Semiclassical calculation of (n-point) spectral correlation functions for chaotic systems

Sebastian Müller (Bristol) and Marcel Novaes (Uberlândia)


Natal, March 2019
Universal spectral statistics

Classical chaos $\Rightarrow$ universal statistics of quantum energy levels, in agreement with random matrix theory (BGS conjecture)
Correlation functions

- **level density** \( \rho(E) = \sum_i \delta(E - E_i) \)

- **\( n \)-point correlation function**

\[
R_n(\epsilon_1, \epsilon_2, \ldots \epsilon_n) = \langle \rho(E + \epsilon_1) \rho(E + \epsilon_2) \cdots \rho(E + \epsilon_n) \rangle_E
\]

(we take \( \bar{\rho} = 1 \))

- for chaotic systems **without time reversal invariance:** agreement with prediction from **Gaussian Unitary Ensemble** (average over hermitian matrices)

\[
R_n(\epsilon_1, \epsilon_2, \ldots \epsilon_n) = \text{det} \left[ \frac{\sin(\pi(\epsilon_j - \epsilon_k))}{\pi(\epsilon_j - \epsilon_k)} \right]_{j,k}
\]
Correlation functions

for chaotic systems **with time reversal invariance:** agreement with prediction from **Gaussian Orthogonal Ensemble**
(average over real symmetric matrices)

\[
R_n(\epsilon_1, \epsilon_2, \ldots, \epsilon_n) = \text{Pf} \left( \begin{array}{cc}
D(\epsilon_i - \epsilon_j) & S(\epsilon_i - \epsilon_j) \\
-S(\epsilon_i - \epsilon_j) & I(\epsilon_i - \epsilon_j)
\end{array} \right)
\]

\[
S(x) = \frac{\sin(\pi x)}{\pi x}
\]

\[
D(x) = \int_0^1 du \, u \sin(\pi ux)
\]

\[
I(x) = -\int_1^\infty \frac{du}{u} \sin(\pi ux)
\]
2-point function

\[ R_2(\epsilon_1, \epsilon_2) = \langle \rho (E + \epsilon_1) \rho (E + \epsilon_2) \rangle \]

\[ = \text{Re} \left( \sum_m c_m \left( \frac{1}{\epsilon_1 - \epsilon_2} \right)^m + \sum_m d_m \left( \frac{1}{\epsilon_1 - \epsilon_2} \right)^m e^{2\pi i(\epsilon_1 - \epsilon_2)} \right) \]

c_m, d_m predicted by random matrix theory, depend only on symmetry
2-point function

use Gutzwiller's trace formula

\[ \rho(E) \approx \bar{\rho} + \frac{1}{\pi \hbar} \text{Re} \sum_{\text{per. orbits } p} T_p^{\text{prim}} F_p e^{iS_p(E)/\hbar} \]

\[ \bar{\rho} = 1 \quad \text{(for convenience)} \quad T_p^{\text{prim}} = \text{primitive period} \]

\[ F_p = \frac{1}{\sqrt{|\det(M_p - I)|}} e^{-i\mu_p \frac{\pi}{2}} \]

\[ S_p = \text{action} \]

\[ R_2(\epsilon_1, \epsilon_2) \approx 1 + \frac{1}{(\pi \hbar)^2} \text{Re} \sum_{p,q} \left\langle T_{p}^{\text{prim}} F_{p} T_{q}^{\text{prim}} F_{q}^{*} e^{i(S_p(E+\epsilon_1) - S_q(E+\epsilon_2))/\hbar} \right\rangle_E \]

⇒ need pairs of orbits with small action difference action correlations (Argaman et al 1993)
2-point function

- Diagonal approximation:
  \( q = p \) or time reversed of \( p \)
  \[ \Rightarrow \frac{1}{(\epsilon_1 - \epsilon_2)^2} \] term
  (Hannay & Ozorio de Almeida, Berry)

- Sum rule: \( \sum_p T_p^2 |F_p|^2 \delta(T_p - T) \approx T \)

- Sieber-Richter pairs
  \[ \Rightarrow \frac{1}{(\epsilon_1 - \epsilon_2)^3} \] term for time rev. inv. systems
2-point function

\[ \Rightarrow \frac{1}{(\epsilon_1 - \epsilon_2)^4} \text{ term} \]

etc ...

for oscillatory terms: need improved semiclassical approximation (Riemann-Siegel lookalike formula, Berry & Keating)

Agreement with random matrix theory 😊

\[ n\text{-point functions} \]

use

\[ \rho(E) \approx \bar{\rho} + \frac{1}{\pi \hbar} \text{Re} \sum_{\text{per. orbits } \rho} T^\text{prim}_\rho F_\rho e^{iS_\rho(E)/\hbar} \]

for factors in

\[ R_n(\epsilon_1, \epsilon_2, \ldots \epsilon_n) = \langle \rho(E + \epsilon_1)\rho(E + \epsilon_2) \ldots \rho(E + \epsilon_n) \rangle_E \]

two kinds of orbits:

- \( p \)-orbits contribute with \( e^{iS_\rho(E+\epsilon_j)/\hbar} \)
- \( q \)-orbits contribute with \( e^{-iS_q(E+\eta_k)/\hbar} \)

(after relabeling energy increments)

need small action difference

\[ \Delta S = \sum_{j=1}^{J} S_{\rho_j} - \sum_{k=1}^{K} S_{q_k} \]
\textit{n-point functions}

use

$$\rho(E) \approx \bar{\rho} + \frac{1}{\pi \hbar} \text{Re} \sum_{\text{per. orbits } p} T^{\text{prim}}_p F_p e^{iS_p(E)/\hbar}$$

for factors in

$$R_n(\epsilon_1, \epsilon_2, \ldots \epsilon_n) = \langle \rho(E + \epsilon_1)\rho(E + \epsilon_2)\ldots \rho(E + \epsilon_n) \rangle_E$$

further book-keeping:

