Restriction on the initial state for the true gauge invariance

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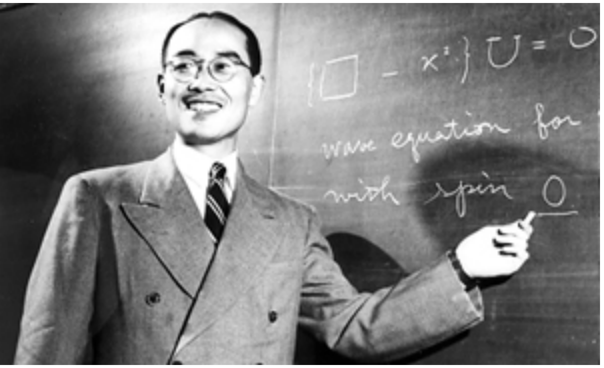

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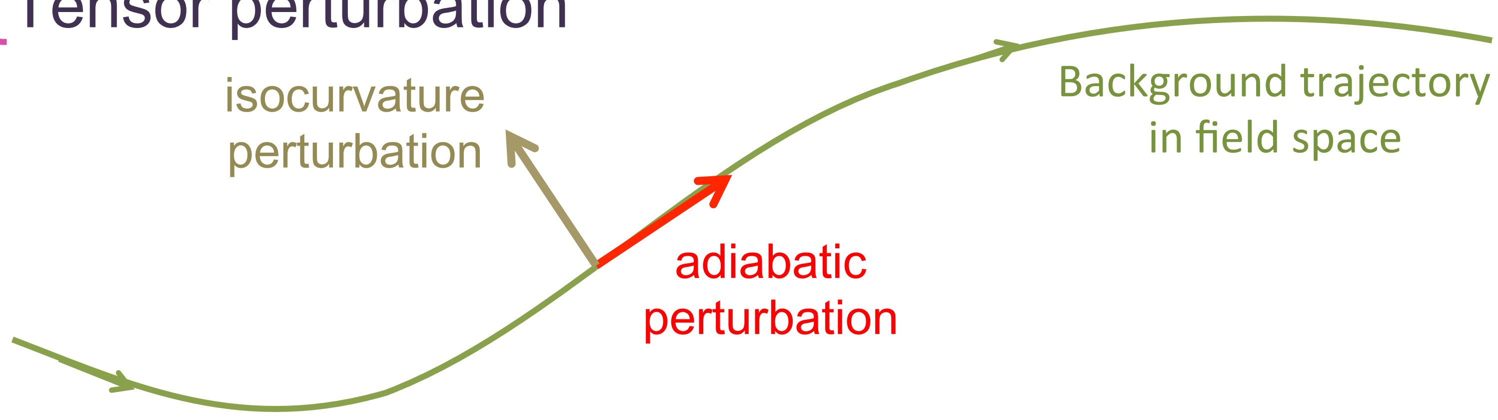
Various IR issues

IR divergence coming from k-integral
Secular growth in time $\propto (Ht)^n$

Adiabatic perturbation,
which can be locally absorbed by the choice of time slicing.

Isocurvature perturbation
≈ field theory on a fixed curved background

Tensor perturbation



§ Isocurvature perturbation

≈ field theory in de Sitter space

$m^2 > 0$: de Sitter invariant vacuum state even with interaction exists.

If we choose de Sitter invariant vacuum at the beginning, the state remains unchanged.

So, there is no secular time evolution in this case!

However, if the initial state is different, secular time evolution will happen.

However, the secular growth can be not an instability but just a relaxation process to the de Sitter invariant vacuum state.

Clustering property: when $\{x_1,\ldots,x_m\}$ are far apart from $\{x_{m+1},\ldots,x_n\}$

$$\langle\phi(x_1)\cdots\phi(x_m)\phi(x_{m+1})\cdots\phi(x_n)\rangle \to \langle\phi(x_1)\cdots\phi(x_m)\rangle\langle\phi(x_{m+1})\cdots\phi(x_n)\rangle$$

⇒ Perturbative stability of de Sitter invariant state

(Marolf and Morrison (1010.5327))

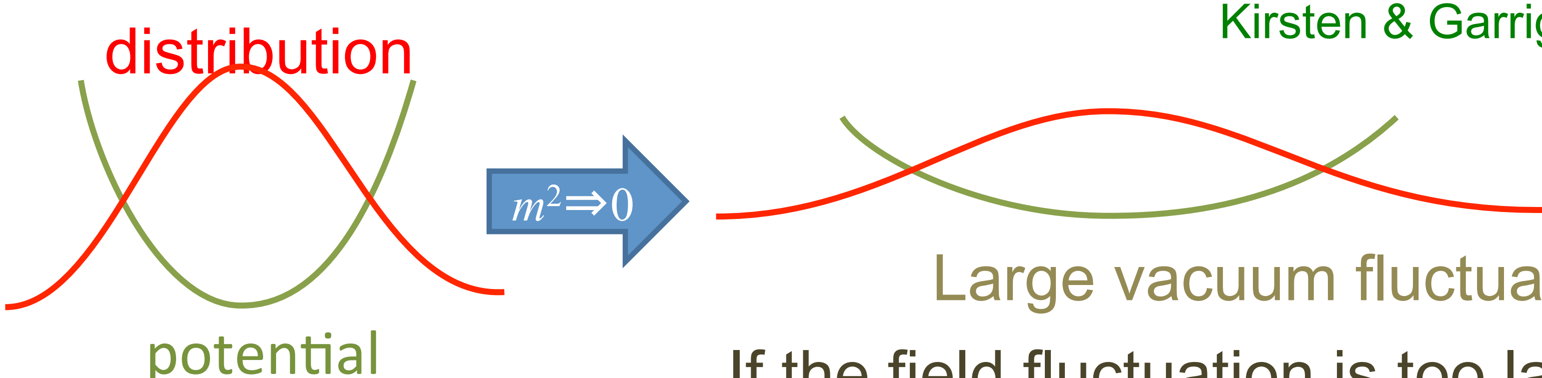
More subtle issue arises in the small mass limit.

