# Black Hole Scattering, Isomonodromy and Hidden Symmetries

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"The main of life is composed ... of meteorous pleasures which dance before us and are dissipated" - Samuel Johnson

### Main Message of My Work

Scattering amplitudes of conformal fields in Kerr-NUT-(A)dS black holes can be calculated using monodromy data of wave solutions and these monodromies can be generally obtained by Painlevé asymptotics and isomonodromic flows.

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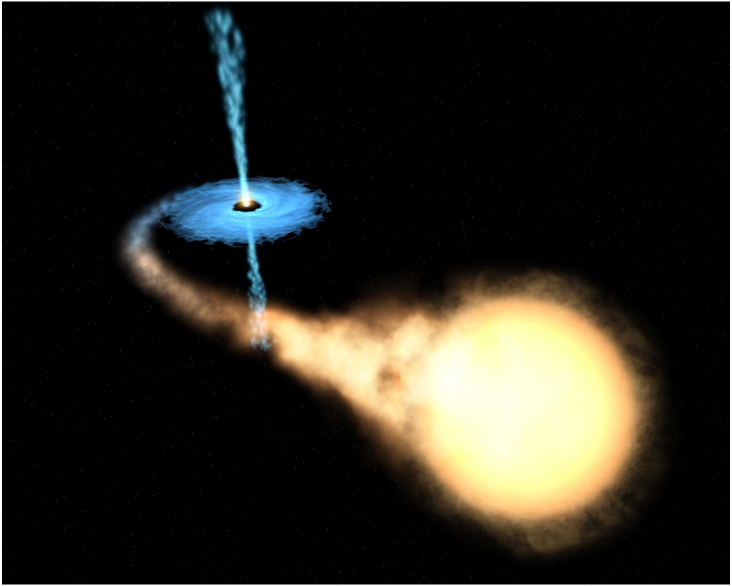
What is the importance of all that?

# Importance of Black Hole Scattering Theory

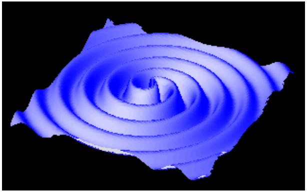
- Astrophysical phenomena: detection of gravitational waves
- Stability criteria of gravitational solutions
- AdS/CFT applications: quark-gluon plasma and condensed matter systems
- Quantum description of black holes

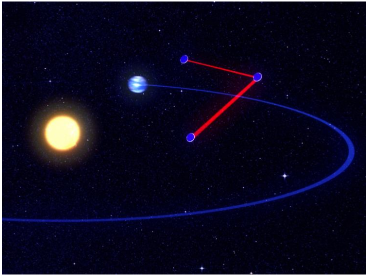
# Matter Accretion Disk around a Black Hole



# Detection of Gravitational Waves

- Hulse -Taylor Binary Pulsar
- BICEP-2 and Inflation: tensor-to-scalar ratio
- aLIGO, aVIRGO: ground-based interferometers
  (4 km L-shaped arms.
  High-frequencies)
- eLISA: coalescing binary black holes and EMRIs.
  ($10^6$ km arms, free-falling test masses.
  $10^4 - 10^7 M_\odot$).
- Pulsar Time Arrays: $10^9\ M_\odot$





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# Linear Perturbation of Gravitational Systems

- Gravity $(M, g)$ + Matter field $\Phi$

$$S = \frac{1}{16\pi G}\int_M d^D x\sqrt{-g}\ (R - 2\Lambda) + \int_M d^D x\sqrt{-g}\ \mathcal{L}_m(\Phi, \nabla\Phi)$$

- Linear perturbation of equations of motion

$$g_{ab} = g_{ab}^{BG} + h_{ab}, \quad \Phi = \Phi^{BG} + \phi$$

# Scalar Field Perturbation

- Scalar and gravitational perturbations decouple
- Non-minimally coupled massless scalar field $\phi(x)$

$$(\nabla^2 + \xi R)\phi(x) = 0, \quad \nabla^2\phi \equiv \frac{1}{\sqrt{-g}}\partial_a(\sqrt{-g}g^{ab}\partial_b\phi)$$

- Separable solutions: $\phi(t, r, \theta, \varphi) = e^{-i\omega t}e^{im\varphi}S_{\omega\ell m}(\theta)\phi_{\omega\ell m}(r)$
- Radial and Angular equations

$$\partial_r(P_r(r)\partial_r\phi_{\omega\ell m}) - Q_r(r)\phi_{\omega\ell m} = 0$$

$$\partial_\theta(P_\theta(\theta)\partial_\theta S_{\omega\ell m}) - Q_\theta(\theta)S_{\omega\ell m} = 0$$

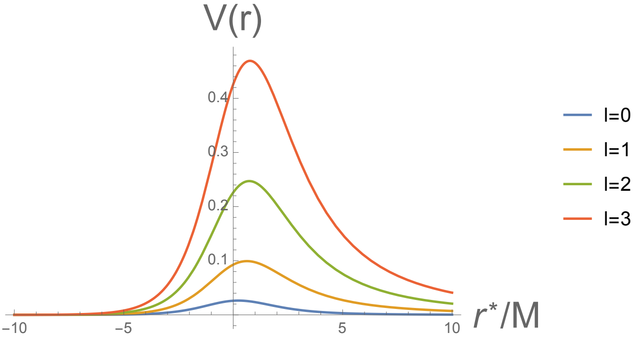
- Angular eigenvalues from angular equation

# Scalar Scattering by Black Holes

- One-dimensional Schrödinger scattering

$$\frac{d^2\phi_\omega}{dr^{*2}} + (\omega^2 - V(r))\phi_\omega = 0$$

- Typical Schwarzschild potentials



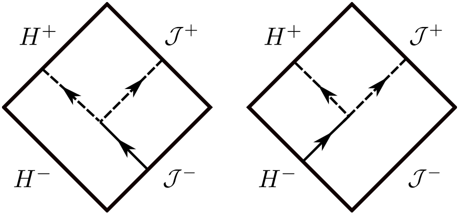
# Ingoing and Outgoing Boundary Conditions

