# Relaxation laws in classical and quantum long-range lattices

R. Bachelard

*Grupo de Óptica – Instituto de Física de São Carlos – USP*

*"Quantum Non-Equilibrium Phenomena"*

*Natal RN– 13/06/2016*

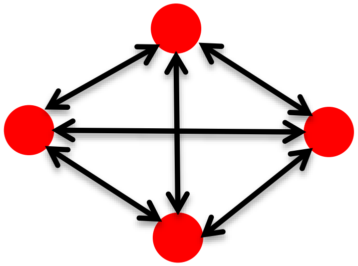
# Lattice systems with long-range interactions

$\varphi_i$ $\qquad \varphi_j$ variables

What is a lattice? $\qquad i$  $j$

$\vec{r}_i$ $\qquad \vec{r}_j$ *fixed* positions

Power-law interaction: all-to-all coupling



$$V_{i,j} \sim \frac{v(\varphi_i, \varphi_j)}{\left|\vec{r}_i - \vec{r}_j\right|^{\alpha}} \quad \rightarrow \quad H = \sum_{i,j\neq i} V_{i,j} \qquad \left( + \sum_i \frac{p_i^2}{2} \right)$$

# Lattice systems with long-range interactions

$\alpha > d$ for **short-range** interactions

The system is additive and extensive

$$H = \underbrace{\sum_{i,j\neq i} V_{i,j}}_{\sim N} \left( + \underbrace{\sum_i \frac{p_i^2}{2}}_{\sim N} \right)$$

$N$: number of particles

$d$: dimension of the space

# Lattice systems with long-range interactions

$\alpha > d$ for **short-range** interactions

The system is **additive** and **extensive**

$$H = \underbrace{\sum_{i,j\neq i} V_{i,j}}_{\sim N} \left( + \underbrace{\sum_i \frac{p_i^2}{2}}_{\sim N} \right)$$

| |
|---|
| $N$: number of particles |
| $d$: dimension of the space |

$\alpha < d$ for **long-range** interactions

The system is **non-additive** and **non-extensive**

$$H = \underbrace{\sum_{i,j\neq i} V_{i,j}}_{\sim N \mathrm{x} N^{1-\alpha/d}} \left( + \underbrace{\sum_i \frac{p_i^2}{2}}_{\sim N} \right)$$

**non-extensive**

$\longrightarrow$

$$\left.\begin{aligned} t &\rightarrow t N^{(1-\alpha/\mathrm{d})/2} \\ H &\rightarrow H / N^{(1-\alpha/\mathrm{d})} \\ p &\rightarrow p / N^{(1-\alpha/\mathrm{d})/2} \end{aligned}\right\}$$

The system becomes **extensive**

# Why studying long-range systems?

Rich nonequilibrium physics due to its slow relaxation:

- Ensemble inequivalence
- Size-dependent equilibration time
- The $N \to \infty$ and $t \to \infty$ limits do not commute!
- Out-of-equilibrium phase transitions and critical points
- Phase reentrance

# Long-range $\alpha XY$ chain

A chain of rotators on a lattice, with Hamiltonian dynamics

$$H = \sum_i \frac{p_i^2}{2} - \frac{J}{2} \sum_{i,j \neq i} \frac{\cos(\varphi_i - \varphi_j)}{|i-j|^\alpha}$$

$\varphi_i$ : phase

$p_i$ : momentum

# Long-range $\alpha XY$ chain

A chain of rotators on a lattice, with Hamiltonian dynamics

$$H = \sum_i \frac{p_i^2}{2} - \frac{J}{2} \sum_{i,j \neq i} \frac{\cos(\varphi_i - \varphi_j)}{|i-j|^{\alpha}}$$