summing up only long wavelength modes beyond the Horizon scale

$$\left\langle \phi^2 \right\rangle_{reg} \approx \int_0^{aH} d^3k \, \frac{H^2}{k^3} \left(\frac{k}{aH} \right)^{\frac{2m^2}{3H^2}} \approx \frac{H^4}{m^2}$$

$\Leftrightarrow$ De Sitter inv. vac. state does not exist in the massless limit.

Allen & Folacci(1987)
Kirsten & Garriga(1993)



If the field fluctuation is too large, it is easy to imagine that a naïve perturbative analysis will break down once interaction is turned on.

Stochastic interpretation

(Starobinsky & Yokoyama (1994))

- Let's consider local average of ϕ :

$$\bar{\phi} = \int_0^{aH} d^3k\, \phi_k e^{ikx}$$

More and more short wavelength modes participate in $\bar{\phi}$ as time goes on.

Equation of motion for $\bar{\phi}$:

$$\frac{d^2\bar{\phi}}{dt^2} + 3H^2 \frac{d\bar{\phi}}{dt} = -V'(\bar{\phi}) + f$$

Newly participating modes act as random fluctuation

$$\langle \phi_k \phi_{-k} \rangle \approx H^2/k^3$$

$$\Rightarrow \quad \langle f(t) f(t') \rangle \approx H\delta(t - t')$$

In the case of massless $\lambda\phi^4$: $\langle \bar{\phi}^2 \rangle \rightarrow \dfrac{H^2}{\sqrt{\lambda}}$

Namely, in the end, thermal equilibrium is realized : $V \approx T^4 \approx H^4$

Wave function of the universe ~parallel universes

- Distant universe is quite different from ours.

- Each small region in the above picture gives one representation of many parallel universes.
- However: wave function of the universe = “a superposition of all the possible parallel universes”

  must be so to keep translational invariance of the wave fn. of the universe
- Do “simple expectation values give really observables for us?”

- Do “simple expectation values give really observables for us?”

“Maybe No”

In the case of massless $\lambda\phi^4$

$$\langle\phi^2\rangle \to \frac{H^2}{\sqrt{\lambda}}$$

As a result, the effective mass squareddevelops as

$$\langle m^2\rangle \to \sqrt{\lambda}H^2$$

However, the success of stochastic approach teaches us that $\langle m^2\rangle$ is just the ensemble average of many parallel universes, and the mass squared observed in each horizon patch is just $\sim\lambda\phi^2$.

“No observer can see the secular growth of mass unrelated to the change of the local averaged field value.”

Identifying the dominant component of IR fluctuation

If we subtract the local average

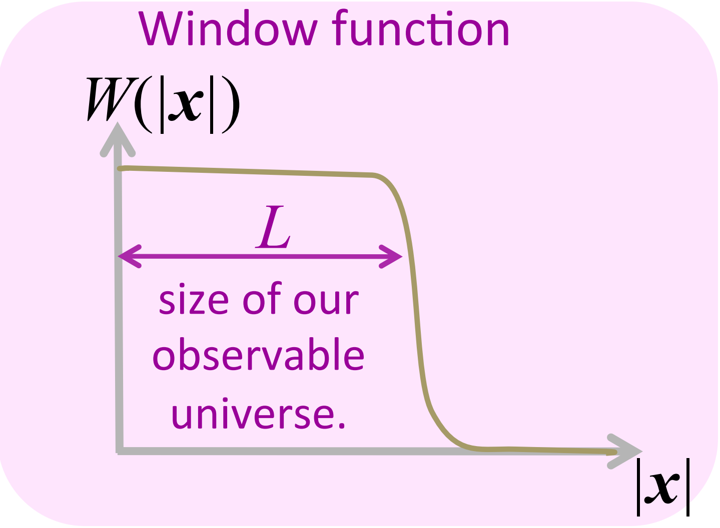
$$\delta\phi := \phi(x) - \bar{\phi}$$

with $\bar{\phi} = \int d^3x\, W(\boldsymbol{x})\phi(\boldsymbol{x})$

then,

$$\delta\phi = \int d^3k \underline{\left(e^{i\boldsymbol{k}\boldsymbol{x}} - W_{\boldsymbol{k}}\right)}\tilde{\phi}_{\boldsymbol{k}}$$