- Classical scattering (IN mode)

$$\phi_\omega^{IN} \sim e^{-i\omega r^*} - \mathcal{R} e^{i\omega r^*}, \qquad r^* \to +\infty,$$
$$\phi_\omega^{IN} \sim \mathcal{T} e^{-i\omega r^*}, \qquad r^* \to -\infty,$$

- Semiclassical scattering (OUT or UP mode)

$$\phi_{-\omega}^{UP} \sim \mathcal{T}' e^{i\omega r^*}, \qquad r^* \to +\infty,$$
$$\phi_{-\omega}^{UP} \sim e^{i\omega r^*} - \mathcal{R}' e^{-i\omega r^*}, \qquad r^* \to -\infty,$$



# Radiation Flux and Greybody Factor

- Radiation Flux (Wronskian)

$$J := \frac{1}{2i}\left(\phi_{-\omega}\frac{d\phi_\omega}{dr^*} - \phi_\omega\frac{d\phi_{-\omega}}{dr^*}\right)$$

- Flux Conservation and Greybody factor

$$\mathcal{R}\tilde{\mathcal{R}} + \mathcal{T}\tilde{\mathcal{T}} = 1, \quad \gamma_\ell(\omega) = \frac{J_{hor}}{J_{in}} = \mathcal{T}(\omega)\tilde{\mathcal{T}}(\omega)$$

- Relation between basis

$$\begin{pmatrix}\phi_\omega^{UP}\\ \phi_{-\omega}^{UP}\end{pmatrix} = \begin{pmatrix}\frac{1}{\mathcal{T}} & \frac{\mathcal{R}}{\mathcal{T}}\\ \frac{\tilde{\mathcal{R}}}{\tilde{\mathcal{T}}} & \frac{1}{\tilde{\mathcal{T}}}\end{pmatrix}\begin{pmatrix}\phi_\omega^{IN}\\ \phi_{-\omega}^{IN}\end{pmatrix},$$

# Scattering for real frequencies

- For real frequency $\omega$, flux conservation implies that

$$|\mathcal{R}|^2 + |\mathcal{T}|^2 = 1$$

and the greybody factor

$$\gamma_\ell(\omega) = |\mathcal{T}(\omega)|^2$$

- Mean number of emitted particles:

$$\langle n(\omega) \rangle = \frac{\gamma_\ell(\omega)}{e^{\omega/T_H} - 1}$$

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## Complex ODEs and Monodromy

- *Self-adjoint* radial equation

$$\partial_z(U(z)\partial_z\phi(z)) - V(z)\phi(z) = 0, \quad z \in \mathbb{CP}^1$$

- Regular singular points $\{z_i\}$, $i = 1, ..., n$.
- Ingoing and Outgoing solutions

$$\phi_i^{\pm}(z) = (z - z_i)^{\pm\theta_i/2}\big(1 + \mathcal{O}(z - z_i)\big)$$

- Singular points = Branch points $\Rightarrow$ Monodromy

$$\phi_i^{\pm}(ze^{2\pi i}) = e^{\pm i\pi\theta_i}\phi_i^{\pm}(z)$$

# Monodromies and Gauge Connection

- Gauge connection formulation

$$(\partial_z - A(z))\Phi(z) = 0 \,,$$

$$A(z) = \begin{pmatrix} 0 & U^{-1} \\ V & 0 \end{pmatrix} \quad , \quad \Phi(z) = \begin{pmatrix} \phi_1 & \phi_2 \\ U\partial_z\phi_1 & U\partial_z\phi_2 \end{pmatrix}$$

- Monodromy matrix

$$\Phi_\gamma(z) = \mathcal{P}\exp\left(\oint_\gamma A\right)\Phi(z) =: \Phi(z)M_\gamma$$

# Monodromies and Frobenius solutions

- Loop around only one pole $z = z_i \quad \Rightarrow \quad \Phi_{\gamma_i} = \Phi M_i$
- Loop enclosing all poles gives monodromy identity

$$M_1 M_2 ... M_n = \mathbb{1}$$

- General Frobenius solution

$$\begin{aligned} \Phi(z) &= \Phi_i(z)\, g_i \\ &= \left(\Phi_0^i + \mathcal{O}(z - z_i)\right) \begin{pmatrix} (z - z_i)^{\theta_i/2} & 0 \\ 0 & (z - z_i)^{-\theta_i/2} \end{pmatrix} g_i \end{aligned}$$

- Monodromy matrix in arbitrary basis

$$M_i = g_i^{-1} \begin{pmatrix} e^{i\pi\theta_i} & 0 \\ 0 & e^{-i\pi\theta_i} \end{pmatrix} g_i$$

# Scattering Amplitudes and Connection Matrix

- Change of basis matrix = Connection matrix

$$\mathcal{M}_{ij} = \Phi_i^{-1}\Phi_j = g_i\, g_j^{-1}$$

$$\mathcal{M}_{ij} = \begin{pmatrix} \frac{1}{\mathcal{T}} & \frac{\mathcal{R}^*}{\mathcal{T}^*} \\ \frac{\mathcal{R}}{\mathcal{T}} & \frac{1}{\mathcal{T}^*} \end{pmatrix} , \qquad |\mathcal{R}|^2 + |\mathcal{T}|^2 = 1$$

- Kerr scattering = 2 regular and 1 irregular singular point = 3-point monodromy group [Castro et al 13']

# Transmission between Two Regular Singular Points

- For higher-point monodromy groups (Cunha and Novaes arXiv:1404.5188)

