The Boundary State from Wilson Loops

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Based on two papers to appear.

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Main Message:

Boundary State = Matrix Product State

Introduction

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In (open) string field theory, they are defined by solutions Ψ_* to EOM:

$$Q_B\Psi_* + \Psi_* * \Psi_* = 0\,.$$

It is not obvious how to write down $|B_{\Psi_*}\rangle$ for any given Ψ_*.

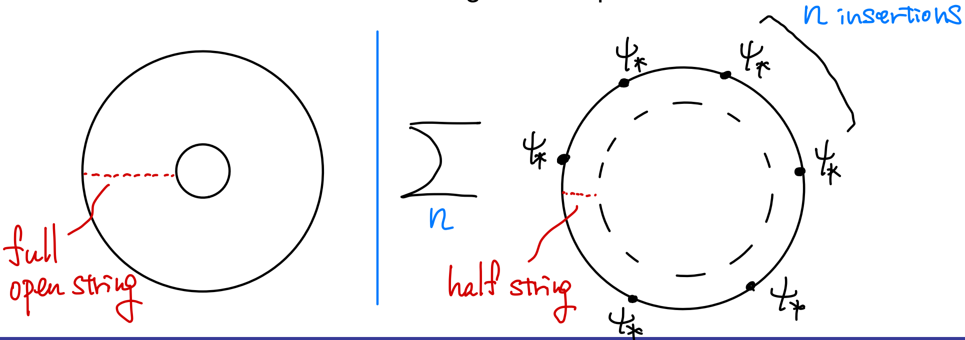
Introduction

Two proposals on $|B_{\Psi_*}\rangle$ in OSFT:

- [Kiermaier, Okawa, Zwiebach 2008]
  Wilson-loop like expression for the boundary state:

$$|B_{\Psi_*}\rangle \sim \text{“Tr"}\left[\mathrm{P}\exp\left(-\int dt[\mathcal{L}_R(t)+\{\mathcal{B}_R,\Psi_*\}]\right)\right]$$

  “Tr" is taken over the half-string Hilbert space.



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- [Kudrna, Maccaferi, Schnabl 2012]
  Overlap with closed string states from Ellwood invariants:

$$\langle \mathcal{V} | B_{\Psi_*} \rangle \sim \langle I | \mathcal{V}(z=i) | \Psi_* \rangle$$

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Question

Can we express the Wilson loop in $\mathcal{N} = 4$ SYM (at weak coupling) as a boundary state in the "closed-string" Hilbert space?

Strategy

Compute the analog of $\langle \mathcal{V} | B_{\Psi_*} \rangle$ (cf [Kudrna, Maccaferi, Schnabl]) and arrive at an expression similar to [Kiermaier, Okawa, Zwiebach].

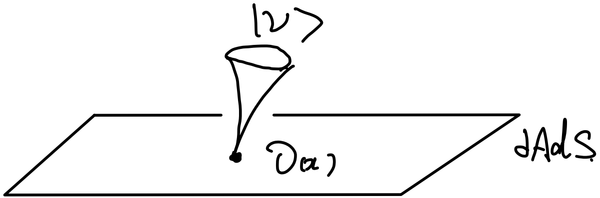
For the 1/2 BPS circular Wilson loop, one can uplift the solution to finite coupling using integrability bootstrap.

Weak Coupling

Single-Trace = $|\mathcal{V}\rangle$, Wilson Loop = $|B\rangle$

Single-trace local operators are dual to on-shell closed strings in $AdS_5 \times S^5$ spacetime.

$$\mathcal{O}(x) = \mathrm{Tr}\left[XXZZXZ\cdots\right] + \cdots \quad \leftrightarrow \quad |\mathcal{V}\rangle$$



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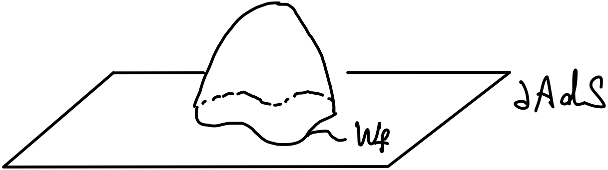
The (locally supersymmetric) Wilson loop in the fundamental rep. is dual to a disk worldsheet with the boundary ending on the contour of the WL.

$$\mathcal{W}_f = \mathrm{Tr}_f\left[P\exp\left(\oint d\tau(iA_\mu \dot{x}^\mu + \theta^I \Phi_I |\dot{x}|)\right)\right] \quad \leftrightarrow \quad |B\rangle$$



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The analog of $\langle \mathcal{V}|B\rangle$ is the correlation function of $\mathcal{O}$ and $\mathcal{W}_f$:

$$\langle \mathcal{O}(x)\mathcal{W}_f\rangle \quad \leftrightarrow \quad \langle \mathcal{V}|B\rangle .$$

In what follows, I will compute $\langle \mathcal{O}(x) \mathcal{W}_f \rangle$ at weak coupling in **4 steps**.