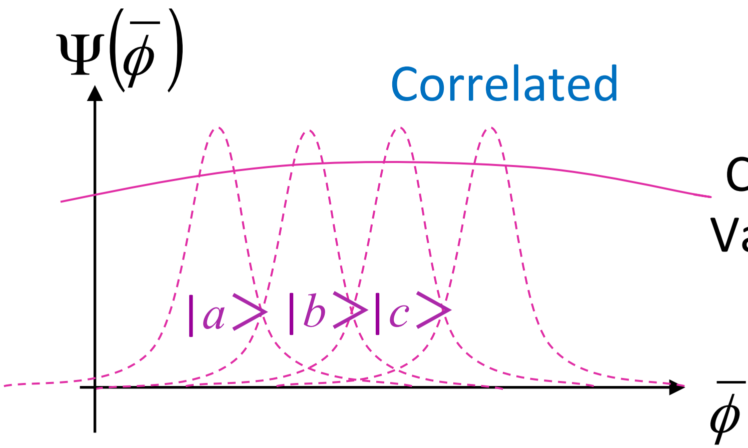
$O(k)$ unless $|\boldsymbol{x}| \gg L$



Dominant IR fluctuation is concentrated on $\bar{\phi}$

Decoherence of the wave function of the universe

<u>*Before*</u>



Superposition of wave packets

$$\rho = \left|\Psi(\bar{\phi})\right\rangle\left\langle\Psi(\bar{\phi})\right|$$
$$= \left(|a\rangle + |b\rangle + |c\rangle + \cdots\right)\left(\langle a| + \langle b| + \langle c| + \cdots\right)$$

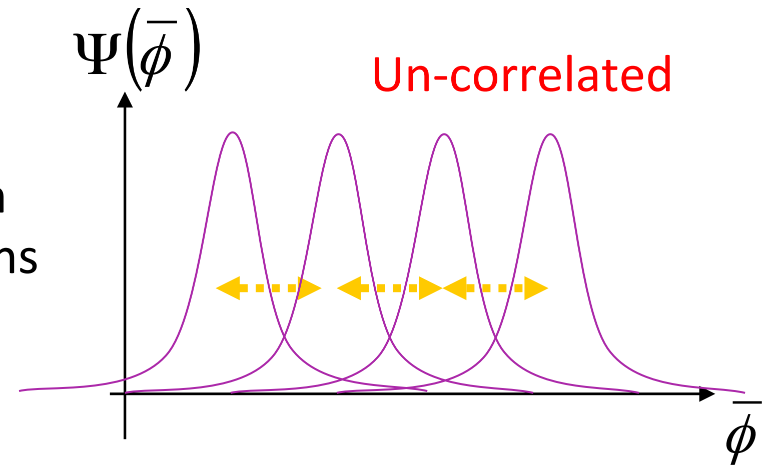
Cosmic expansion
Various interactions

Decoherence

Coarse graining
Unseen d.o.f.

Sorry, but this process is too complicated.

<u>*After*</u>



Statistical ensemble

$$\rho = |a\rangle\langle a| + \underbrace{|b\rangle\langle b|} + |c\rangle\langle c| + \cdots$$

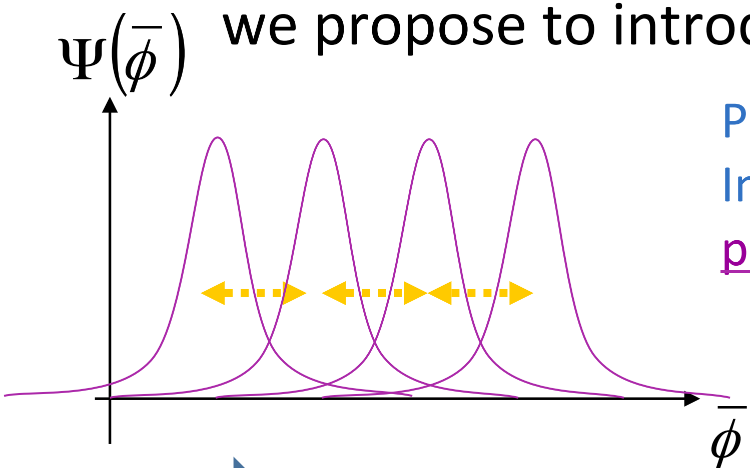
Our classical observation picks up one of the decohered wave packets.

Substitute of picking up one decohered history

(Urakawa & Tanaka PTP122:1207)

- Discussing quantum decoherence is annoying.
  - Which d.o.f. to coarse-grain?
  - What is the criterion of classicality?
- To avoid subtle issues about decoherence,

we propose to introduce a "projection operator".



Picking up one history is difficult.
Instead, we throw away the other histories presumably uncorrelated with ours.