Let $g_i \in \mathrm{SL}(2,\mathbb{C})$. If we define

$$m_{ij} = \mathrm{Tr}\, M_i M_j = 2\cos\pi\sigma_{ij}$$

then

$$|\mathcal{T}|^2 = \left|\frac{\sin\pi\theta_i\sin\pi\theta_j}{\sin\frac{\pi}{2}(\sigma_{ij}+\theta_i-\theta_j)\sin\frac{\pi}{2}(\sigma_{ij}-\theta_i+\theta_j)}\right|$$

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# Killing-Yano Tensors and Separability

- Most general spacetime with separable equations?
- $D = 2n + \epsilon$ Kerr-NUT-(A)dS spacetime ($\epsilon = \pm 1$)

  (Frolov and Kubzniak 07')
- Closed Conformal Killing-Yano tensor $h_{ab} \quad \Rightarrow$

  $(n-1)$ Killing Tensors $+$ $(n+\epsilon)$ Killing vectors $+$ $1$ metric

  $= 2n + \epsilon$ conserved quantities
- Separability of Klein-Gordon, Dirac and Gravitational equations (Frolov; Oota and Yasui)

# $D = 4$ Kerr-NUT-(A)dS Black Hole

$$ds^2 = -\frac{Q(r)}{r^2+p^2}(dt + p^2 d\phi)^2 + \frac{P(p)}{r^2+p^2}(dt - r^2 d\phi)^2 + \frac{r^2+p^2}{Q(r)}dr^2 + \frac{r^2+p^2}{P(p)}dp^2$$

$$P(p) = -\frac{\Lambda}{3}p^4 - \epsilon p^2 + 2np + k$$

$$Q(r) = -\frac{\Lambda}{3}r^4 + \epsilon r^2 - 2Mr + k$$

# Separation of Variables in KG equation

- Separation of variables: $\phi(t,r,\varphi,\theta) = e^{-i\omega t} e^{im\varphi} R(r) S(\theta)$
- Radial and angular equations

$$\partial_p(P(p)\partial_p S(p)) + \left(-4\Lambda\xi p^2 - \frac{(\Psi_0 p^2 - \Psi_1)^2}{P(p)}\right) S(p) = -C_\ell S$$

$$\partial_r(Q(r)\partial_r R(r)) + \left(-4\Lambda\xi r^2 + \frac{(\Psi_0 r^2 + \Psi_1)^2}{Q(r)}\right) R(r) = C_\ell R$$

- Each equation has 5 regular singular points

# Kerr-NUT-(A)dS Radial Equation

- Radial equation

$$\partial_r(Q(r)\partial_r R(r)) + \left(-4\Lambda\xi r^2 + \frac{(\Psi_0 r^2 + \Psi_1)^2}{Q(r)}\right) R(r) = C_\ell R$$

- Frobenius coefficients

$$\rho_i^{\pm} = \pm i\left(\frac{\Psi_0 r_i^2 + \Psi_1}{Q'(r_i)}\right), \quad i = 1, ..., 4$$

$$\rho_\infty^{\pm} = \frac{3}{2} \pm \frac{1}{2}\sqrt{9 - 48\xi}$$

- $\theta_\infty = \sqrt{9 - 48\xi}$ is an integer when $\xi = 0, \frac{5}{48}, \frac{1}{6}, \frac{3}{16}$

# Removable Singularity for Conformally Coupled Case

- For $\xi = \frac{1}{6}$, $r = \infty$ is a *removable singularity*
- Homographic transformation:

$$z = \frac{r - r_1}{r - r_4}\frac{r_2 - r_4}{r_2 - r_1}, \quad (r_1, r_2, r_3, r_4, \infty) \;\mapsto\; (0, 1, t_0, \infty, z_\infty)$$

- Homotopic transformation:

$$R(z) = z^{-\theta_0/2}(z-1)^{-\theta_1/2}(z-t_0)^{-\theta_t/2}(z-z_\infty)y(z)$$

# Heun Equation for Conformally Coupled Case

- Heun equation (4 regular singular points)

$$y'' + \left(\frac{1-\theta_0}{z} + \frac{1-\theta_1}{z-1} + \frac{1-\theta_{t_0}}{z-t_0}\right) y' + \\ + \left(\frac{1+\theta_\infty}{z(z-1)} - \frac{t_0(t_0-1)K_0}{z(z-1)(z-t_0)}\right) y = 0$$

- Frobenius coefficients

$$\theta_k = 2i\left(\frac{\Psi_0 r_k^2 + \Psi_1}{Q'(r_k)}\right), \quad k = 0, 1, t_0, \infty$$

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# Kerr-dS Black Hole

$$ds^2 = -\frac{\Delta_r(r)}{(r^2+p^2)\chi^4}\left(dt - \frac{(a^2-p^2)}{a}d\phi\right)^2 +$$

$$+\frac{\Delta_p(p)}{(r^2+p^2)\chi^4}\left(dt - \frac{(r^2+a^2)}{a}d\phi\right)^2 + \frac{r^2+p^2}{\Delta_p(p)}dp^2 + \frac{r^2+p^2}{\Delta_r(r)}dr^2$$

$$\Delta_p(p) = -\frac{1}{L^2}p^4 - (1-\frac{a^2}{L^2})p^2 + a^2,$$

$$\Delta_r(r) = -\frac{1}{L^2}r^4 + (1-\frac{a^2}{L^2})r^2 - 2Mr + a^2$$

# Kerr-dS Black Hole

- 5 singular points $(r_{--}, r_{-}, r_H, r_C, \infty)$
- $r = \infty$ removable by conformal coupling
- Horizon angular velocity

