# D-instanton calculus in C=1 string theory

based on work with
B. Balthazar, V. Rodriguez
1907.07688, 1912.07170

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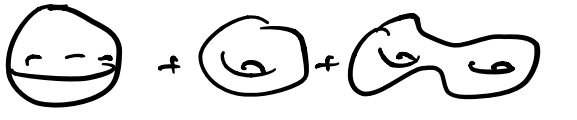
world sheet formulation of string theory beyond perturbation theory?

perturbative expasion



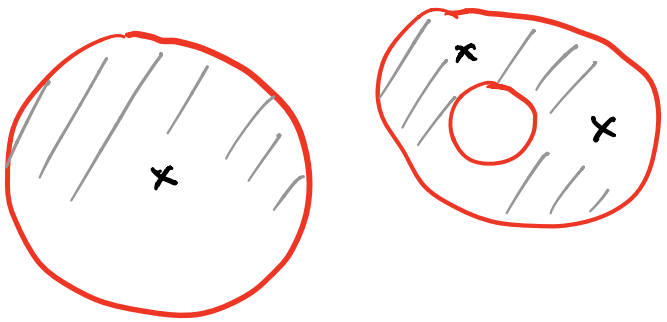
(resummation?)

"+"
instantons?

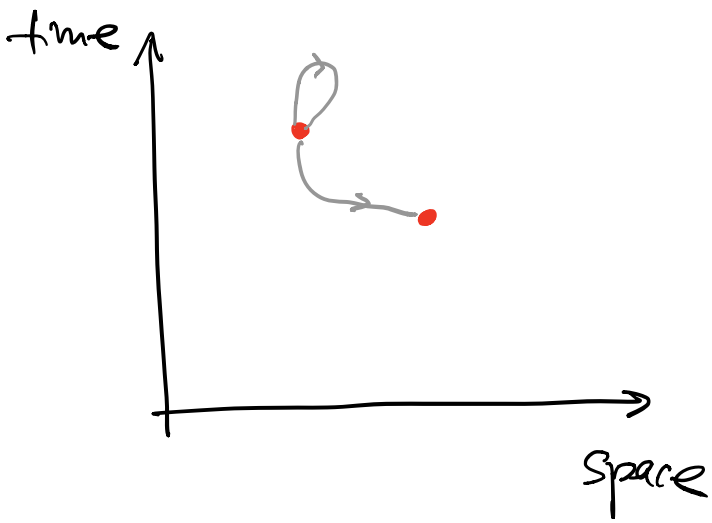
↳ D-instantons

Polchinski '94
Green-Gutperle '97

world sheet

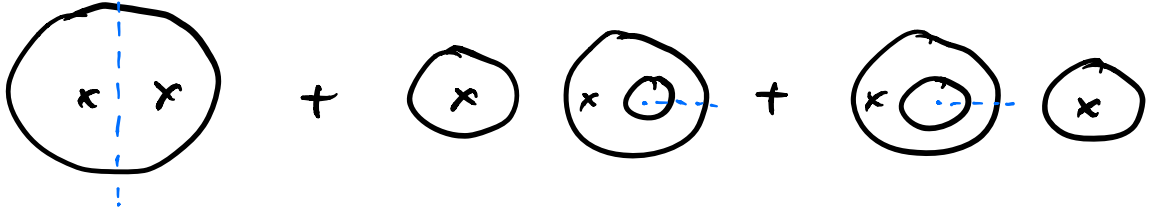


target spacetime




- No first principle derivation of D-instanton prescription (but many consistency checks!)
- Overall normalization fixed by duality arguments
  (S-duality in IIB superstring, MQM in C=1 string)
- Fischler-Susskind-Polchinski mechanism for canceling divergence due to open string collective modes

e.g.



- Full prescription requires open + closed SFT

  (See Sen's lecture series)

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## $c=1$ string theory

- the only known bosonic string theory with worldsheet description that admits space-time interpretation and consistent pert. theory at the quantum level.