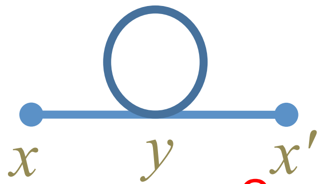
⇨ over-estimate of fluctuations

We compute

$$\langle PO \rangle / \langle P \rangle \quad \text{with} \quad P = \exp\left(- \frac{\bar{\phi}^2}{2\sigma^2} \right)$$

IR finiteness

Projection acts only on the external lines.
How the contribution from the IR modes at $k \approx k_{\min}$ is suppressed?



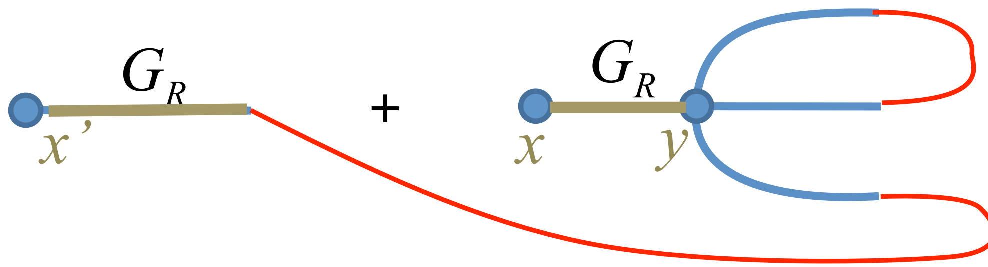
$$\approx \int d^4 y\, G(x,y) G(x',y) \underline{G(y,y)}$$

One of Green fns. is retarded, G_R.

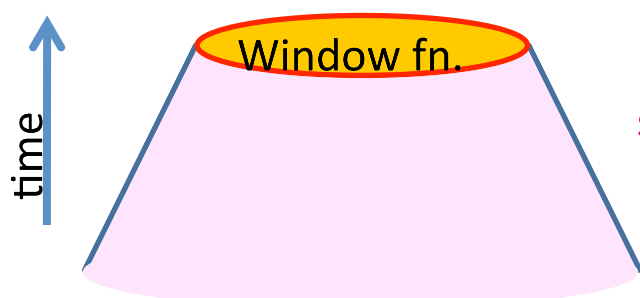
$\langle \phi_{\rm int}(y) \phi_{\rm int}(y) \rangle$

∵ Expansion in terms of interaction-picture fields:

$$\phi(x) = \phi_{\rm int}(x) + \int d^4 y \lambda_{4-point}(y) G_R(x,y) \phi_{\rm int}(y)^3 + \cdots$$



Integration over the vertex y is restricted to the region within the past light-cone.

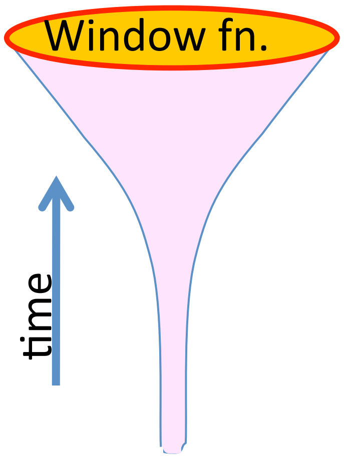


For each η_y, IR fluctuation of $\phi_{\rm int}(y)$ is suppressed since $\overline{\phi}(\eta_{\rm fin})$ is restricted.

OK to any order of loop expansion!

IR finiteness

~ secular growth in time



Past light cone during inflation shrinks down to horizon size.

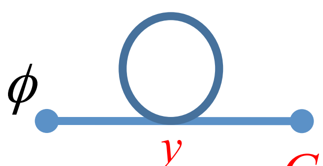
$$ds^2_{de\ Sitter} \approx \frac{1}{(-H\eta)^2}\left(-d\eta^2 + dr^2 + \cdots\right)$$

$-\eta \approx \Delta\eta = \Delta r$:past light cone

$$R_{light\ cone} = \frac{\Delta r}{-H\eta} \to \frac{1}{H}$$

However, for $\eta_y \to -\infty$, the suppression due to constraint on $\bar{\phi}$ gets weaker.

$\infty\infty\infty\infty\infty\infty$Then, $G(y, y) \approx \langle \phi_{\rm int}(y)\phi_{\rm int}(y)\rangle$ becomes large.



$$\phi \approx \int d^4x\, G_R(x, y) G(x', y) G(y, y)$$

$G_R(x, y) \to$ constant for $\eta_y \to -\infty$

η_y-integral looks divergent, but homogeneous part of ϕ is constrained by the projection.

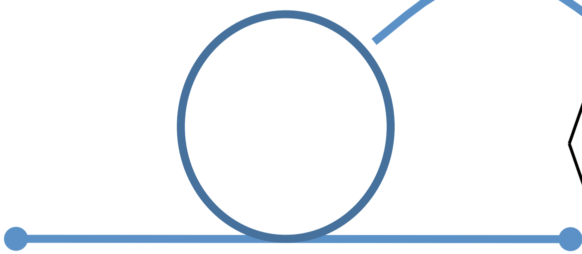
$\partial_x G_R(x, y) \to 0$ faster than $G_R(x, y)$ for $\eta_y \to -\infty$

looks OK at one-loop level but not promising in general.