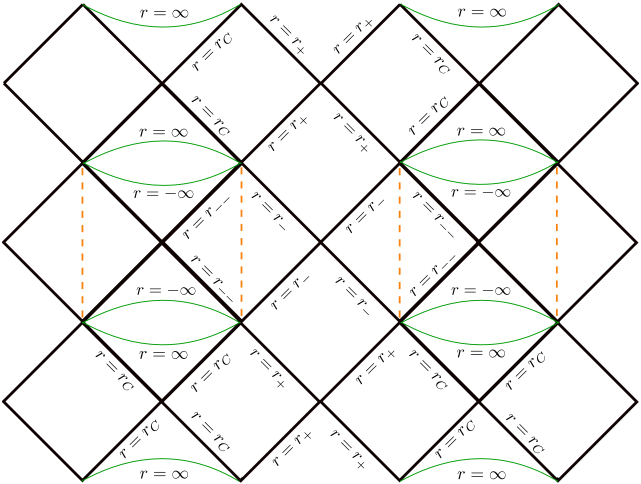
$$\Omega_{H,C} = \frac{a}{r_{H,C}^2 + a^2}$$

- Horizons temperatures

$$T_H = \frac{|\Delta_r'(r_H)|}{4\pi\chi^2(r_H^2 + a^2)}, \quad T_C = \frac{|\Delta_r'(r_C)|}{4\pi\chi^2(r_C^2 + a^2)}$$

- Regular black hole if $T_H \geq 0$ and $a < L$

# Causal Diagram of Kerr-dS



# (A)dS Spheroidal Harmonics

- Set $p = au$

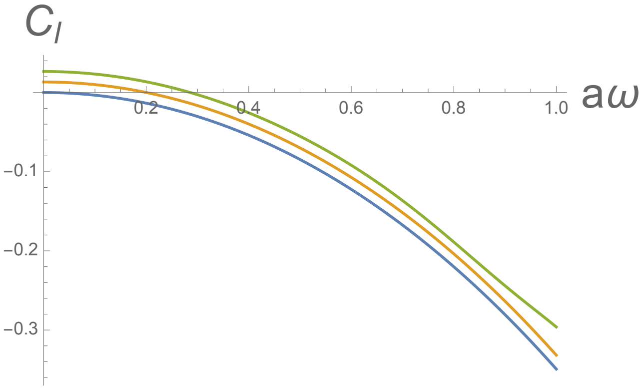
$$\partial_u(1-u^2)(1-\hat{a}^2u^2)\partial_u S + \left(Au^2 + B - \frac{m^2(1-\hat{a}^2)^2}{(1-u^2)(1-\hat{a}^2u^2)}\right)S = -C_\ell S$$

where $\hat{a} = a/L$

- Connects to angle from $u = \cos\theta$

# Kerr-dS Angular Eigenvalues ($\Lambda > 0$)

- Case $\ell = m = 0$




$\hat{a} = 0$ (blue), $\hat{a} = 0.02$ (yellow), $\hat{a} = 0.04$ (green)

# Kerr-dS Conformally Coupled Radial Equation

$$y'' + \left(\frac{1-\theta_0}{z} + \frac{1-\theta_1}{z-1} + \frac{1-\theta_{t_0}}{z-t_0}\right) y' + \\ + \left(\frac{1+\theta_\infty}{z(z-1)} - \frac{t_0(t_0-1)K_0}{z(z-1)(z-t_0)}\right) y = 0,$$

- Frobenius coefficients are purely imaginary for real $\omega$

$$\theta_k = \pm\frac{i}{2\pi}\left(\frac{\omega-\Omega_k m}{T_k}\right), \quad k = 0, t_0, 1, \infty$$

# Kerr-dS Greybody Factor

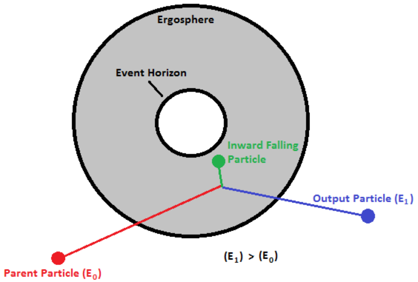
$$\gamma_\ell(\omega, m) = \frac{\sinh(\frac{\omega - \Omega_H m}{2T_H}) \sinh(\frac{\omega - \Omega_C m}{2T_C})}{\cosh\left(\frac{\omega - \Omega_H m}{2T_H} + \frac{\omega - \Omega_C m}{2T_C}\right) - \cosh(\pi \nu_{HC})}$$

where $\nu_{HC} = i\sigma_{HC}(\omega, \ell, m)$

- $\nu_{HC}$ encodes scattering global information
- $\omega = \Omega_H m \Rightarrow$ onset of *superradiance*

# Superradiant Scattering

- Superradiance = wave analog of Penrose process




- In terms of the classical impact parameter $b = \mathcal{L}/\mathcal{E} \sim \ell/\omega$

$$\frac{\omega}{m} = \frac{\omega}{\ell}\frac{\ell}{m} \sim \frac{1}{b}\frac{\mathcal{L}}{\mathcal{L}_z}$$

# Properties of Greybody Factor

- Scattering regimes

$$\begin{cases} \omega > \Omega_H m \quad \text{or} \quad \Omega_C m > \omega & \text{Normal scattering} \\ \Omega_H m > \omega > \Omega_C m & \text{Superradiant scattering} \end{cases}$$

- Poles of scattering matrix (resonances)

$$\omega = \begin{cases} m\Omega_H - 2\pi i n T_H \\ m\Omega_C + 2\pi i n T_C \end{cases} \quad (n \in \mathbb{Z}^+)$$

- We expect that

$$\gamma_l(\omega) \to 1, \qquad \text{as} \quad \omega \to \infty$$

$$\gamma_l(\omega) \to 0 \text{ or constant} \qquad \text{as} \quad \omega \to 0$$

# Quasinormal Modes

- Modes purely ingoing at $r_H$ *and* purely outgoing at $r_C$
- Possible only for complex $\omega$
- In this case,

$$\mathcal{M}_{CH} = \begin{pmatrix} \frac{1}{\mathcal{T}} & \frac{\mathcal{R}'}{\mathcal{T}'} \\ \frac{\mathcal{R}}{\mathcal{T}} & \frac{1}{\mathcal{T}'} \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -1 & \frac{1}{\mathcal{T}'} \end{pmatrix}$$

- Poles of transcendental equation

$$\nu_{HC}(\omega,\ell,m) = \frac{\omega - \Omega_H m}{2T_H} + \frac{\omega - \Omega_C m}{2T_C} + 2\pi i n, \qquad n \in \mathbb{Z}$$

# Summary of Part 2

- Perturbations for Kerr-NUT-(A)dS are separable into angular and radial part
- Angular eigenvalues obtained numerically
- Conformal coupling removes singularity at $r = \infty$
- Greybody factor accounts for superradiance and quasinormal modes

How to find $\sigma_{HC}$?

