Localization in interacting fermionic chains with quasi-random disorder

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- In 1D typically any amount of disorder produces localization, while in 3D the disorder has to be sufficiently strong and a metal to insulator transition is expected varying the strength of the random field. Still open problems in 2D and 3D.

Many Body Localization

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- MBL has dramatic consequences for non equilibrium: Many-body localized systems fail to thermally equilibrate.
- A proof of MBL in generality is a challenging problem (single particle description breaks down, full N-particle Schroedinger)

Localization and MBL

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- With no interaction very good theoretical understanding based on advanced mathematical tools; quest for understanding the role of interaction.

The interacting Aubry-Andre' model

- If a^+_x, a^-_x, $x \in \mathbb{Z} \equiv \Lambda$ are spinless creation or annihilation operators on the Fock space verifying $\{a^+_x, a^-_y\} = \delta_{x,y}$, $\{a^+_x, a^+_y\} = \{a^-_x, a^-_y\} = 0$. The Fock space Hamiltonian is

$$H = -\varepsilon(\sum_{x\in\Lambda}(a^+_{x+1}a_x + a^+_{x-1}a^-_x) +$$

$$\sum_{x\in\Lambda} u\cos(2\pi(\omega x + \theta))a^+_x a^-_x + U\sum_{x,y} v(x-y)a^+_x a^-_x a^+_y a^-_y$$

with $v(x-y) = \delta_{y-x,1} + \delta_{x-y,1}$.

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- Early studies of the extended phase in Mastropietro (1999) and Giamarchi, Mohunna,Vidal (1999)

The Aubry-Andre' model

- In the non interacting case the states are obtained by the antisymmetrization (fermions) of the eigenfunctions of almost Mathieu equation

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- the spectrum is a Cantor set for all irrational ω. For almost every ω, θ the almost Mathieu operator has
  a)for $\varepsilon/u < \frac{1}{2}$ exponentially decaying eigenfunctions (Anderson localization);
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  b)for $\varepsilon/u > \frac{1}{2}$ purely absolutely continuous spectrum (extended quasi-Bloch waves)
- Metal insulator transition (with no interaction) seen experimentally by Roati et al (2008)

The KAM theorem

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- KAM ensures the existence of quasi-periodic solutions of Hamiltonian systems close to integrable one, that is the stability of invariant tori. Applications to the stability of solar system.
- A crucial assumption of KAM and of the analysis of almost mathieu is that the frequency verify a number theoretical condition called Diophantine condition to deal with small divisors. It says that a number is a "good irrational" and is full mesaure.

Diphantine conditions

- We impose a Diophantine condition on the frequency

$$||\omega x|| \geq C_0|x|^{-\tau} \quad \forall x \in \mathbb{Z}/\{0\} \quad (*)$$

$||.||$ is the norm on the one dimensional torus of period 1.

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- The continued fraction representation of a number ω

$$\omega = a_0 + \cfrac{1}{a_1 + \frac{1}{a_2 + \frac{1}{a_3 + \dots}}}$$

The golden ratio $\omega = \frac{\sqrt{5}+1}{2}$ has representation $1; 1, ..1, ..$ and it verifies the Diophantine condition with $\tau = 1$ and $C_0 = \frac{3+\sqrt{5}}{2}$.

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- For ω θ verifying Diophantine conditions, for small $\frac{\varepsilon}{u}$, $\frac{U}{u}$ the fermionic zero temperature grand canonical infinite volume truncated correlations of local operators decays exponentially for large distances.
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- Renormalized expansion around the anti-integrable limit

Main resut

- $$H = -\varepsilon\Big(\sum_{x\in\Lambda}(a^+_{x+1}a_x + a^+_{x-1}a^-_x\Big) + \sum_{x\in\Lambda} u\cos(2\pi(\omega x+\theta))a^+_x a^-_x + U\sum_{x,y} v(x-y)a^+_x a^-_x a^+_y a^-_y$$

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with $v(x-y) = \delta_{y-x,1} + \delta_{x-y,1}$.
- If $a^{\pm}_{\mathbf{x}} = e^{(H-\mu N)x_0} a^{\pm}_x e^{-(H-\mu N)x_0}$, $\mathbf{x} = (x, x_0)$, $N = \sum_x a^+_x a^-_x$ and μ the chemical potential, the Grand-Canonical imaginary time 2-point correlation is
$$< \mathbf{T} a^-_{\mathbf{x}} a^+_{\mathbf{y}} >= \frac{Tr e^{-\beta(H-\mu N)}\mathbf{T}\{a^-_{\mathbf{x}} a^+_{\mathbf{y}}\}}{Tr e^{-\beta(H-\mu N)}}$$
where $\mathbf{T}$ is the time-order product and μ is the chemical potential.

Localized regime

- It is convenient to write the chemical potential as a function of the interaction so that the density has the same value in the free or interacting case. We introduce a counterterm ν so that the interacting chemical potential is $u \cos 2\pi(\omega\hat{x} + \theta)$.