§IR divergence in single field inflation

Setup: 4D Einstein gravity + minimally coupled scalar field

Single field case is special because broadening of averaged field can be absorbed by an appropriate choice of the time coordinate.



Factor coming from this loop:

$$\langle \zeta(y)\zeta(y)\rangle \approx \int d^3k\, \underline{P(k)} \approx \log(aH/k_{\min})$$

$$\boxed{\propto 1/k^3}$$

scale invariant spectrum

- no typical mass scale

curvature perturbation in co-moving gauge.

$$\begin{cases} \gamma_{ij} = e^{2\rho+2\zeta}\exp(h)_{ij} \\ \delta\phi = 0 \end{cases}$$

Transverse traceless

Gauge issue in single field inflation

Yuko Urakawa and T.T., PTP122: 779 arXiv:0902.3209

– In conventional cosmological gauge invariant perturbation theory, gauge is not completely fixed.

Time slicing can be uniquely specified: $\delta\phi=0$ OK!

but spatial coordinates are not.

$$h^j_j = 0 = h^j_{i,j}$$

Residual gauge:

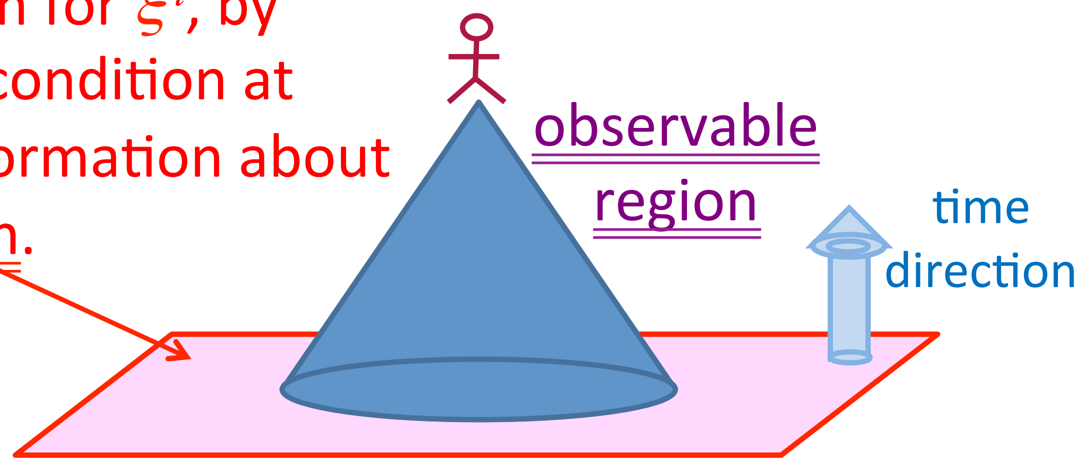
$$\delta_g h_{ij} = \xi_{i,j} + \xi_{j,i}$$

Elliptic-type differential equation for ξ^i.

$$\Delta\xi^i = \cdots$$

Not unique locally!

- To solve the equation for ξ^i, by imposing boundary condition at infinity, we need information about un-observable region.



Why I expect IR finiteness in single field inflation

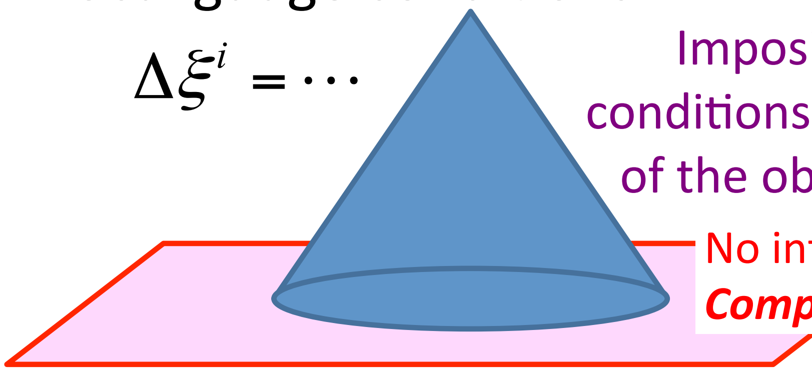
- First of all, our observables in the real world are finite and inflationary universe model works well without drastic modification due to IR effects.
- The local spatial average of ζ can be set to 0 identically by an appropriate gauge choice

  (=time-dependent scale transformation).
- Even if we choose such a local gauge, the evolution equation for ζ stays hyperbolic. So, the interaction vertices are localized inside the past light cone.
- Therefore, IR divergence does not appear as long as we compute ζ in this local gauge. But here we assumed that the initial quantum state is free from IR divergence.

Complete gauge fixing vs. Genuine gauge-invariant quantities

- Local gauge conditions.

$$\Delta \xi^i = \cdots$$



Imposing boundary conditions on the boundary of the observable region

No influence from outside

Complete gauge fixing ☺

⇒ But unsatisfactory?
The results depend on the choice of boundary conditions.
Translation invariance is lost.