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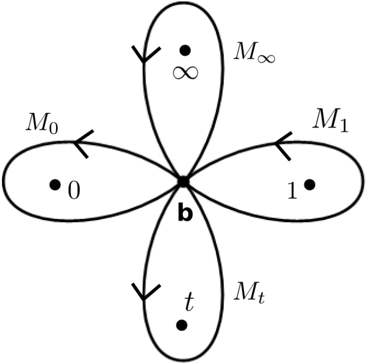
# Four-point Monodromy Group




$$\det M_i = 1, \quad m_i \equiv \mathrm{Tr}\, M_i = 2\cos\pi\theta_i,$$

$$M_\infty M_1 M_t M_0 = \mathbb{1}$$

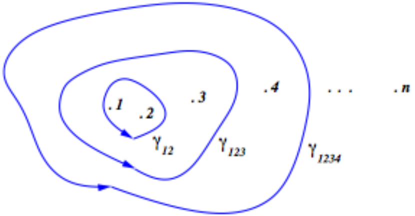
# $\mathrm{SL}(2,\mathbb{C})$ Representations of Free Groups

- Free group with $n-1$ generators

$$G_{n-1} =< M_1, M_2, ..., M_{n-1} >$$

- Dimension of moduli space with fixed monodromies $m_i$ is $2(n-3)$
- Composite trace coordinates map moduli space

$$m_{12...k} = \mathrm{Tr}\, M_1 M_2 ... M_k \ , \quad k < n-1$$



# Representation of Heun Monodromy Group

- 3 composite traces

$$m_{ij} = \mathrm{Tr}(M_i M_j) = 2\cos(\pi\sigma_{ij}), \quad i,j = 0,1,t$$

- Only 2 are independent because of Fricke-Jimbo relation

$$\begin{aligned} W_4(m_1, m_2, m_3, m_{13}, m_{23}, m_{12}, m_4) \equiv \\ m_{13}m_{23}m_{12} + m_{13}^2 + m_{23}^2 + m_{12}^2 \\ -m_{13}(m_2m_4 + m_1m_3) - m_{23}(m_1m_4 + m_2m_3) - m_{12}(m_3m_4 + m_1m_2) \\ + m_1^2 + m_2^2 + m_3^2 + m_4^2 + m_1m_2m_3m_4 - 4 = 0 \end{aligned}$$

- Monodromy representations are parametrized by two composite traces $(\sigma_{0t}, \sigma_{1t})$, $(\sigma_{0t}, \sigma_{01})$ or $(\sigma_{1t}, \sigma_{01})$

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# How to obtain the composite monodromies $\sigma_{ij}$?

**Miwa, Jimbo and Ueno 1980, Jimbo 1982**

Painlevé VI asymptotics depend explicitly on monodromy data of a 4-point Fuchsian system

# Garnier System and Apparent Singularity

- Fuchsian System with $4$ singular points

$$\partial_z \mathcal{Y}(z) = A(z)\mathcal{Y}(z), \quad A(z) = \sum_{i=1}^{3} \frac{A_i}{z - z_i},$$

with $\mathcal{Y}(z) = \begin{pmatrix} y_1(z) & y_2(z) \end{pmatrix}^T$

- Component $y_1$ obeys the ODE

$$\partial_z^2 y - (\partial_z \log A_{12} + \operatorname{Tr} A(z))\partial_z y \\ + (\det A(z) - \partial_z A_{11} + A_{11}\partial_z \log A_{12})y = 0$$

# Garnier System and Apparent Singularity

- Apparent singularity at $z = \lambda$ if

$$A_{12}(z) = k \frac{z-\lambda}{z(z-1)(z-t)}, \quad k \in \mathbb{C}$$

- *Deformed* Heun equation with one apparent singularity

$$\partial_z^2 y + \left( \frac{1-\theta_0}{z} + \frac{1-\theta_1}{z-1} + \frac{1-\theta_t}{z-t} - \frac{1}{z-\lambda} \right) \partial_z y$$

$$+ \left( \frac{\kappa}{z(z-1)} - \frac{t(t-1)K}{z(z-1)(z-t)} + \frac{\lambda(\lambda-1)\mu}{z(z-1)(z-\lambda)} \right) y = 0$$

- $\lambda(t_0) = t_0$ and $\mu_0 = -K_0/\theta_t$ for our Heun

# Garnier Hamiltonian System

- $z = \lambda$ is an apparent singularity if

$$K(\lambda, \mu, t) = \frac{1}{t(t-1)}[\lambda(\lambda-1)(\lambda-t)\mu^2 - \{\theta_0(\lambda-1)(\lambda-t) + \theta_1\lambda(\lambda-t) + (\theta_t - 1)\lambda(\lambda-1)\}\mu + \kappa(\lambda-t)]$$