$\varphi_i$ : phase

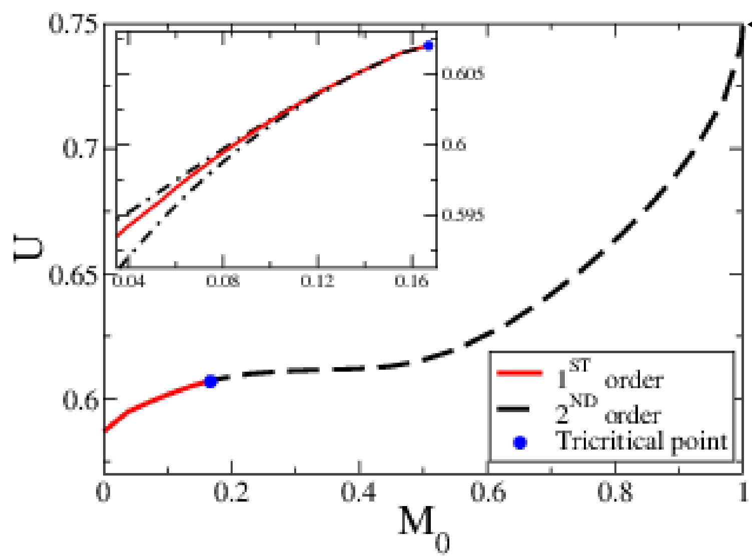
$p_i$ : momentum

<u>Mean-field case: $\alpha$=0</u>

Nonequilibrium phase diagram

**Nonequilibrium: $M = 0$**

**Equilibrium: $M \neq 0$**



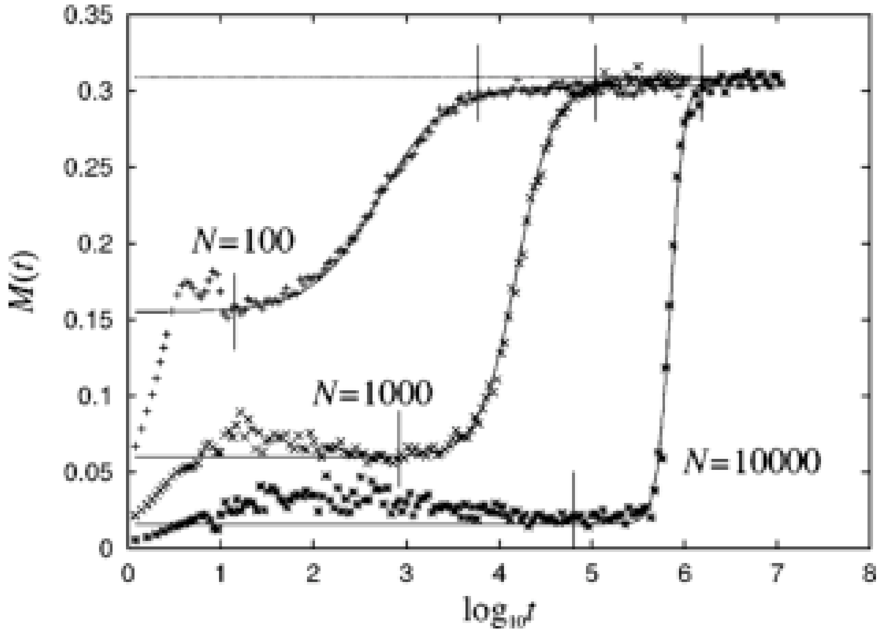

Equilibrium phase transition

[Antoniazzi *et al.*, Phys Rev Lett. **99**, 040601 (2007)]

# Long-range $\alpha XY$ chain

Just below the transition point,

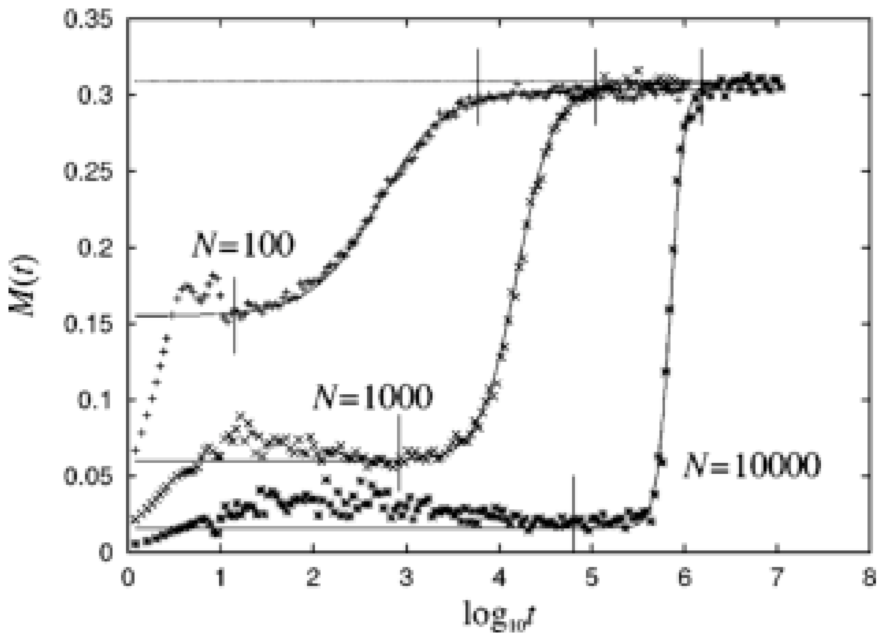
the system slowly goes from an unmagnetized phase to a magnetized one



