanayake and M. J. Duff and W. Rubens, Phys. Rev. Phys.Rev.D80 (2009) 026003.

$$S_4 = \frac{1}{\alpha} S_{5(bs)} = \frac{1}{\beta} S_{5(bh)}$$

Now we see that

$$2J_{\alpha=1} = \xi = \beta - (A, B) + 2\mathrm{Det}B, \qquad Q_m = -X$$

We see that J and Q_m are related to coefficients of cluster operators of entanglement, namely triples $\hat{T}_3$ and doubles $\hat{T}_2$.

The fermionic Fock space

Let V be the N-dimensional complex vector space corresponding to the space of single particle states or modes. Take the 2^N dimensional space

$$\wedge^{\bullet} V^* = \mathbb{C} \oplus V^* \oplus \wedge^2 V^* \oplus \cdots \oplus \wedge^N V^*. \tag{2}$$

as it is well-known there is a Fock space description of this space and to an element $\varphi \in \wedge^{\bullet} V^*$ one can associate a Fock space element $|\varphi\rangle$ of the form

$$|\varphi\rangle = (\varphi^{(0)} + \varphi^{(1)}_a \hat{f}^{\dagger a} + \frac{1}{2}\varphi^{(2)}_{ab} \hat{f}^{\dagger a} \hat{f}^{\dagger a} \ldots)|0\rangle \in \mathcal{F}$$

On this space the group $Sin(2N, \mathbb{C})$ acts and has two invariant subspaces of positive and negative chirality

$$\mathcal{F} = \mathcal{F}^+ \oplus \mathcal{F}^-. \tag{3}$$

Generalized SLOCC transzformations

Note that $G = Spin(2N, \mathbb{C})$ is also describing particle creation and annihilation. It is easy to see that if the underlying V vector space is also equipped with a Hermitian scalar product, then G-transformations which are also respecting this extra structure are the Bogoliubov transformations well-known to physicists. Moreover, G also contains the SLOCC group $GL(N, \mathbb{C})$ describing transformations with fixed fermion number as a subgroup. Hence it is natural to consider G as a group of **generalized SLOCC transformations**.

Finding the **orbits** under the group $GL(1, \mathbb{C}) \times G$ is known in the mathematics literature as the problem of classification of spinors. Since these orbits are just our entanglement classes, then we can simply use the se well-known results. However the classification problem is a hard one and very little is known for $N > 7$. The classification up to $N = 6$ is a result due to Igusa 1970. The $N = 7$ case was tackled by Popov.

Separable states as pure spinors

Let $\{e_i\}$ and $\{e^i\}$, be the basis vectors of V and V^*. Let us define the $2N$ dimensional vector space

$$\mathcal{V} = V \oplus V^*, \qquad x = v + \alpha = v^i e_i + \alpha_j e^j \in \mathcal{V}$$

Let us also define the $(\cdot,\cdot) : \mathcal{V} \times \mathcal{V} \to \mathbb{C}$ symmetric bilinear form as

$$(x, y) = (v + \alpha, w + \beta) \equiv \alpha^i w_i + \beta^i v_i$$

A subspace of $\mathcal{V}$ is called **totally isotropic** if for $\forall u, v \in \mathcal{V}$ we have $(u, v) = 0$. Note that due to the structure of the bilinear form the **maximal** dimension of such subspaces is N. In the Fock space description we can associate to x an operator $\hat{x} = \alpha^i \hat{f}_i + v_i \hat{f}^{\dagger i}$ Then a **spinor is pure** if its

$$E_\varphi \equiv \{x \in \mathcal{V} | \qquad \hat{x}|\varphi\rangle = 0\}$$

annihilator subspace is **maximally totally isotropic**.

Pure spinors. Example.

Let us consider the subspace : $\mathrm{span}\{e^1, \dots e^k, e_{k+1}, \dots, e_N\}$. This space is clearly a maximally totally isotropic one. The corresponding operators

$$\{\hat{f}^{\dagger 1}, \dots, \hat{f}^{\dagger k}, \hat{f}_{k+1}, \dots, \hat{f}_N\}$$

annihilate the state

$$|\varphi\rangle = \hat{f}^{\dagger 1}\hat{f}^{\dagger 2}\cdots\hat{f}^{\dagger k}|0\rangle$$

Hence $|\varphi\rangle$ is a pure spinor. Since it is of the form of a **single Slater determinant** this state is **separable**. Hence it is natural to regard pure spinors as representatives of separable states under the generalized SLOCC group $GL(1, \mathbb{C}) \times \mathrm{Spin}(2N, \mathbb{C})$. Note, however that apart from this example of fixed fermion number there are other states to be regarded as separable in this generalized sense. These are superpositions of Slater determinants with different numbers of fermions.