- Garnier System

$$\frac{d\lambda}{dt} = \frac{\partial K}{\partial \mu}, \quad \frac{d\mu}{dt} = -\frac{\partial K}{\partial \lambda}$$

generates isomonodromic flow $(\lambda(t), \mu(t), K(\lambda, \mu, t))$

- Second-order equation for $\lambda(t) =$ Painlevé VI

# Schlesinger System and Painlevé VI

- Schlesinger system

$$\partial_z Y(z,t) = A(z,t)Y(z,t), \quad A(z,t) = \frac{A_0(t)}{z} + \frac{A_1(t)}{z-1} + \frac{A_t(t)}{z-t},$$
$$\partial_t Y(z,t) = B(z,t)Y(z,t), \quad B(z,t) = -\frac{A_t(t)}{z-t}$$

- Integrability condition

$$\partial_t A - \partial_z B + [A,B] = 0$$

is equivalent to *Schlesinger equations*

$$\frac{dA_0}{dt} = \frac{[A_t, A_0]}{t}, \quad \frac{dA_1}{dt} = \frac{[A_t, A_1]}{t-1}, \quad \frac{dA_t}{dt} = \frac{[A_0, A_t]}{t} + \frac{[A_1, A_t]}{t-1}$$

# Schlesinger System Asymptotics

- Near $t = 0$

$$A_0 \approx t^{\Lambda} A_0^0 t^{-\Lambda} \quad \text{and} \quad A_t \approx t^{\Lambda} A_t^0 t^{-\Lambda}, \text{ where } \Lambda = A_0^0 + A_t^0$$

- Schlesinger system degenerates into two hypergeometric systems

$$\frac{dY_0}{dz} = \left( \frac{\Lambda}{z} + \frac{A_1^0}{z-1} \right) Y_0, \quad \frac{dY_1}{dz} = \left( \frac{A_0^0}{z} + \frac{A_t^0}{z-1} \right) Y_1$$

# Painlevé VI Asymptotics

- Using that

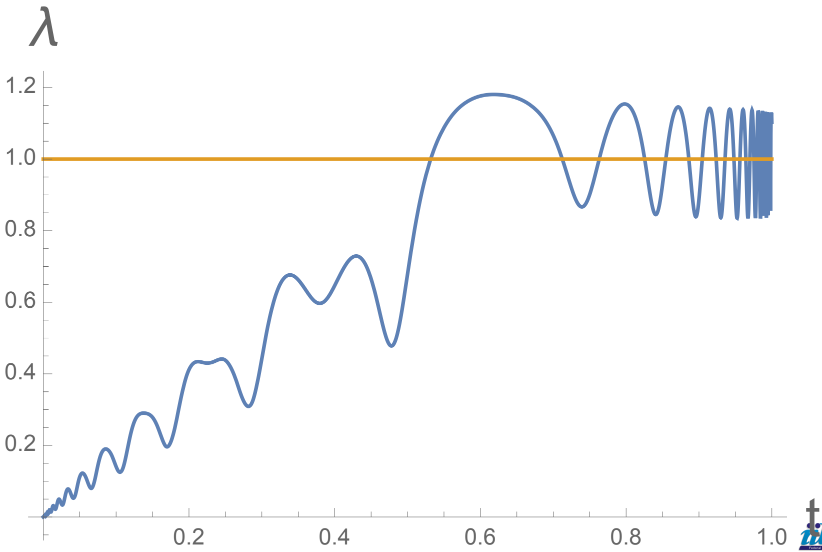
$$A_{12}(z) = k\frac{z-\lambda}{z(z-1)(z-t)}$$

and homographic transformations, we get $P_{VI}$ asymptotics for $0 < \operatorname{Re}\sigma_{ij} < 1$

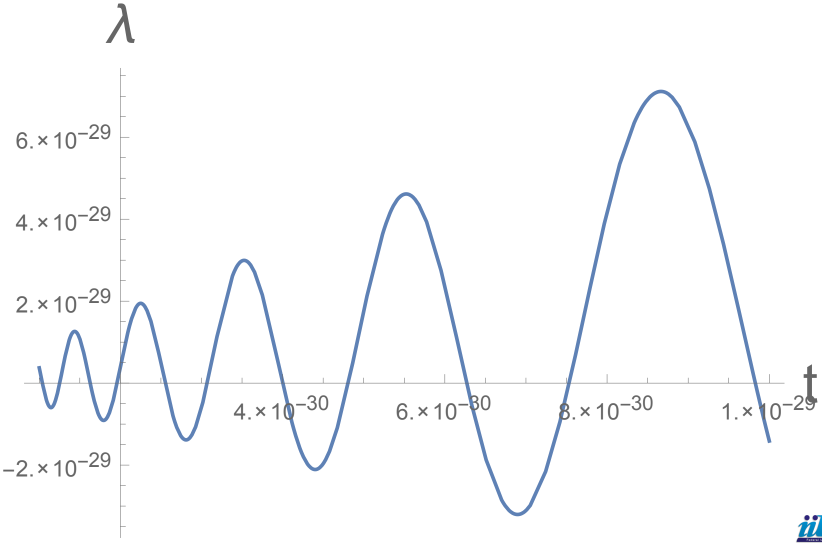
$$\lambda(t) = \begin{cases} a_0 t^{1-\sigma_{0t}}(1+O(t^{\delta})), & |t| < r, \\ 1 + a_1(1-t)^{1-\sigma_{t1}}(1+O((1-t)^{\delta}), & |t-1| < r, \\ a_{\infty} t^{\sigma_{01}}(1+O(t^{-\delta})), & |1/t| < r, \end{cases}$$

where $a_i$ are functions of monodromy data and $r, \delta > 0$.