# Long-range $\alpha XY$ chain

Just below the transition point,

the system slowly goes from an unmagnetized phase to a magnetized one

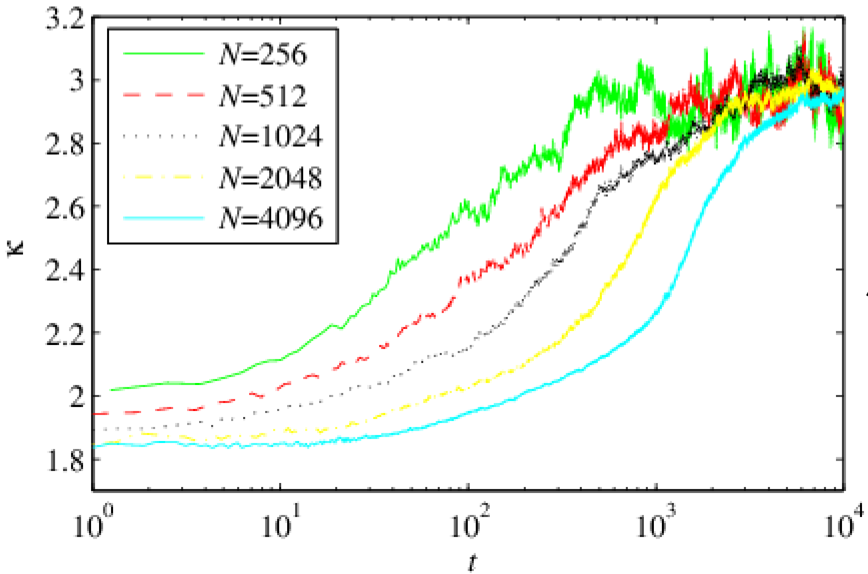



A **size-dependent relaxation time** was observed

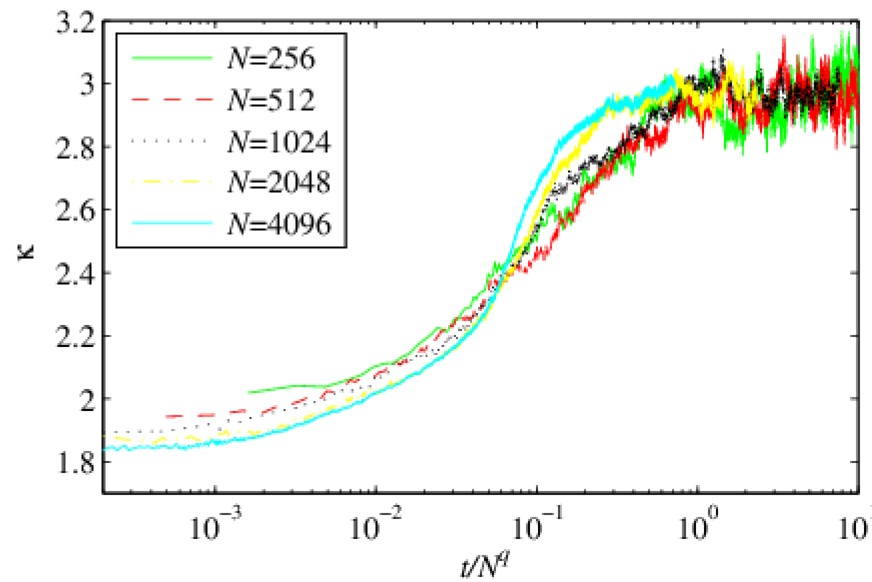
A law $\boldsymbol{\tau} \sim \boldsymbol{N^q}$, $with\ q = 1.2$ was reported in the mean-field case

[Y. Yamaguchi..Phys Rev E **68**, 066210 (2003)]