Key ideas

1. Use the “generating function" of the WLs.
2. Express the WL as 1d fermion.

Step 1: Wilson Loop as Fermion

Generating function of the Wilson loops in the anti-symmetric representations:

$$Z(a) = \mathrm{Det}\left[\mathbf{1}_{N\times N} + e^{ia} P \exp\left(\oint d\tau (iA_\mu \dot{x}^\mu + \theta^I \Phi_I |\dot{x}|)\right)\right]$$
$$= 1 + e^{ia}\mathcal{W}_f + e^{2ia}\mathcal{W}_{\mathrm{A}_2} + e^{3ia}\mathcal{W}_{\mathrm{A}_3} + \cdots$$

One can recover the fundamental rep. by $\mathcal{W}_f = \int da\, e^{-ia} Z(a)$.

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$Z(a)$ can be expressed as a path integral of 1d fermion χ living on the contour of WL [cf. Gomis, Passerini 2006]:

$$Z(a) = \int \mathcal{D}\chi^\dagger \mathcal{D}\chi\, e^{-S_\chi} = \mathrm{Tr}_\chi \left[e^{ia\chi^\dagger\chi} e^{\chi^\dagger (iA_\mu \dot{x}^\mu + \theta^I \Phi_I |\dot{x}|)\chi}\right]$$

$$S_\chi = \int_0^1 d\tau \underbrace{\chi^\dagger}_{\overline{\square}} \left[\partial_\tau - ia - \left(iA_\mu \dot{x}^\mu + \theta^I \Phi_I |\dot{x}|\right)\right] \underbrace{\chi}_{\square}$$

Step 2: Integrating out $\mathcal{N} = 4$ SYM

Next integrate out (free) $\mathcal{N} = 4$ SYM. For simplicity we first consider the expectation value of $Z(a)$:

$$\langle Z(a) \rangle = \int \mathcal{D}A_\mu \mathcal{D}\Phi \int \mathcal{D}\chi^\dagger \mathcal{D}\chi \, e^{-(S_\chi + S_{\mathcal{N}=4})}.$$

$$S_\chi + S_{\mathcal{N}=4} = \int_0^1 d\tau \chi^\dagger (\partial_\tau - ia - iA)\chi +$$

$$\frac{1}{g_{\rm YM}^2} \int d^4x \, {\rm Tr} \left[\partial_\mu \Phi^I \partial^\mu \Phi_I - g_{\rm YM}^2 \Phi_I \underbrace{\theta^I \int d\tau \, \chi\chi^\dagger(\tau) |\dot{x}| \delta^4(x - x(\tau))}_{\text{source term}} \right]$$

WL is a source term for the $\mathcal{N} = 4$ SYM fields.

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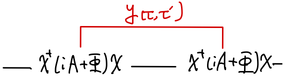
Integrating out $\mathcal{N} = 4$ SYM using $\int d\varphi \; e^{-\varphi K \varphi + J\varphi} = e^{JK^{-1}J}$,

$$\langle Z(a)\rangle = \int \mathcal{D}\chi^\dagger \mathcal{D}\chi \, e^{-\widetilde{S}_\chi}$$

$$\widetilde{S}_\chi = g_{\rm YM}^2 \int d\tau d\tau' y(\tau,\tau') \left(\chi^\dagger(\tau)\chi(\tau')\right)\left(\chi^\dagger(\tau')\chi(\tau)\right) + \int d\tau \chi^\dagger(\partial_\tau - ia)\chi.$$

$$y(\tau,\tau') = \frac{\theta^I\theta_I|\dot{x}(\tau)||\dot{x}(\tau')| - \dot{x}^\mu(\tau)\dot{x}_\mu(\tau')}{(x(\tau) - x(\tau'))^2}$$



Step 3: Hubbard-Stratonovich Transf.

$$\widetilde{S}_{\chi} = \frac{\lambda}{N_c} \int d\tau d\tau' y(\tau,\tau') \left(\chi^{\dagger}(\tau)\chi(\tau')\right) \left(\chi^{\dagger}(\tau')\chi(\tau)\right) + \int d\tau \chi^{\dagger}(\partial_{\tau} - ia)\chi$$

Integrating in a bilocal field $\rho(\tau,\tau') \left(\sim \chi^{\dagger}(\tau)\chi(\tau')\right)$,

$$\widetilde{S}_{\chi,\rho} \equiv -\frac{N_c}{\lambda} \int_0^1 d\tau d\tau' \frac{\rho(\tau,\tau')\rho(\tau',\tau')}{y(\tau,\tau')} + \int d\tau d\tau' \chi^{\dagger}(\tau) \left[\delta_{\tau,\tau'}(i\partial_{\tau} - a) - \rho(\tau,\tau')\right] \chi(\tau')$$

Now there is a piece of the action which scales as N_c in the 't Hooft limit.