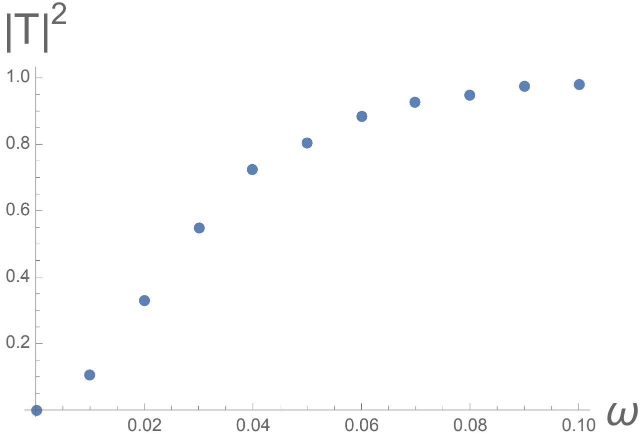
# Numerical Integration of $P_{VI}$



# Numerical Integration of $P_{VI}$ near $t = 0$



# Kerr-dS Greybody Factor



# Transcendental Solution from $\tau$-function

- Definition of $\tau$-function

$$K(\lambda,\mu,t) = \frac{d}{dt}\log(t^{A}(t-1)^{B}\tau(t,\{\theta_i\}))$$

- $\tau$-function asymptotics

$$\tau(t) \propto t^{\sigma^2/4-(\theta_0-\theta_t)^2/4}[1+\mathcal{O}(t^{1\pm\sigma},t)]$$

# Near-extremal Case and Tau Function

- Initial conditions can be inverted to obtain $\sigma$

$$\frac{d}{dt}\log\tau(t,\{\theta_i\})\Big|_{t=t_0} = \frac{\theta_0\theta_t}{t_0} + \frac{\theta_1\theta_t}{t_0-1} + K_0$$

$$\frac{d^2}{dt^2}\log\tau(t,\{\theta_i\})\Big|_{t=t_0} = -\frac{\theta_0\theta_t}{t_0^2} - \frac{\theta_1\theta_t}{(t_0-1)^2} + \frac{\kappa_1\theta_t}{t_0(t_0-1)} - \frac{2t_0-1}{t_0(t_0-1)}K_0.$$

- Near-Extremal case

$$\sigma = \theta_0 + \theta_t + \frac{2K_0 - \theta_1\theta_t}{\theta_0+\theta_t}t_0 + \mathcal{O}(t_0^2)$$

# Contents

# Near-Horizon Extremal Kerr (NHEK) Metric

- Extremal Kerr Properties

$$r_{\pm} = a = M, \quad S = 2\pi M^2 = 2\pi J, \quad T_H = 0, \quad \Omega_H = \frac{1}{2M}$$

- Near-horizon limit

$$r = \frac{\hat{r} - M}{\lambda M}, \quad t = \frac{\lambda \hat{t}}{2M}, \quad \phi = \hat{\phi} - \frac{\hat{t}}{2M},$$

- When $\lambda \to 0$, we get $AdS_2 \ltimes S^1$

$$ds^2 = 2\Omega^2 J \left[ \frac{dr^2}{r^2} + d\theta^2 - r^2 dt^2 + \Lambda^2 (d\phi + r dt)^2 \right]$$

# Kerr/CFT Correspondence

- ASG = Allowed diffeos / Trivial diffeos (Brown, Henneaux 1986)
- Most general perturbations $\delta g_{ab} = h_{ab}$ preserving metric boundary conditions are generated by a Virasoro algebra

$$[L_n, L_m] = (n-m)L_{n+m} + Jm(m^2-1)\delta_{n+m} \, .$$

- Corresponds to a chiral thermal CFT with temperature $T_L = \frac{1}{2\pi}$ and central charge $c = 12J$
- Cardy formula for CFT entropy reproduces black hole entropy (Guica et al 2009)

$$S_{\mathsf{CFT}} = \frac{\pi^2}{3} c_L T_L = 2\pi J = S_{BH}$$

# Kerr/CFT Away From Extremality

- Wave equation for $M\omega \ll 1$ and $r\omega \ll 1$ also presents hidden conformal symmetry (Castro et al 2010)
- Hypergeometric scattering amplitudes match $\mathrm{SL}(2,\mathbb{C})$ symmetry of dual CFT
- For the Kerr black hole, we can write (Castro et al 2013)

$$\mathcal{T}\mathcal{T}' = \frac{\sinh 2\pi(\omega_L + \omega_R)\sinh(2\pi\alpha_{\mathsf{irr}})}{\sinh \pi(\omega_L - \alpha_{\mathsf{irr}})\sinh \pi(\omega_R + \alpha_{\mathsf{irr}})}$$

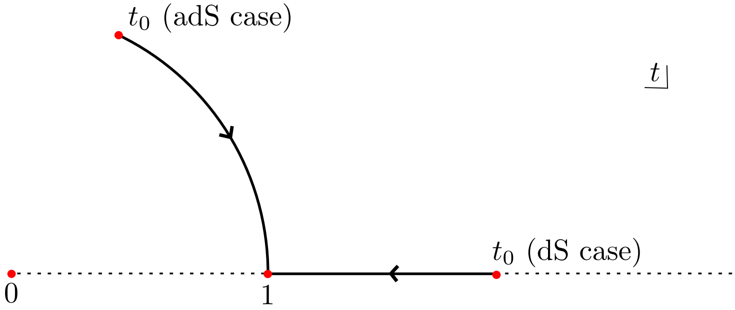
- For low-frequencies and $\ell \neq 0$

$$i\alpha_{\mathsf{irr}} = \ell - 2M^2\omega^2 f(\ell) + O(\omega^3)$$

suggesting no simple CFT description

# Lessons from Isomonodromic Flow

- Confluence of $P_{VI}$ gives more information on $\alpha_{\mathsf{irr}}$ (Bruno's talk)
- This suggests that scattering data of non-extremal black holes is equivalent in some sense to extremal black hole scattering



# Summary of Part 3

- Composite traces $\sigma_{ij}$ map monodromy moduli space
- $P_{VI}$ asymptotics depend explictly on $\sigma_{ij}$
- $\sigma_{ij}$ can be obtained either numerically in general or analytically in the near-extremal case
- Our results can maybe shed some light on the Kerr/CFT duality away from the near-horizon infrared limit