# Long-range $\alpha XY$ chain: $\alpha>0$




Time rescaling $t \to t/N^q$



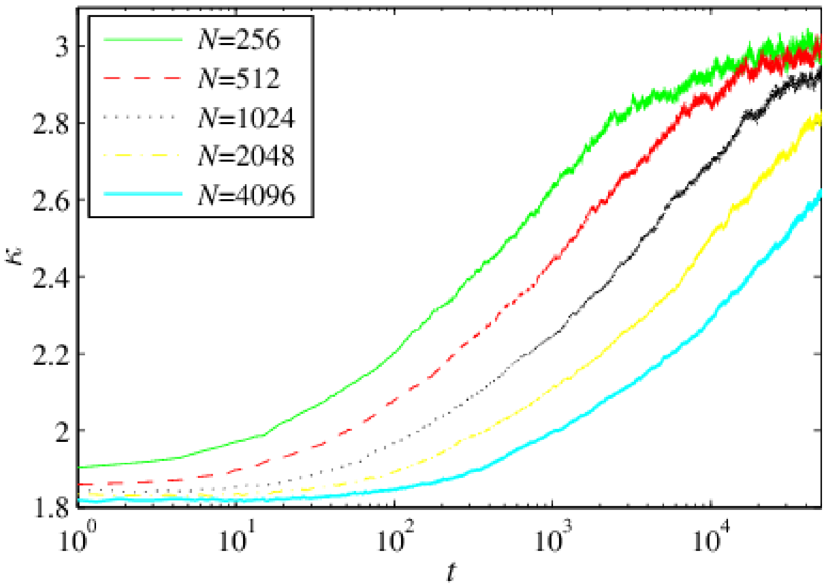

Just below the transition, for any $0 \leq \alpha < d$

we observe relaxation times that scale as a power-law of $N$

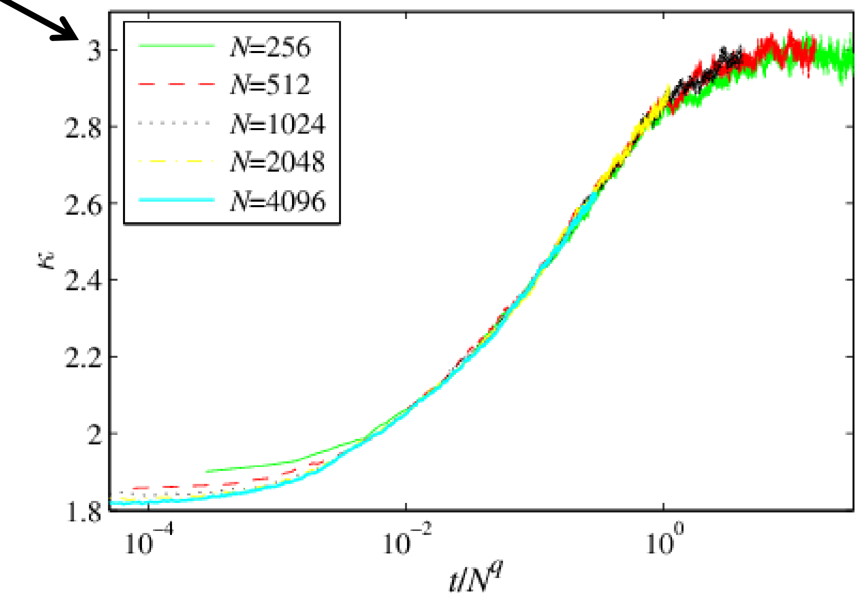
# Long-range $\alpha XY$ chain: $\alpha > 0$

high energy regime: $U \gg U_c$

*(the system remains unmagnetized at any time)*




Time rescaling $t \to t/N^q$

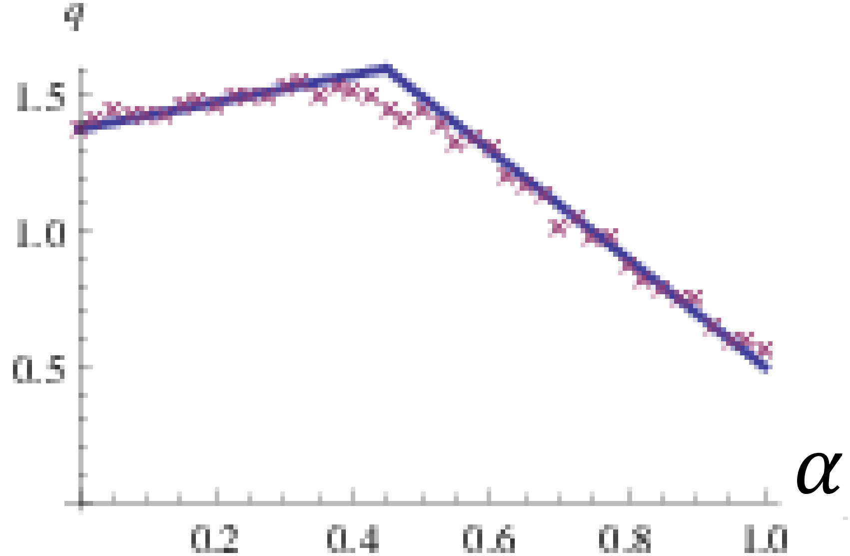



For any energy, for any $0 \leq \alpha < d$

we observe relaxation times that scale as a power-law of $N$

# Long-range $\alpha XY$ chain

$$\tau_{relax} \sim N^q$$




Size-dependent slow relaxation time (~power-law)

# Long-range quantum Ising model

$$H = -\frac{J}{2}\sum_{i,j\neq i}\frac{\sigma_i^z\sigma_j^z}{\left|i-j\right|^\alpha} - h\sum_i \sigma_i^z$$

| | |
|---|---|
| $\sigma_i^z$ | $z$-component of the Pauli spin operator |
| $h$ | external magnetic field |
| $x_i = i$ | particle position |