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- We impose a Diophatine condition on the frequency (*)
- A condition on the phase is also imposed

$$||\omega x \pm 2\theta|| \geq C_0 |x|^{-\tau} \quad \forall x \in \mathbb{Z}/\{0\} \quad (**)$$

Localized regime

Theorem

Under conditions (*) and (**), $u = 1$ $\mu = \cos 2\pi(\omega\hat{x} + \theta) + \nu$ *there exists an* ε_0 *such that, for* $|\varepsilon|, |U| \leq \varepsilon_0$*,it is possible to choose* ν *so that the limit* $\beta \to \infty$

$$|< \mathbf{T} a_{\mathbf{x}}^- a_{\mathbf{y}}^+ >| \leq C e^{-\xi|x-y|} \log(1+\min(|x||y|))^\tau \frac{1}{1 + (\Delta|x_0 - y_0)|)^N} (***)$$

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- (**) could be replaced by a condition on the density

Extended regime

- Different behavior is found close to the integrable limit. Fix $\varepsilon = 1, \theta = 0$, U, u small, $\mu = \cos p_F$, $||2\pi\omega n||_{2\pi} \geq C|n|^{-\tau}$, $n \neq 0$, then (M, arxiv 1604.08264, PRB 2016) :

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- 1)If $||2p_F + 2\pi n\omega||_{2\pi} \geq C|n|^{-\tau}$ a decay of the two point function $O(|x-y|^{-1-\eta})$, $\eta = aU^2 + O(U^3)$ (metallic Luttinger liquid behavior).

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- 2) If $p_F = n\omega\pi$ a faster than any power decay with rate

$$\Delta_{n,U} \sim [u^{2n}(a_n + F)]^{X_n}$$

with $F = O(|U| + |\lambda|)$, a_n non vanishing and $X_n = X_n(U) = 1 + bU + O(U^2)$; the decay rate is of the order of the interacting gap. Dense set of gaps.

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- All gaps are renormalized via a critical exponent

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- In the case of a Fibonacci quasi-periodic potential there is evidence that the interaction closes the smallest gaps, Giamarchi (1999), causing an insulating to metal transition.
- In the case of Aubry-Andre' potential all gaps persists instead; no quantum phase transition at small coupling.

Integrable limit

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  $\mu = \cos p_F$. $\pm p_F$ Fermi momenta. GS occupation number $\chi(\cos k - \mu \leq 0)$.

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- Close to the singularity

$$\cos(k' \pm p_F) - \mu \sim \pm \sin p_F k' + O(k'^2)$$

  linear dispersion relation.

Anti-integrable limit

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- The denominator can be arbitrarily large; for $x \neq \rho\hat{x}$ by (*),(**) ,$||\omega x'|| = ||\omega(x - \rho\hat{x}) + 2\delta_{\rho,-1}\theta|| \geq C|x - \rho\hat{x}|^{-\tau}$. $(\omega x')_{\mathrm{mod}.1}$ can be very small for large x (infrared-ultraviolet mixing)

Anti-integrable limit; proof of localization

The 2-point function is given by $\frac{\partial^2}{\partial \phi^+_{\mathbf{x}} \partial \phi^-_{\mathbf{y}}} W|_0$

$$e^{W(\phi)} = \int P(d\psi) e^{-V(\psi) - \mathcal{B}(\psi,\phi)}$$

with $P(d\psi)$ a gaussian Grassmann integral with propagator $\delta_{x,y}\bar{g}(x, x_0 - y_0)$

$$V(\psi) = U \int d\mathbf{x} \sum_{\alpha=\pm} \psi^+_{\mathbf{x}} \psi^-_{\mathbf{x}} \psi^+_{\mathbf{x}+\alpha \mathbf{e}_1} \psi^-_{\mathbf{x}+\alpha \mathbf{e}_1}$$

$$+\varepsilon \int d\mathbf{x}(\psi^+_{\mathbf{x}+\mathbf{e}_1} \psi^-_{\mathbf{x}} + \psi^+_{\mathbf{x}-\mathbf{e}_1} \psi^-_{\mathbf{x}}) + \nu \int d\mathbf{x} \psi^+_{\mathbf{x}} \psi^-_{\mathbf{x}}$$

where $\int d\mathbf{x} = \sum_{x \in \Lambda} \int_{-\frac{\beta}{2}}^{\frac{\beta}{2}} dx_0$, Finally $B = \int d\mathbf{x}(\phi^+_{\mathbf{x}} \psi^-_{\mathbf{x}} + \psi^+_{\mathbf{x}} \phi^-_{\mathbf{x}})$

Small divisors

- In absence of many body interaction there are only chain graphs, $\alpha_i = \pm$

$$\varepsilon^n \sum_{x_1} \int dx_{0,1} \dots dx_{0,n} \bar{g}(x_1, x_0 - x_{0,1}) \bar{g}(x_1 + \sum_{i \leq n} \alpha_i, (x_{0,n} - y_0))$$

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- When $U \neq 0$ there also loops producing additional divergences, absent in classically.
- To establish localization in presence of interaction one has to prove that such small divisors are harmless. Sort of quantum KAM. Constructive RG approach.