Fermionic systems with six modes

Let us define: $\hat{p}^a \equiv \hat{f}^{\dagger a}$, $\qquad \hat{n}_a \equiv \hat{f}_a$, where

$$\{\hat{p}^a, \hat{n}_b\} = \delta^a{}_b, \qquad \{\hat{p}^a, \hat{p}^b\} = \{\hat{n}_a, \hat{n}_b\} = 0$$

$$|\varphi\rangle = (\eta + \frac{1}{2!} y_{ab} \hat{p}^a \hat{p}^b + \frac{1}{2!4!} x^{ab} \varepsilon_{abijkl} \hat{p}^i \hat{p}^j \hat{p}^k \hat{p}^l + \xi \hat{p}^1 \hat{p}^2 \hat{p}^2 \hat{p}^3 \hat{p}^4 \hat{p}^5 \hat{p}^6)|0\rangle$$

The group of generalized SLOCC transformations is $GL(1, \mathbb{C}) \times Spin(12, \mathbb{C})$. An element $\hat{G} = e^{\hat{S}} \in Spin(12, \mathbb{C})$ is generated by

$$\hat{S} = -\hat{B} - \hat{\beta} + \hat{A} - \frac{1}{2}(\mathrm{Tr} A)\hat{1}$$

where

$$\hat{A} = A^i{}_j \hat{p}^j \hat{n}_i, \quad \hat{B} = \frac{1}{2} B_{ij} \hat{p}^i \hat{p}^j, \quad \hat{\beta} = \frac{1}{2} = \beta^{ij} \hat{n}_i \hat{n}_j$$

The relative invariant for the even chirality case

$$J_4(\varphi) = (\eta\xi - (x, y))^2 + 4\eta \mathrm{Pf}(x) + 4\xi \mathrm{Pf}(y) - 4\mathrm{Tr}(\tilde{x}\tilde{y})$$

where

$$\mathrm{Pf}(x) = \frac{1}{3!2^3}\varepsilon_{abcdef}x^{ab}x^{cd}x^{ef}$$

$$(x, y) = -\frac{1}{2}\mathrm{Tr}(xy)$$

$$\tilde{x}_{ab} = \frac{1}{8}\varepsilon_{abijkl}x^{ij}x^{kl}$$

Generalized SLOCC classes for the even chirality case

Type	$\mathcal{J}_4(\varphi)$	$K_\varphi \varphi$	K_φ	φ
I.	$\neq 0$	$\neq 0$	$\neq 0$	$1 + e^{1234} + e^{3456} + e^{1256}$
II.	0	$\neq 0$	$\neq 0$	$1 + e^{1234} + e^{3456}$
III.	0	0	$\neq 0$	$1 + e^{1234}$
IV.	0	0	0	1
V.	0	0	0	0

Table: Canonical forms, invariants and covariants in six dimension.

Notice that the structure of these classes is qualitatively the same as the one for three-qubits, and three-fermions with six modes. The IV.th class is the separable one, with the vaccum states as a pure spinor.

The odd chirality case

$$|\psi\rangle = (u_a\hat{p}^a + \frac{1}{3!}\mathcal{P}_{abc}\hat{p}^a\hat{p}^b\hat{p}^c + \frac{1}{5!}v^a\varepsilon_{abcdef}\hat{p}^b\hat{p}^c\hat{p}^d\hat{p}^e\hat{p}^f)|0\rangle$$

One can calculate a quartic relative invariant under generalized SLOCC.

$$I_4(\psi) = (v^a u_a)^2 - \frac{1}{3}u_a * \mathcal{P}^{aij}\mathcal{P}_{bij}v^b + \mathcal{D}(\mathcal{P})$$

where

$$*\mathcal{P}^{abc} = \frac{1}{3!}\varepsilon^{abcijk}\mathcal{P}_{ijk}$$

$\mathfrak{J}$	Inv($\mathfrak{M}$)	dim$\mathfrak{M}$	Hitchin functional
$\mathcal{H}_3(\mathbb{R})$	$Sp(6, \mathbb{C})$	14	Constrained Hitchin
$\mathcal{H}_3(\mathbb{C})$	$SL(6, \mathbb{C})$	20	Hitchin
$\mathcal{H}_3(\mathbb{H})$	$Spin(12, \mathbb{C})$	32	Generalized Hitchin
$\mathcal{H}_3(\mathbb{O})$	$E_7(\mathbb{C})$	56	Generalized Exceptional

Table: Freudenthal triple systems ($\mathfrak{M}(\mathfrak{J})$) over cubic Jordan algebras ($\mathfrak{J}$), their automorphism groups (Inv$\mathfrak{M}(\mathfrak{J})$) and the corresponding Hitchin functional.

Speculations on the physical basis of the BHQC

According to the OSV conjecture $Z_{BH} = |Z_{TOP}|^2$.
Now it is conjectured that $Z_{GH} = Z_{BH}$.
Since Z_{GH} is based on entanglement measures we can generalize the BHQC substantially.
In the BHQC the entangled states are associated to cohomology classes like $H^3(M, \mathbb{R})$. These are just classical phase spaces. Moreover they parametrize locally the moduli space $\mathcal{M}$ of M which is a complex space. Embedding to $H^3(M, \mathbb{C})$ gives rise to a Hilbert space with the complex polarization coming from the Hodge star.
From OSV we know that geometric quantization on $H^3(M, \mathbb{R})$ yields another wave function which is just the partition function for topological strings.
How to connect these wave functions? This could be the clue for a physical basis of the BHQC.

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5. Via the OSV conjecture this interpretation also hints that one can use the BHQC beyond the semiclassical level.
6. This approach suggests that one does not have to assume the underlying manifold to be furnished with a special holonomy structure from the start. These structures are arising as critical points of functionals coming from measures of entanglement.