*Integrable system, but that provides a dynamics for* $\sigma_i^x(t)$

# Long-range quantum Ising model

$$H = -\frac{J}{2}\sum_{i,j\neq i}\frac{\sigma_i^z\sigma_j^z}{|i-j|^\alpha} - h\sum_i \sigma_i^z$$

| | |
|---|---|
| $\sigma_i^z$ | $z$-component of the Pauli spin operator |
| $h$ | external magnetic field |
| $x_i = i$ | particle position |

*Integrable system, but that provides a dynamics for* $\sigma_i^x(t)$

$$\langle\sigma_i^x\rangle(t) = \langle\sigma_i^x\rangle(0)\cos(2ht)\prod_{j\neq i}\cos\left(\frac{2Jt}{|i-j|^\alpha}\right)$$

# Thermodynamic limit: $N \to \infty$

$$\left|\left\langle \sigma_i^x \right\rangle(t)\right| \le \left|\left\langle \sigma_i^x \right\rangle(0)\cos\left(2ht\right)\right| \begin{cases} \exp\left[-\dfrac{2^{5+2\alpha-d}\pi^{d/2-2}}{(d-2\alpha)\Gamma(d/2)} J^2 t^2 N^{1-2\alpha/d}\right] & \text{for } 0 \le \alpha < \dfrac{d}{2} \\ \exp\left[-\dfrac{2^{1+d-2\alpha}\pi^{d/2}}{(2\alpha-d)\Gamma(d/2)}\left|\dfrac{4Jt}{\pi}\right|^{d/\alpha}\right] & \text{for } \alpha > \dfrac{d}{2} \end{cases}$$

# Thermodynamic limit: $N\to\infty$

$$\left|\left\langle \sigma_i^x \right\rangle(t)\right| \le \left|\left\langle \sigma_i^x \right\rangle(0)\cos(2ht)\right| \begin{cases} \exp\left[-\dfrac{2^{5+2\alpha-d}\pi^{d/2-2}}{(d-2\alpha)\Gamma(d/2)} J^2 t^2 N^{1-2\alpha/d}\right] & \text{for } 0\le\alpha<\dfrac{d}{2} \\ \exp\left[-\dfrac{2^{1+d-2\alpha}\pi^{d/2}}{(2\alpha-d)\Gamma(d/2)}\left|\dfrac{4Jt}{\pi}\right|^{d/\alpha}\right] & \text{for } \alpha>\dfrac{d}{2} \end{cases}$$

- For $\alpha < d/2$, relaxation to equilibrium is Gaussian in time,

  with time scale $\boldsymbol{\tau \sim N^{\alpha/d-1/2}}$ (shrinks to zero in the large $N$ limit)

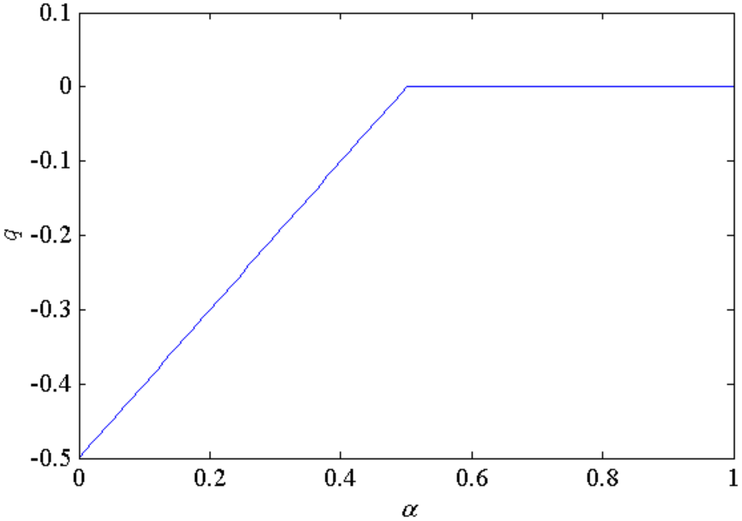
- For $\alpha > d/2$, compressed/stretched exponential in $t$,

  with time scale $\boldsymbol{\tau \sim cst}$ in the $N\to\infty$ limit

# Relaxation of the quantum Ising model

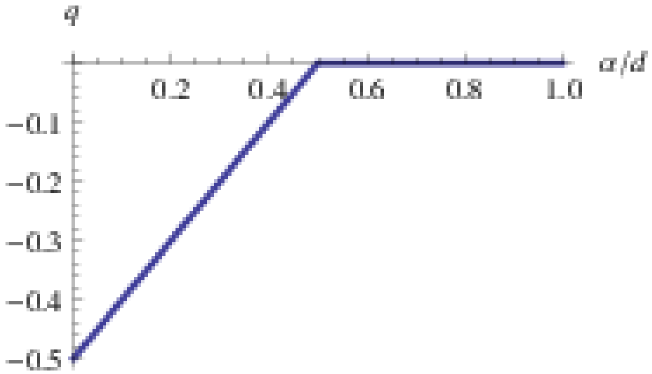
For $\alpha < d/2$ we have relaxation times that scale as a power-law of the system size N