# Conclusions

- Monodromy technique is the most powerful way to treat scattering problems
- Insights on CFT description of black holes
- General formula for scattering amplitudes between two regular singular points
- Conformally coupled case is easier
- Valid for higher-dimensional Kerr-NUT-(A)dS black holes

# Perspectives

- Higher-spin modes, gravitational stability and astrophysical problems
- Higher-dimensional Kerr-(A)dS and SUGRA backgrounds
- Recover literature via $\Lambda \to 0$ confluence. Irregular singular points ($P_V$ and $P_{III}$ for extremal BH)
- Quasinormal modes and plasma thermalization
- Twistorial and geometrical interpretation of isomonodromic symmetry

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## Relation with Fuchsian Equation

- Fuchsian ODE normal form with $n$ finite singular points

$$\psi''(z) + T(z)\psi(z) = 0, \quad T(z) = \sum_{i=1}^{n} \left( \frac{\delta_i}{(z - z_i)^2} + \frac{c_i}{z - z_i} \right),$$

$$\sum_{i=1}^{n} c_i = 0\,, \quad \sum_{i=1}^{n} (c_i z_i + \delta_i) = 0\,, \quad \sum_{i=1}^{n} (c_i z_i^2 + 2\delta_i z_i) = 0$$

- Local monodromies: $\delta_i = (1 - \theta_i^2)/4$
- Accessory parameters $c_i$ have global properties
- $2(n-3)$ independent parameters: $(c_i, z_i)$

# Symplectic Structure of Flat $\mathrm{SL}(2,\mathbb{C})$ Connections

- Moduli space of flat connections $A \sim$ moduli space of monodromy group
- Atiyah-Bott symplectic structure

$$\Omega = \sum_{i=1}^{n-3} dc_i \wedge dz_i = \sum_{i=1}^{n-3} d\nu_i \wedge d\mu_i$$

  where $(\nu_i, \mu_i)$ are trace coordinates (Nekrasov et al 2011)
- Canonical transformation connects both set of coordinates
- Suggests analytical approach to find composite monodromies
- Relation with classical conformal blocks of 2D CFT

# Recurrence Relations

- Taylor solution $y(z) = \sum_{n=0}^{\infty} g_n z^{n/2}$, $|z| < 1$

$$-(Q_0 + q)g_0 + R_0 g_1 = 0,$$

$$P_n g_{n-1} - (Q_n + q)g_n + R_n g_{n+1} = 0, \quad (n > 0)$$

$$\begin{aligned} P_n &= (n - 1 + \alpha_+)(n - 1 + \alpha_-), \\ Q_n &= n((t + 1)(n - 1 + \gamma) + t\delta + \epsilon), \\ R_n &= t(n + 1)(n + \gamma) \end{aligned}$$

- Solved using Leaver's continued-fraction method (Leaver 1985, Berti, Cardoso and Will (2006))

# Continued-fraction Method

- Augmented convergence for $|z| \geq 1$ if

$$\lim_{n\to\infty}\left|\frac{g_{n+1}}{g_n}\right| = |t|^{-1} = \hat{a}^2 \;\Rightarrow\; a < L$$

- Recurrence relation in terms of $v_n = g_{n+1}/g_n$

$$v_{n-1} = \frac{P_n}{(Q_n + q) - R_n v_n}$$

- Equivalent to continued-fraction

$$(Q_0 + q) - \frac{R_0 P_1}{(Q_1 + q)-}\,\frac{R_1 P_2}{(Q_2 + q)-}\ldots = 0$$

- Solve numerically with $v_N = \hat{a}^2$ for some large integer $N$

## Schlesinger System Asymptotics

- Using that $\det A_i^0 = -\theta_i^2/4$ and $\det \Lambda = -\sigma_{0t}^2/4$

$$\Lambda + \frac{1}{2}\sigma\mathbb{1} = \frac{1}{4\theta_\infty}\begin{pmatrix} (-\theta_\infty-\theta_1+\sigma)(\theta_\infty-\theta_1-\sigma) & (-\theta_\infty-\theta_1+\sigma)(\theta_\infty+\theta_1+\sigma) \\ (\theta_\infty-\theta_1+\sigma)(\theta_\infty-\theta_1-\sigma) & (\theta_\infty-\theta_1+\sigma)(\theta_\infty+\theta_1+\sigma) \end{pmatrix}$$

$$A_1^0 + \frac{1}{2}\theta_1\mathbb{1} = \frac{1}{4\theta_\infty}\begin{pmatrix} -(\theta_\infty-\theta_1)^2+\sigma^2 & (\theta_\infty+\theta_1)^2-\sigma^2 \\ -(\theta_\infty-\theta_1)^2+\sigma^2 & (\theta_\infty+\theta_1)^2-\sigma^2 \end{pmatrix}$$

$$A_0^0 + \frac{1}{2}\theta_0\mathbb{I} = G_1\frac{1}{4\sigma}\begin{pmatrix} (\theta_0-\theta_t+\sigma)(\theta_0+\theta_t+\sigma) & (\theta_0-\theta_t+\sigma)(-\theta_0-\theta_t+\sigma) \\ (\theta_0-\theta_t-\sigma)(\theta_0+\theta_t+\sigma) & (\theta_0-\theta_t-\sigma)(-\theta_0-\theta_t+\sigma) \end{pmatrix} G_1^{-1}$$

$$A_t^0 + \frac{1}{2}\theta_t\mathbb{I} = G_1\frac{1}{4\sigma}\begin{pmatrix} (\theta_t+\sigma)^2-\theta_0 & -(\theta_t-\sigma)^2+\theta_0^2 \\ (\theta_t+\sigma)^2-\theta_0 & -(\theta_t-\sigma)^2+\theta_0^2 \end{pmatrix} G_1^{-1}.$$