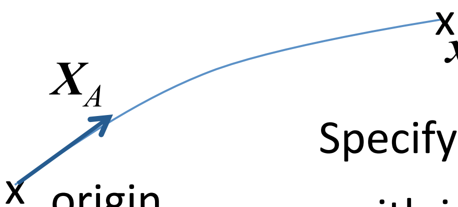
◆ ***Genuine*** coordinate-independent quantities.

Correlation functions for 3-d scalar curvature on ϕ =constant slice.

$\langle R(\boldsymbol{x}_1)\, R(\boldsymbol{x}_2)\rangle$ Coordinates do not have gauge invariant meaning.

(Giddings & Sloth 1005.1056)
(Byrnes et al. 1005.33307)

Use of geodesic coordinates:



$$\boldsymbol{x}(\boldsymbol{X}_A, \lambda=1) = \boldsymbol{X}_A + \delta \boldsymbol{x}_A$$

Specify the position by solving geodesic eq. $D^2 x^i / d\lambda^2 = 0$

with initial condition $\left. Dx^i / d\lambda \right|_{\lambda=0} = X^i$

$${}^gR(\boldsymbol{X}_A) := R(\boldsymbol{x}(\boldsymbol{X}_A, \lambda=1)) = R(\boldsymbol{X}_A) + \delta \boldsymbol{x}_A \boldsymbol{\nabla} R(\boldsymbol{X}_A) + \ldots$$

$\langle {}^gR(\boldsymbol{X}_1)\, {}^gR(\boldsymbol{X}_2)\rangle$ should be truly coordinate independent.

Extra requirement for IR regularity

In $\delta\phi$ =0 gauge, EOM is very simple

$$\left\lfloor \partial_t^2 + \left(3+\varepsilon_2\right)\dot{\rho}\,\partial_t - e^{-2(\rho+\zeta)}\Delta \right\rfloor \zeta \approx 0$$

Only relevant terms in the IR limit were kept.

Non-linearity is concentrated on this term.

Formal solution in IR limit can be obtained as

$$\zeta = \zeta_I - 2\zeta_I \quad {}^{-1} e^{-2\rho}\Delta\zeta_I + \cdots$$

$$\varepsilon_2 = -\frac{d^2}{d\rho^2}\log H$$

with L^{-1} being the formal inverse of

$$= \partial_t^2 + \left(3+\varepsilon_2\right)\dot{\rho}\,\partial_t - e^{-2\rho}\Delta$$

$${}^{g}R \approx -4e^{-2\rho}\Delta\left\lfloor \zeta_I - \underline{\zeta_I\left(2 \quad {}^{-1}e^{-2\rho}\Delta + \boldsymbol{x}\cdot\partial_{\boldsymbol{x}}\right)\zeta_I} + \cdots \right\rfloor$$

$$\left\langle {}^{g}R(\boldsymbol{x}_1)\,{}^{g}R(\boldsymbol{x}_2)\right\rangle \ni \underbrace{\left\langle \zeta_I^2 \right\rangle}_{\text{IR divergent factor}} \left\langle \Delta\left(2 \quad {}^{-1}e^{-2\rho}\Delta + \boldsymbol{x}\cdot\partial_{\boldsymbol{x}}\right)\zeta_I(\boldsymbol{x}_1)\times\Delta\left(2 \quad {}^{-1}e^{-2\rho}\Delta + \boldsymbol{x}\cdot\partial_{\boldsymbol{x}}\right)\zeta_I(\boldsymbol{x}_1)\right\rangle$$

IR regularity may require $\left\lfloor 2 \quad {}^{-1}e^{-2\rho}\Delta + \left(\boldsymbol{x}\cdot\nabla\right)\right\rfloor \zeta_I = 0$

IR regularity may require

$$\left[2\quad{}^{-1}e^{-2\rho}\Delta+\left(\boldsymbol{x}\cdot\nabla\right)\right]\zeta_I=0$$

However, L^{-1} should be defined for each Fourier component.

$$^{-1}f\left(t,\boldsymbol{x}\right)=\int d^3k\,e^{i\boldsymbol{k}\cdot\boldsymbol{x}}\quad{}_k^{-1}\tilde{f}_k\left(t\right)$$ for arbitrary function $f\,(t,\boldsymbol{x})$

with $\quad{}_k=\partial_t^2+\left(3+\varepsilon_2\right)\dot{\rho}\,\partial_t+e^{-2\rho}k^2$

Then, satisfying $2\quad{}^{-1}e^{-2\rho}\Delta\zeta_I+\left(\boldsymbol{x}\cdot\nabla\right)\zeta_I=0$ is impossible,

because for $\zeta_I\equiv\int d^3k\,(e^{ikx}\,v_{\boldsymbol{k}}(t)\,a_{\boldsymbol{k}}+h.c.)$,

$$^{-1}e^{-2\rho}\Delta\zeta_I\propto e^{i\boldsymbol{k}\cdot\boldsymbol{x}}a_{\boldsymbol{k}}$$ while $$\left(\boldsymbol{x}\cdot\nabla\right)\zeta_I\propto i\boldsymbol{k}\cdot\boldsymbol{x}\,e^{i\boldsymbol{k}\cdot\boldsymbol{x}}a_{\boldsymbol{k}}$$