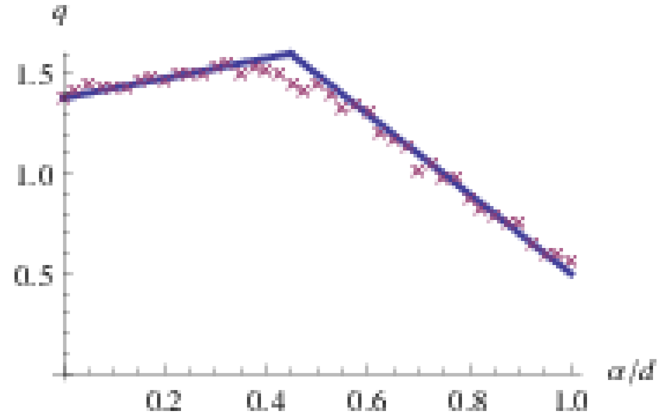
$$\tau \sim N^{q}$$



# Quantum Ising model vs. $\alpha XY$ chain

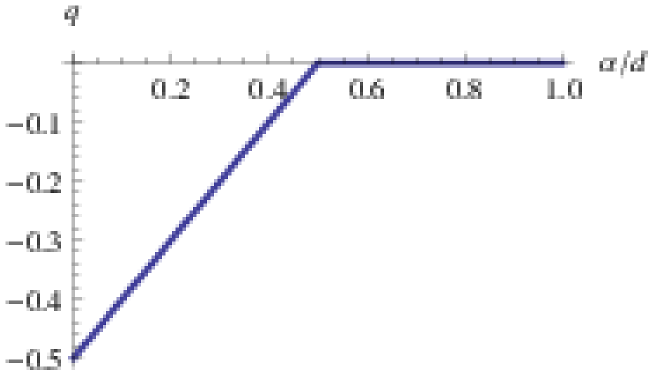
$$\tau_{relax} \sim N^q$$

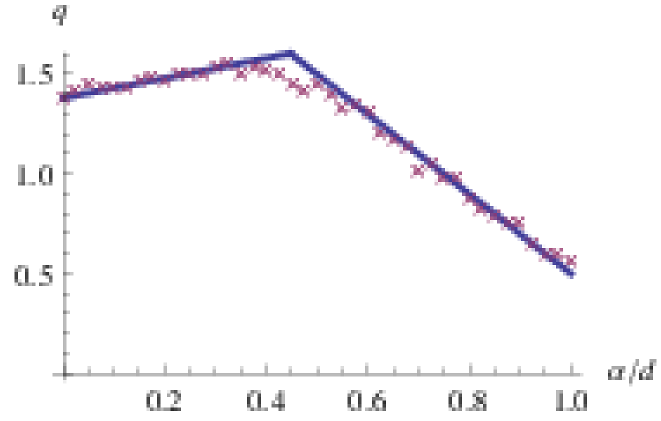


# Quantum Ising model vs. $\alpha XY$ chain

$$\tau_{relax} \sim N^{q}$$

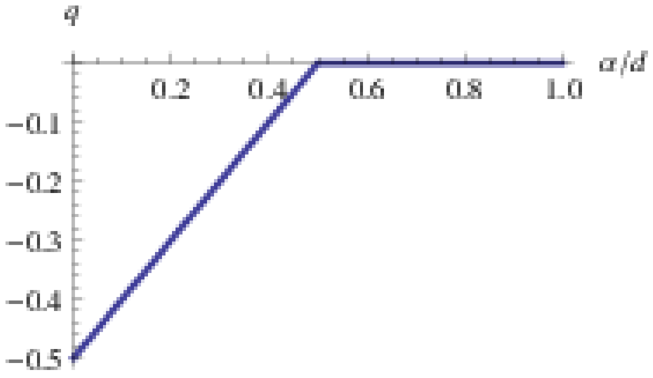


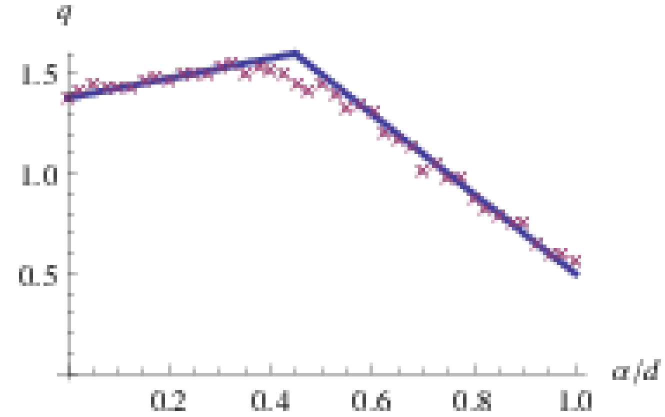




- Both systems exhibit size-dependent relaxation times
- For both systems (one quantum and one classical), a (universal) threshold at $\alpha = d/2$ is observed

# Quantum Ising model vs. $\alpha XY$ chain

$$\tau_{relax} \sim N^{q}$$




- Both systems exhibit size-dependent relaxation times
- For both systems (one quantum and one classical), a (universal) threshold at $\alpha = d/2$ is observed

- Why at $\alpha = d/2$ ? At the moment, I have no clue... (but it is a **dynamical** threshold)
- The scaling laws look quite different in the two cases (diverging vs. vanishing time scales)