- Taylor expand action using $\frac{dS}{dE} = T$
- get period factors using derivatives of actions
- contributions with Weyl term related to lower-order correlations
Contributing orbits

ccontributions with small small action difference

\[ \Delta S = \sum_{j=1}^{J} S_{p_j} - \sum_{k=1}^{K} S_{q_k} \]

- **diagonal approximation**: \( p \)- and \( q \)-orbits coincide pairwise
- \( p \)- and \( q \)-orbits coincide up to connections in **encounters**
- can also have mix of both mechanisms
Contributing orbits

3-point function: reconnections in one orbit give 2 orbits
Contributing orbits

Some contributions to 4-point function:
Semiclassical calculation

- action difference: e.g. for 2-encounter product of stable and unstable deviations between encounter stretches (Turek & Richter 2003, Spehner 2003)
- ergodicity: Hannay-Ozorio de Almeida sum rule, probability for encounters
- each link gives \((\epsilon_j - \eta_k)^{-1}\) \(j = \) index of \(p\)-orbit, \(k = \) index of \(q\)-orbit
- encounter contributions cancel some link contributions

Result proportional to:

\[
\prod_j \frac{\partial}{\partial \epsilon_j} \prod_j \frac{\partial}{\partial \eta_k} \sum_{\text{diagrams}} (-1)^{\# \text{enc}} \prod_{\text{links (uncancelled)}} (\epsilon_j - \eta_k)^{-1}
\]
general diagrammatic rule for non-oscillatory contributions
for systems with and without time-reversal invariance:
agreement with RMT up to 5-point correlation function for leading few orders,
for systems without time-reversal invariance:
proof that encounter contributions cancel in all orders, for arbitrary $n$–point functions
based on mapping to matrix model
(diagonal approximation evaluated in Nagao & S.M. 2009)
Matrix model

Encounter contributions proportional to

\[
\prod_{j=1}^{J} \frac{\partial}{\partial \epsilon_j} \prod_{k=1}^{K} \frac{\partial}{\partial \eta_k} \left[ r^{J+K} \right] \int d\mu(Z) \exp \left( - \sum_{q \geq 2} \text{Tr}[X(ZZ^\dagger)^q - (Z^\dagger Z)^q Y] \right)
\]

where

\[
d\mu(Z) = \exp \left( -\text{Tr}[XZZ^\dagger - Z^\dagger ZY] \right) dZ
\]

\[
X \propto \text{diag}(\epsilon_1, \ldots, \epsilon_1, \epsilon_2, \ldots, \epsilon_2, \ldots)
\]

\[
Y \propto \text{diag}(\eta_1, \ldots, \eta_1, \eta_2, \ldots, \eta_2, \ldots)
\]
Matrix model: Motivation

\[
\prod_{j=1}^{J} \frac{\partial}{\partial \epsilon_j} \prod_{k=1}^{K} \frac{\partial}{\partial \eta_k} \left[ r^{J+K} \right] \int d\mu(Z) \exp \left( - \sum_{q \geq 2} \text{Tr}[X(ZZ^\dagger)^q - (Z^\dagger Z)^q Y] \right)
\]

- expansion of exponential and Wick's theorem lead to terms like

\[
\int d\mu(Z) \text{Tr}[X(ZZ^\dagger ZZ^\dagger)] \text{Tr}[X(ZZ^\dagger ZZ^\dagger)]
\]

- contraction lines analogous to links, give factors \((\epsilon_j - \eta_k)^{-1}\)

- traces analogous to encounters, with \(Z_{jk}\) and \(Z_{jk}^*\) corresponding to 'ports' at the ends of encounter stretches and \(j, k\) corresponding to orbits:
can do integral exactly:

\[
\frac{\det(e^{X_j-Y_k}\text{Ei}(2N, X_j-Y_k))}{\det((X_j-Y_k)^{-1})}
\]

all terms in result vanish either due to \([r^J+K]\) or due to derivatives

off-diagonal contributions to all correlation functions cancel (for time-reversal invariant systems)
Conclusions

- $n$-point correlations of chaotic systems determined by multiple sums over orbits
- contributions arise if orbits are identical (up to time reversal) or differ in encounters
- $n$-point correlation functions agree with RMT with time-reversal invariance: checked leading few orders up to $n = 5$ (for non-oscillatory terms)
- without time-reversal invariance: cancellation of off-diagonal contributions shown using matrix integral
Oscillatory terms

Need improved semiclassical approximation: **Riemann-Siegel lookalike formula** (Berry, Keating 1990)

\[
\rho(E) = -\frac{1}{2\pi} \text{Im} \left. \frac{\partial}{\partial E'} \frac{\det(E - H)}{\det(E' - H)} \right|_{E' = E}
\]

\[
\det(E - H) = e^{-i\pi E} \times \sum_A F_A e^{iS_A(E)/\hbar} + \text{c.c.}
\]

Derivation:
- Gutzwiller formula for \( \text{tr} \frac{1}{E - H} \)
- \( \det(E - H) = \exp \text{tr} \ln(E - H) = \exp \left( \int \text{tr} \frac{1}{E - H} \right) \)
- expand exponential
- get relation between short and long orbits from sum over sets of classical periodic orbits shorter than \( T_H/2 \)