Instead, one can impose

$$\left[2\quad{}^{-1}e^{-2\rho}\Delta+(\boldsymbol{x}\cdot\nabla)\right]\zeta_I=\int d^3k\left(a_{\boldsymbol{k}}D_k e^{i\boldsymbol{kx}}v_{\boldsymbol{k}}(t)+h.c.\right)$$

with $D_k\equiv k^{-3/2}e^{-i\phi(\boldsymbol{k})}\dfrac{d}{d\log k}k^{3/2}e^{i\phi(\boldsymbol{k})}$,

which reduces to conditions on the mode functions.

$$-2k^2\quad{}_k^{-1}e^{-2\rho}v_k=D_k v_k$$

- extension to the higher order:

$$\left[\left(2\quad{}^{-1}e^{-2\rho}\Delta\right)^2+\frac{1}{2}(2+\boldsymbol{x}\cdot\nabla)\boldsymbol{x}\cdot\nabla\right]\zeta_I=\int d^3k\left(a_{\boldsymbol{k}}D_k^2 e^{i\boldsymbol{kx}}v_{\boldsymbol{k}}(t)+h.c.\right)$$

With this choice, IR divergence disappears.

$$\left\langle {}^gR(\boldsymbol{X}_1)\,{}^gR(\boldsymbol{X}_2)\right\rangle^{(4)}\propto\underbrace{\left\langle\zeta_I^2\right\rangle}_{\text{IR divergent factor}}\times\underbrace{\int d(\log k)\,\partial_{\log k}^2\left(k^7|v_k|^2e^{i\boldsymbol{k}(\boldsymbol{X}_1-\boldsymbol{X}_2)}\right)}_{\text{total derivative}}$$

Physical meaning of IR regularity condition

In addition to considering ${}^{g}R$, we need additional conditions

$$-2k^2 \quad {}_k^{-1} e^{-2\rho} v_k = D_k v_k$$ and its higher order extension.

What is the physical meaning of these conditions?

Background gauge: $\tilde{\boldsymbol{x}} = e^{s}\boldsymbol{x} \qquad \tilde{\zeta}(\tilde{\boldsymbol{x}}) = \zeta(\boldsymbol{x})$

$$ds^2 = -dt^2 + e^{2\rho} d\boldsymbol{x}^2 \Longrightarrow d\tilde{s}^2 = -dt^2 + e^{2\rho - 2s} d\tilde{\boldsymbol{x}}^2$$

$$H = H_0[\zeta] + H_{\mathrm{int}}[\zeta] \Longrightarrow \underline{\tilde{H} = H_0\left[\tilde{\zeta}\right] + H_{\mathrm{int}}\left[\tilde{\zeta} - s\right]}$$

- Quadratic part in $\tilde{\zeta}$ and s is identical to $s = 0$ case.
- Interaction Hamiltonian is obtained just by replacing the argument ζ with $\tilde{\zeta} - s$.

Therefore, one can use

1) common mode functions for ζ_I and $\tilde{\zeta}_I$

$$\zeta_I \equiv \int d^3k\, (e^{i\boldsymbol{k}\boldsymbol{x}} v_{\boldsymbol{k}}(t)\, a_{\boldsymbol{k}} + h.c.) \Longrightarrow \tilde{\zeta}_I \equiv \int d^3k\, (e^{i\boldsymbol{k}\boldsymbol{x}} v_{\boldsymbol{k}}(t)\, \tilde{a}_{\boldsymbol{k}} + h.c.)$$

2) common iteration scheme.

$$\zeta = \zeta_I + \delta\zeta[\zeta_I] \Longrightarrow \tilde{\zeta} = \tilde{\zeta}_I + \delta\zeta\left[\tilde{\zeta}_I - s\right]$$

We may require

$$\langle 0|\zeta(\boldsymbol{x}_1)\zeta(\boldsymbol{x}_2)\cdots\zeta(\boldsymbol{x}_n)|0\rangle = \langle\tilde{0}|\tilde{\zeta}(\tilde{\boldsymbol{x}}_1)\tilde{\zeta}(\tilde{\boldsymbol{x}}_2)\cdots\tilde{\zeta}(\tilde{\boldsymbol{x}}_n)|\tilde{0}\rangle$$

$\Rightarrow$ $$-2k^2 \quad {}_k^{-1} e^{-2\rho} v_k = D_k v_k$$

the previous condition compatible with Fourier decomposition

Retarded integral with $\zeta(\eta_0)=\zeta_I(\eta_0)$ guarantees the commutation relation of ζ

$\Rightarrow$ $D_k v_k(\eta_0)=0$: incompatible with the normalization condition.

It looks quite non-trivial to find consistent IR regular states.

However, the Euclidean vacuum state (defined by the regularity $\eta_0 \to \pm i\infty$) satisfies this condition.

Why the Euclidean vacuum state is special.

We define Euclidean vacuum by the conditions

$$\left\langle T_c \zeta(x_1)\cdots\zeta(x_n)\right\rangle < \infty \quad \text{for} \quad \eta(t_a) \to -\infty(1 \pm i\varepsilon)$$

These conditions are equivalent even if they are written in terms of $\tilde{\xi}$.