Step 4: Integrating out χ

$$\widetilde{S}_{\chi,\rho} = \cdots + \int d\tau d\tau' \chi^{\dagger}(\tau) \left[\delta_{\tau,\tau'}(i\partial_\tau - a) - \rho(\tau,\tau')\right] \chi(\tau')$$

Since the action of χ is Gaussian, we can integrate them out to get

$$\left[\mathrm{Det}\left(\delta_{\tau,\tau'}(i\partial_\tau - a) - \rho(\tau,\tau')\right)\right]^{N_c}$$

As a result, we get an effective action of ρ:

$$S_{\rho}^{\mathrm{eff}} = -N_c \left(\int d\tau d\tau' \frac{\rho(\tau,\tau')\rho(\tau',\tau)}{\lambda\, y(\tau,\tau')} + \mathrm{Tr}\log\left[\delta_{\tau,\tau'}(i\partial_\tau - a) - \rho(\tau,\tau')\right] \right)$$

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In the large N_c limit, we can use the saddle-point approximation:

Saddle point eq: $$\frac{\rho_*(\tau,\tau')}{y(\tau,\tau')} = \frac{1}{\delta_{\tau,\tau'}(i\partial_\tau - a) - \rho_*(\tau,\tau')}.$$

Including $\mathcal{O}$

We can repeat the 4 steps for the correlation function $\langle Z(a)\mathcal{O}\rangle$:

$$\mathcal{O}(x) = \mathrm{Tr}\left[\Phi_{I_1}\Phi_{I_2}\cdots\right](x)$$

Steps 1 and 2: χ's are the sources for $\mathcal{N}=4$ SYM fields. Thus integrating out $\mathcal{N}=4$ SYM replaces Φ_I's with

$$\mathcal{O}(x) \mapsto \mathrm{Tr}\left[\Phi^{\chi}_{I_1}\Phi^{\chi}_{I_2}\cdots\right](x), \qquad \Phi^{\chi}_{I}(x) \equiv -g^2_{\mathrm{YM}}\theta^I \int_0^1 d\tau \frac{|\dot{x}(\tau)|}{(x-x(\tau))^2}\chi\chi^{\dagger}(\tau)$$

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Step 3 and 4: Integrate out χ's by Wick contracting χ's in Φ^{χ}:

$$\mathcal{O}(x) \sim \int d\tau_0 d\tau_1 d\tau_2 \cdots \left(\underbrace{\chi^{\dagger}(\tau_0)\chi}(\tau_1)\right)\left(\underbrace{\chi^{\dagger}(\tau_1)\chi}(\tau_2)\right)\cdots$$

In the large N_c limit, the contractions between the neighboring fields are dominant: $(\chi^{\dagger}\chi) = \chi^{\dagger}_a\chi^a \propto \delta^a_a = N_c$.

Including $\mathcal{O}$

$$\mathcal{O}(x) = \mathrm{Tr}\left[\Phi_{I_1}\Phi_{I_2}\cdots\right](x)$$

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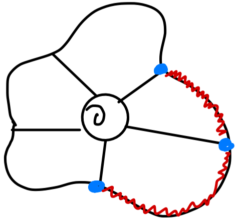
Contracting χ's using $\langle\chi^\dagger(\tau_1)\chi(\tau_2)\rangle \sim \frac{N_c}{\delta_{\tau_1,\tau_2}(i\partial_{\tau_2}-a)-\rho_*(\tau_1,\tau_2)}$, we get

$$\langle Z(a)\mathcal{O}(x)\rangle \sim \left(\prod_{\ell=1}^{L}\int_0^1 d\tau_\ell\right) M^{I_1}_{\tau_0,\tau_1} M^{I_2}_{\tau_1,\tau_2}\cdots = \mathrm{Tr}_\tau\left[\hat{M}^{I_1}\hat{M}^{I_2}\cdots\right]$$

$$M^{I}_{\tau,\tau'} \equiv \frac{\theta^I|\dot{x}(\tau)|}{\underline{(x-x(\tau))^2}}\,\frac{1}{\underline{\delta_{\tau,\tau'}(i\partial_\tau - a) - \rho_*(\tau,\tau')}}$$

Tr_τ is the operator trace over functions of τ.