# Normalization of the energy scale

$$H = -\frac{J}{2}\sum_{i,j\neq i}\frac{\sigma_i^z\sigma_j^z}{|i-j|^\alpha} - h\sum_i \sigma_i^z \rightarrow \tilde{H} = -\frac{\tilde{J}}{2}\sum_{i,j\neq i}\frac{\tilde{\sigma}_i^z\tilde{\sigma}_j^z}{|i-j|^\alpha} - h\sum_i \tilde{\sigma}_i^z$$

$$H = \sum_i \frac{p_i^2}{2} - \frac{J}{2}\sum_{i,j\neq i}\frac{\cos(\varphi_i-\varphi_j)}{|i-j|^\alpha} \rightarrow \tilde{H} = \sum_i \frac{\tilde{p}_i^2}{2} - \frac{\tilde{J}}{2}\sum_{i,j\neq i}\frac{\cos(\varphi_i-\varphi_j)}{|i-j|^\alpha}$$

$\tilde{J}$ and *time* renormalized by a power of $N$ so $\tilde{H}$ is extensive ($\tilde{H} \sim N$)

[“Kac prescription”]

This normalization

- is necessary to have a well-defined thermodynamic limit

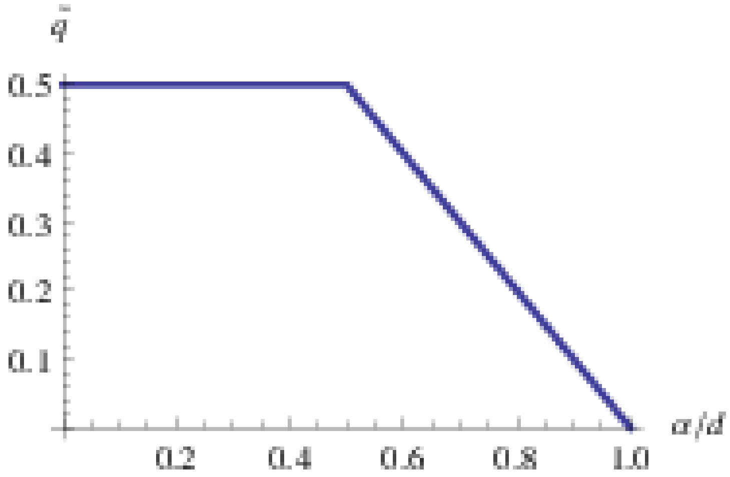
  (very useful in statistical physics)

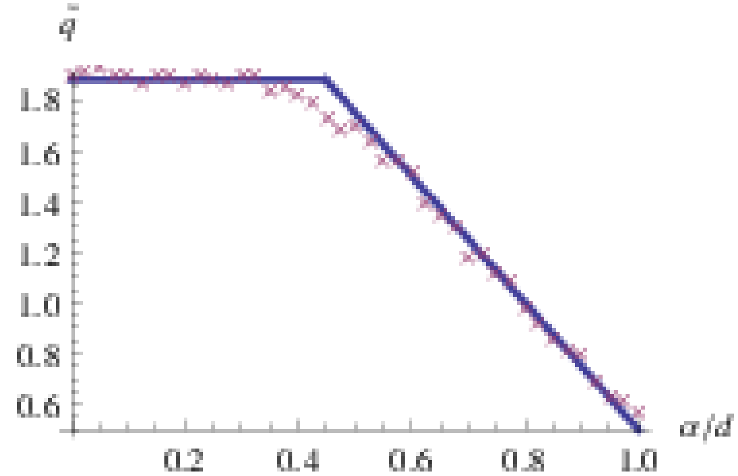
- provides a size-independent time scale for short-time dynamics