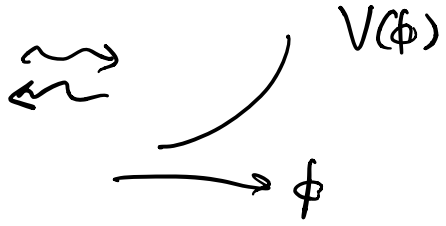
- worldsheet CFT

  $X^0 \oplus (c=25 \text{ Liouville}) \oplus bc$ system

$$S_L = \frac{1}{4\pi}\int (\partial\phi)^2 + QR\phi + \mu e^{2b\phi}$$

$Q = b + \frac{1}{b}, \quad b \to 1 \quad (c \to 25)$

$$V_p \underset{\phi \to -\infty}{\sim} S(p)^{-\frac{1}{2}} e^{(Q+2ip)\phi} + S(p)^{\frac{1}{2}} e^{(Q-2ip)\phi}$$

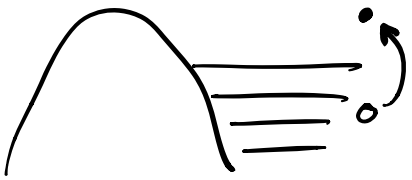


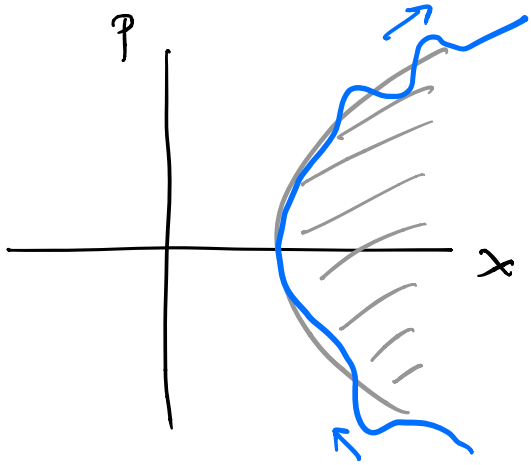

closed string states

- vertex operator

$$V_\omega^{\pm} = g_s e^{\pm i\omega X^0} V_{p=\frac{\omega}{2}}$$

- Matrix Quantum Mechanics dual





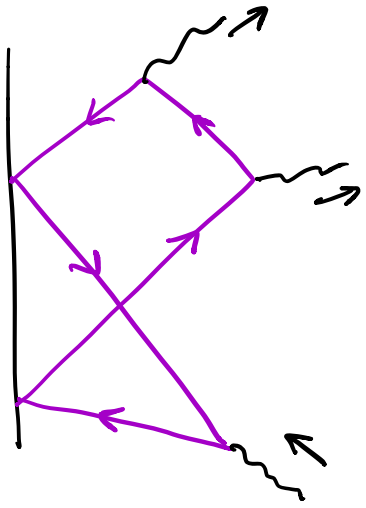

closed string

= particle/hole pair

= collective excitation of fermi surface

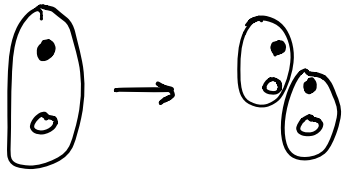
fermions are free

but collective excitations are effectively interacting



$$g_s = \frac{1}{2\pi\mu} \qquad (\hbar = 1)$$

Moore - Plesser - Ramgoolam '91



Non-perturbative completion of MQM

$|E\rangle_R$

↑

fill up to $E \leq -\mu$

$|E\rangle_L$

↑

do not fill

Structure of amplitude (exactly computable)

$$\mathcal{A}_{1\to k} = \sum_{g=0}^{\infty} \left(\frac{1}{\mu}\right)^{k-1+2g} \mathcal{A}_{1\to k}^{\text{pert},[g]}$$

↖ Borel-resummed

Borel-resummed

$$+ \sum_{n=1}^{\infty} e^{-2\pi n \mu} \sum_{L=0}^{\infty} \left(\frac{1}{\mu}\right)^{L} \mathcal{A}_{i \to k}^{n\text{-inst},(L)}$$