Thus, we find

$$\left\langle T_c \zeta(x_1)\cdots\zeta(x_n)\right\rangle = \left\langle T_c \tilde{\xi}(t_1, e^s x_1)\cdots\tilde{\xi}(t_n, e^s x_n)\right\rangle$$

is satisfied when the both sides are evaluated in the Euclidean vacuum.

Sketch of our proof.

Gauge transformation: $\tilde{\boldsymbol{x}} = e^{s}\boldsymbol{x}$

$$ds^2 = -dt^2 + e^{2(\rho+\zeta)}d\boldsymbol{x}^2 = -dt^2 + e^{2(\rho+\tilde{\zeta}-s)}d\tilde{\boldsymbol{x}}^2$$

$$\Longrightarrow \quad \zeta'(\tilde{\boldsymbol{x}}) = \tilde{\zeta}(\tilde{\boldsymbol{x}}) - s = \zeta(\boldsymbol{x}) - s$$

For constant s, the form of the Hamiltonian does not change. Setting $s = \overline{\zeta(t_{fin}, \boldsymbol{x})}$, we can suppress the IR contribution at $t=t_{fin}$, but $\zeta'(\tilde{\boldsymbol{x}}) = \zeta(\boldsymbol{x}) - \overline{\zeta(\boldsymbol{x})}$, which appears in the past vertices, is not IR-suppressed.

Thus, we need to make s to be time-dependent.

$$\tilde{H} = H_0\left[\tilde{\zeta}, \tilde{\pi}\right] + H_{\rm int}\left[\tilde{\zeta} - s, \tilde{\pi}\right] - \dot{s}\tilde{\pi}\,\boldsymbol{x}\cdot\nabla\tilde{\zeta}$$

$\tilde{\zeta}$ appears only in the form $\tilde{\zeta} - s(t), \nabla\tilde{\zeta}, \partial_t\tilde{\zeta}$

$\rightarrow$ All the propagators are associated with an IR suppressing factor.

Further, we replace $s(t)$ with the operator $\bar{\zeta}$ by inserting its complete set, using the property of Euclidian vacuum is independent of $s(t)$.

Sketch of our proof.

IR suppressed free propagator in Euclidean vacuum is regular, except for the case that the two points are light-like.

no singularity in the complex k plane since the mode equation and the boundary conditions are analytic in k.

Mode sum is finite for both IR and UV limits.

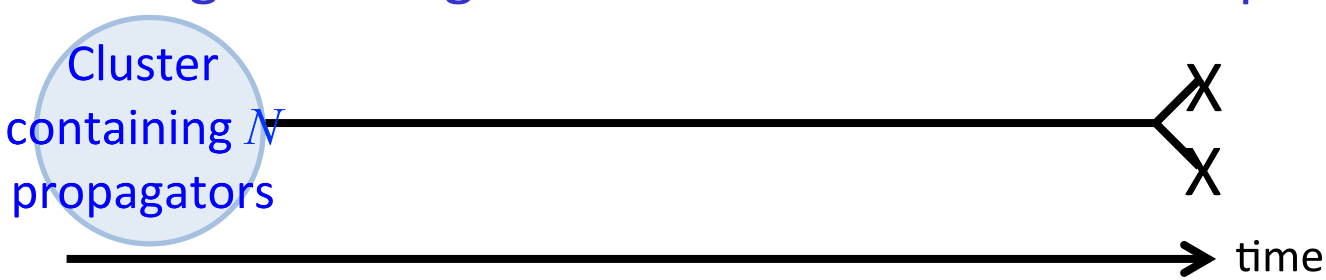
IR) owing to IR suppressing operators

UV) owing to $i\varepsilon$ prescription, peculiar to the Euclidean vacuum

When two times in the arguments of the Wightman function are far apart, there is suppression from one side.

To solve the constraint equations, we use the local boundary conditions

$\rightarrow$ Integration region is restricted to the causal past.



$$\int \frac{d\eta}{\eta} a(\eta) H^{2N}$$

Summary

There are conditions required for the absence of IR divergences.

"Wave function must be homogeneous in the direction of background scale transformation"

If we naively set initial condition by choosing a free field vacuum at a finite time, these IR regularity conditions are not satisfied.

However, Euclidean vacuum (selected by $i\varepsilon$-prescription) and its excited states satisfy the IR regular condition.

Discussions are similar but more complicated when we include the graviton loops:

PTEP2014 073E01 :arXiv:1402.2076

How we avoided the effect of UV mode to the geodesic distance.

We did not compute the geodesic in the full perturbed metric including all UV modes.

Instead, we consider a finite volume and define an averaged metric perturbation in that volume:

$$\bar{\zeta} \approx \int d^3x\, W(\boldsymbol{x})\,\zeta(\boldsymbol{x})$$

which depends on the choice of the window function.

Therefore, the discussed quantities are not manifestly gauge invariant, although one can say quantities under a completely specified gauge conditions are gauge invariant.

Actual definition of $\bar{\zeta}$ is a little more complicated because $W(x)$ should also be defined with respect to the geodesic distance.