# Normalization of the energy scale

$$H=-\frac{J}{2}\sum_{i,j\neq i}\frac{\sigma_i^z\sigma_j^z}{|i-j|^\alpha}-h\sum_i\sigma_i^z \rightarrow \tilde{H}=-\frac{\tilde{J}}{2}\sum_{i,j\neq i}\frac{\tilde{\sigma}_i^z\tilde{\sigma}_j^z}{|i-j|^\alpha}-h\sum_i\tilde{\sigma}_i^z \longrightarrow \tilde{q}=q+\left(1-\frac{\alpha}{d}\right)$$

$$H=\sum_i\frac{p_i^2}{2}-\frac{J}{2}\sum_{i,j\neq i}\frac{\cos(\varphi_i-\varphi_j)}{|i-j|^\alpha}\rightarrow \tilde{H}=\sum_i\frac{\tilde{p}_i^2}{2}-\frac{\tilde{J}}{2}\sum_{i,j\neq i}\frac{\cos(\varphi_i-\varphi_j)}{|i-j|^\alpha} \longrightarrow \tilde{q}=q+\frac{1}{2}\left(1-\frac{\alpha}{d}\right)$$




Using the "normalized Hamiltonian", the similarity between the scaling laws of the two models is striking

R. Bachelard and M. Kastner, Phys. Rev. Lett. **110**, 170603 (2013)

# Conclusion/perspectives

- Long-range classical and quantum systems have similar relaxations, with relaxation times that exhibit dependence on the system size (power-laws)

- Theory is still missing... Especially for the $\alpha=d/2$ threshold

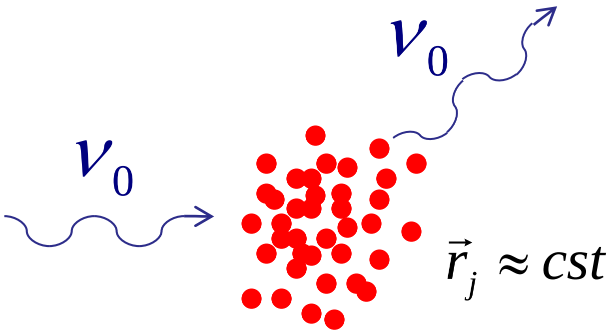
Realized in collaboration with **Michael Kastner**

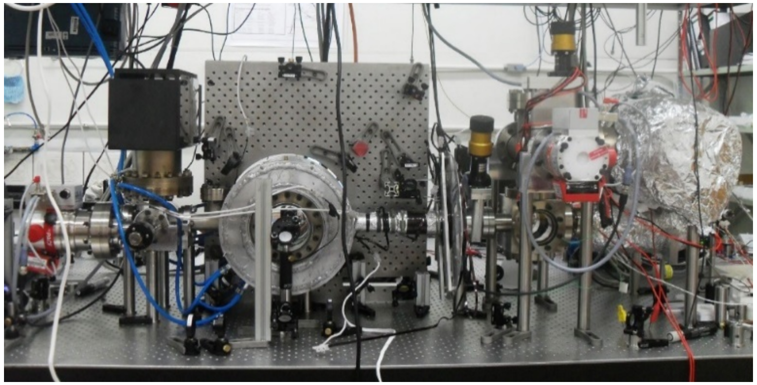
*NITheP, Stellenbosch, South Africa*

R. Bachelard and M. Kastner, Phys. Rev. Lett. **110**, 170603 (2013)

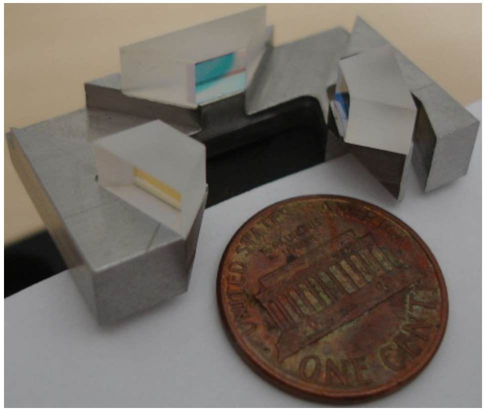
# Example: Light scattering by cold atomic clouds

Light induces a $1/r$ interaction between the atoms in a $d=3$ space and a mean-field interaction in a d=1 space (optical cavity)



3d experiment@São Carlos SP



1d experiment@São Carlos SP

(under construction)

Don’t hesitate to visit us in São Carlos!

www.ifsc.usp.br/~strontium