↑ D-instanton expansion

↑ open string expansion

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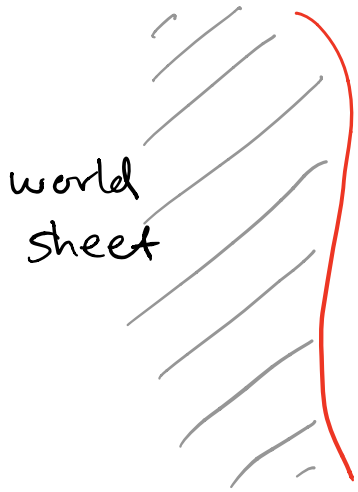
We now turn to the worldsheet formulation of the D-instanton-mediated closed string amplitudes




↖ D-instanton boundary condition

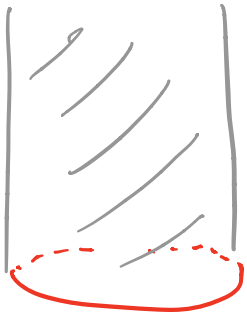
In $c=1$ string theory, the simplest class of D-instantons are "ZZ-instantons"

(Dirichlet in $X^0$)

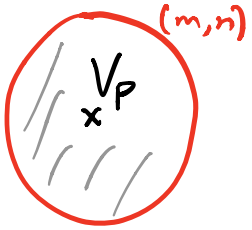
$\otimes$ (ZZ-brane in Liouville)

ZZ boundary state

$$|ZZ(m,n)\rangle\rangle = \int_0^\infty \frac{dP}{\pi}\, \Psi^{(m,n)}(P)\, |V_P\rangle\rangle$$



ZZ (m,n)




$$\Psi^{(m,n)}(P) = 2^{\frac{5}{4}}\sqrt{\pi}\, \frac{\sinh(2\pi m P)\,\sinh(2\pi n P)}{\sinh(2\pi P)}$$

$\rightsquigarrow$ $(m,n)$ ZZ-instanton $\quad (m,n \in \mathbb{Z}_{\geq 1})$

Instanton action


$$S_{(1,1)\ ZZ\text{-inst}} = \frac{1}{g_s} \qquad (= 2\pi\mu)$$

"fermion bounce"

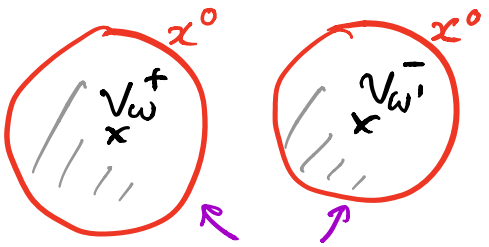
$$S_{(m,n)\ ZZ\text{-inst}} = \lim_{P\to i} \frac{\Psi^{(m,n)}(P)}{\Psi^{(1,1)}(P)}\, S_{(1,1)}$$

$$= \frac{mn}{g_s}$$

# closed string $1 \to 1$ amplitude

1-instanton contribution

- leading order $\sim e^{-\frac{1}{g_s}}$




same ZZ-instanton

$$= \mathcal{N} \int dx^0 \, e^{-S_{ZZ}} \langle V_\omega^+ \rangle^{D^2}_{ZZ,x^0} \langle V_{\omega'}^- \rangle^{D^2}_{ZZ,x^0}$$

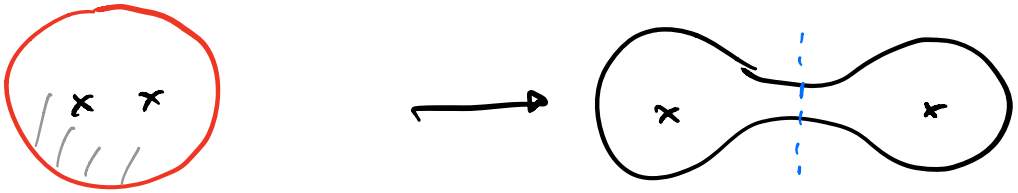
$$= 2\pi \mathcal{N} \, e^{-\frac{1}{g_s}} \delta(\omega - \omega') \cdot 4 \sinh^2(\pi\omega)$$