Some idea of the proof

- We perform an *RG* analysis decomposing the propagator as sum of propagators living at scale $|\phi_x - \phi_{\hat{x}}| \sim \gamma^h$, $h = 0, -1, -2...$, $\phi_x = \cos 2\pi(\omega x + \theta)$; this correspond to two regions, around $\bar{x}_+$ and $\bar{x}_-$.

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- Very similar to what is done in the integrable limit u/ε small (Aubry duality).
- We integrate the fields with decreasing scale; for instance $W(0)$ (the partition function) can be written as

$$\int P(d\psi)e^V = \int P(d\psi^{\leq -1}) \int P(d\psi)e^V = \int P(d\psi^{\leq -1})e^{V^{-1}}...$$

The effective potential V^h, sum of monomials of any order in $\psi_\rho^\pm$.

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- It turns out that the non resonant terms are irrelevant (even if they are relevant according to power counting).
- Roughly speaking, the idea is that if two propagators have similar (not equal) small size (*non resonant subgraphs*) , then the difference of their coordinates is large and this produces a "gain" as passing from x to $x + n$ one needs n vertices. That is if $(\omega x'_1)_{\mathrm{mod}1} \sim (\omega x'_2)_{\mathrm{mod}1} \sim \Lambda^{-1}$ then by the Diophantine condition

$$2\Lambda^{-1} \geq ||\omega(x'_1 - x'_2)|| \geq C_0 |x'_1 - x'_2|^{-\tau}$$

that is $|x'_1 - x'_2| \geq \bar{C} \Lambda^{\tau^{-1}}$

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- As usual in renormalization theory, one needs to introduce clusters v with scale h_v; the propagators in v have divisors smaller than γ^{h_v} (necessary to avoid overlapping divergences). Zimmermann forests or Gallavotti-Nicolo' trees. v' is the cluster containing v.

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- Consider two vertices w_1, w_2 such that x'_{w_1} and x'_{w_2} are coordinates of the external fields, and let be c_{w_1,w_2} the path (vertices and lines) in $\bar{T}_v$ connecting w_1 with w_2; we call $|c_{w_1,w_2}|$ the number of vertices in c_{w_1,w_2}. The following relation holds, if $\delta^i_w = \pm 1$ it corresponds to an ε end-point and $\delta^i_w = (0, \pm 1)$ is a U end-point

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- As $x_i - x_j = M \in \mathbb{Z}$ and $x'_i = x'_j$ then $(\bar{x}_{\rho_i} - \bar{x}_{\rho_j}) + M = 0$, so that $\rho_i = \rho_j$ as $\bar{x}_+ = \hat{x}$ and $\bar{x}_- = -\hat{x} - 2\theta/\omega$ and $\hat{x} \in \mathbb{Z}$.

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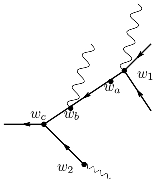

FIG. 1: A tree $\bar{T}_v$ with attached wiggly lines representing the external lines P_v; the lines represent propagators with scale $\geq h_v$ connecting w_1, w_a, w_b, w_c, w_2, representing the end-points following v in τ.

.

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- By the Diophantine condition a) $\rho_{w_1} = \rho_{w_2}$ the (*); b)if $\rho_{w_1} = -\rho_{w_2}$ by (**)

$$2cv_0^{-1}\gamma^{h_{\bar{v}'}} \geq$$
$$||(\omega x'_{w_1})||_1 + ||(\omega x'_{w_2})||_1 \geq ||\omega(x'_{w_1} - x'_{w_2})||_1 \geq C_0(|c_{w_2,w_1}|)^{-\tau}$$

so that $|c_{w_1,w_2}| \geq A\gamma^{\frac{-h_{\bar{v}'}}{\tau}}$. If two external propagators are small but not exactly equal, you need a lot of hopping or interactions to produce them

Ideas of proof

- If $\bar{\varepsilon} = \max(|\varepsilon|, |U|))$ from the $\bar{\varepsilon}^n$ factor we can then extract

$$\bar{\varepsilon}^{\frac{n}{4}} \leq \prod_{v \in L} \varepsilon^{N_v 2^{h_{v'}}}$$

where N_v is the number of points in v; as $N_v \geq |c_{w_1, w_2}| \geq A\gamma^{\frac{-h_{v'}}{\tau}}$ then

$$\bar{\varepsilon}^{\frac{n}{4}} \leq \prod_{v \in L} \bar{\varepsilon}^{A\gamma^{\frac{-h_{v'}}{\tau}} 2^{h_{v'}}}$$

where L are the non resonant vertices. If $\gamma^{\frac{1}{\tau}}/2 > 1$ then $\leq C^n \prod_{v \in L} \gamma^{3h_v S_v^L}$ where S_v^L is the number of non resonant clusters in v.

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- We localize the resonant terms $\mathbf{x} = x_{0,i}, x$ with all x'_i equal

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- The result can be rephrased fixing θ and changing the chemical potential.

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- This is true for quasi-periodic functions with fast decaying Fourier transform; With other quasi-random noise, is believed instead that there are infinitely many rcc.

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- Spin? Coupled chains? other eigenstates? 2 or 3 dimension?