fix once for all by matching w/ MQM

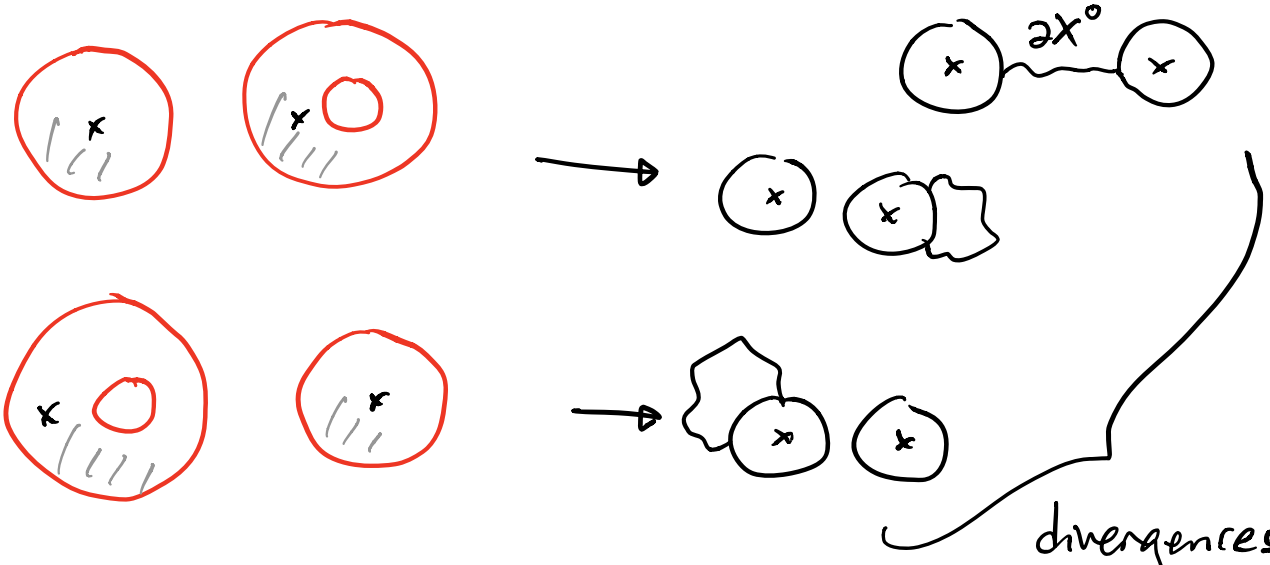
agrees w/ MQM

$$\mathcal{N} = -\frac{1}{8\pi^2} \quad \text{(for (1,1) ZZ-instanton)}$$

- next-to-leading order $\sim e^{-\frac{1}{g_s}} \cdot g_s$

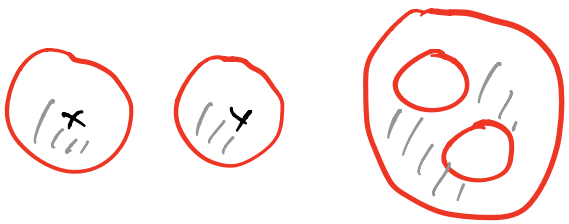


open string divergence




divergences cancel (F-S-P)

a finite ambiguity that can be fixed by SFT (Sen)



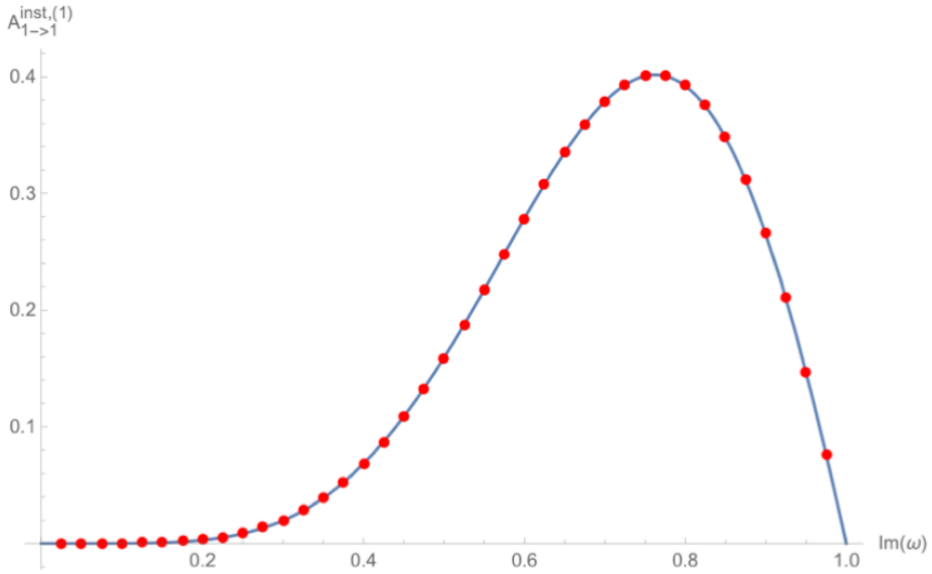
$O(g_s)$ correction to $S_{zz}$.

- assume const $\times g_s$,

Fit to MQM result

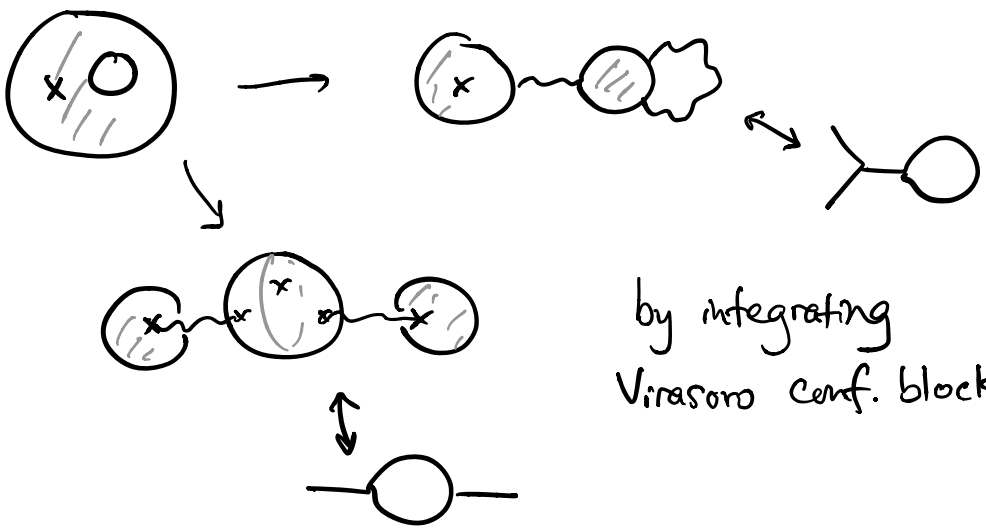
$$\propto \omega\left(\frac{\pi\omega}{\tanh \pi\omega} - 1\right)\sinh^2(\pi\omega)$$

fit 2 parameters




- numerics for imaginary $\omega$

most error comes from evaluating

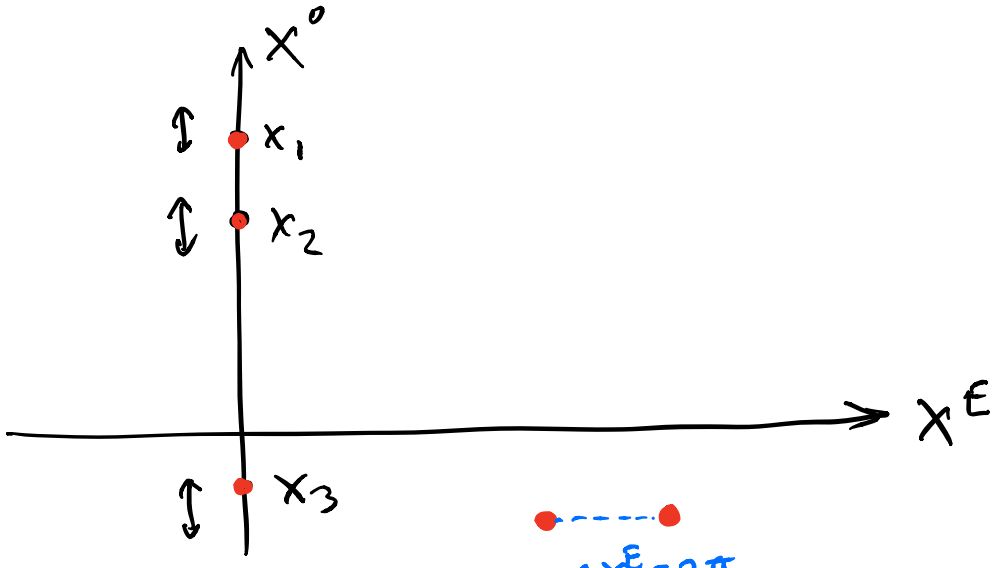


by integrating Virasoro conf. blocks

# Multi-instanton contribution

prescription

- Lorentzian contour in $X^0_i$




- include $(n,1)$ ZZ-brane

with normalization factor

$\mathcal{N}_n$ in measure

↑ will fix by matching w/ MQM

$$\mathcal{N}_n = \frac{(-)^n}{4\pi^2 n} \frac{(2n-1)!!}{(2n)!!}$$

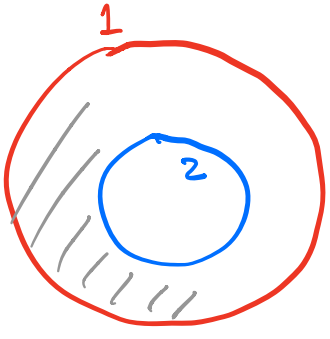
Example: 2-instanton contribution at order $\sim e^{-\frac{2}{g_s}}$

$$N_1^2 \frac{1}{2} \int dx_1\, dx_2 \Big[ \Big( \overset{1}{\overset{\circ}{\omega}} \quad \overset{2}{\overset{\circ}{\omega'}} + 1 \leftrightarrow 2 \Big) e^{2 \bigcirc} + 2 \overset{1}{\overset{\times}{\omega}} \overset{1}{\overset{\times}{\omega'}} \Big( e^{2 \bigcirc} - 1 \Big) \Big]$$

↑ normalize vac. amp.

$$+ N_2 \int dx \quad \overset{(2,1)}{\overset{\times}{\omega}} \quad \overset{(2,1)}{\overset{\times}{\omega'}}$$

Note: cylinder correction to 2-instanton measure



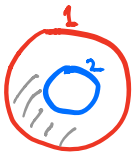

$$= \int_0^\infty \frac{dt}{2t} \frac{e^{2\pi t} - 1}{\eta(it)} \frac{e^{-t\frac{(\Delta x^6)^2}{2\pi}}}{\eta(it)} \eta(it)^2$$

Liouville $X^0$ bc

$$= \frac{1}{2} \log \frac{(\Delta X^E)^2}{(\Delta X^E)^2 - (2\pi)^2}$$

"Vandermonde"

$$e^{2} \quad = \quad \frac{(\Delta X^E)^2}{(\Delta X^E)^2 - (2\pi)^2}$$




"tachyon pole"

$$\longrightarrow \quad \frac{(\Delta X)^2}{(\Delta X)^2 + (2\pi)^2}$$ Lorentzian

with some combinatoric magic ...

match with MQM at order $e^{-\frac{n}{g_s}}$

for all positive integer $n$ !

# Conclusion

- c=1 string theory defined by worldsheet genus expansion

  + D-instanton effect

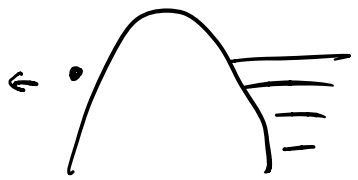
  is non-perturbatively "complete"

- possible thanks to Borel summability of pert. series, and absence of $\mathcal{O}(e^{-1/g_s^2})$ effects

  - a special feature of c=1 string (in singlet sector. i.e. without FZZT-branes) ?

- so far, restricted to exclusive closed string amplitudes, which do not saturate unitarity

To restore unitarity, need to include ZZ-branes w/ OS tachyon rolling to the "wrong side"?



- can generalize to 2D OB string/MQM

- Lessons for critical superstrings?

(W.I.P.)