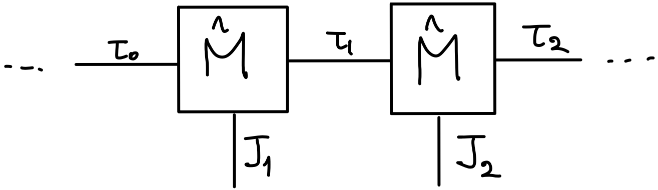


Wilson Loop as Matrix Product State

Alternatively, one can write

$$\langle Z(a)\mathcal{O}(x)\rangle = \langle \mathrm{MPS}|\mathcal{O}\rangle,$$

$$|\mathcal{O}\rangle \equiv |I_1 \cdots I_L\rangle\,, \qquad |\mathrm{MPS}\rangle \equiv \sum_{L=0}^{\infty}\sum_{J_1,\ldots J_L} \mathrm{Tr}_{\mathcal{T}}\left[\hat{M}^{J_1}\cdots\hat{M}^{J_L}\right]|J_1\cdots J_L\rangle$$



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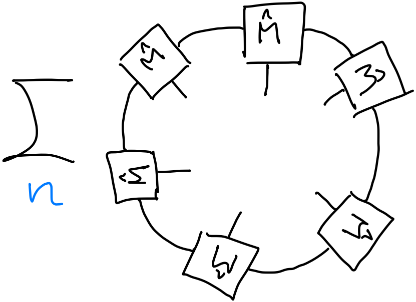
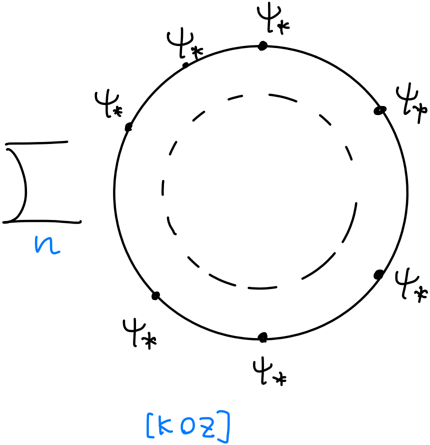
- Continuous, infinite bond dimensions.
- Can be converted into discrete ∞-dim matrices by the mode expansion: $\int_0^1 d\tau \mapsto \sum_n e^{2\pi i n\tau}$.
- $|\mathrm{MPS}\rangle \sim$ "discretized analog" of [Kiermaier, Okawa, Zwiebach].

Wilson Loop as Matrix Product State





A Simple Application

For the 1/2 BPS circular Wilson loop, the saddle-point equation simplifies after the mode expansion, $\rho(\tau,\tau') = \sum_n e^{2\pi i(n+\frac{1}{2})(\tau-\tau')}\rho_n$:

$$\frac{4}{\lambda}\rho_n^* = \frac{1}{2\pi i(n+\frac{1}{2}) - ia - \rho_n^*}$$

$$\Longleftrightarrow \rho_n^* = \frac{i}{2}\left[(2n+1)\pi - a - \sqrt{\lambda + ((2n+1)\pi - a)^2}\right]$$

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$$\frac{4}{\lambda}\rho_n^* = \frac{1}{2\pi i(n+\frac{1}{2}) - ia - \rho_n^*}$$
$$\iff \rho_n^* = \frac{i}{2}\left[(2n+1)\pi - a - \sqrt{\lambda + ((2n+1)\pi - a)^2}\right]$$

If $\mathcal{O}$ **is also BPS**, $\mathcal{O}=\mathrm{Tr}[(\Phi_1+i\Phi_2)^L]$, we get (by the mode expansion)

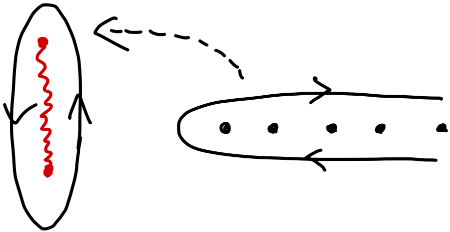
$$\langle Z(a)\mathcal{O}(0)\rangle = \mathrm{Tr}_{n,m}\left[\left(\widetilde{M}^1 + i\widetilde{M}^2\right)^L\right] = \sum_{n=-\infty}^{\infty}\left(\frac{1}{2\pi i(n+\frac{1}{2}) - ia - \rho_n^*}\right)^L$$

$$\widetilde{M}_{n,m} = \frac{1}{2\pi i(n+\frac{1}{2}) - ia - \rho_n^*}\delta_{nm} \qquad (n,m = -\infty\cdots\infty)$$

$$\langle Z(a)\mathcal{O}(0)\rangle = \mathrm{Tr}\left[\left(\widetilde{M}^1 + i\widetilde{M}^2\right)^L\right] = \sum_{n=-\infty}^{\infty}\left(\frac{1}{2\pi i(n+\frac{1}{2}) - ia - \rho_n^*}\right)^L$$

The sum can be performed explicitly using the Sommerfeld-Watson transformation.

$$\langle Z(a)\mathcal{O}(0)\rangle = \oint \frac{dx(1-x^{-2})}{x^L}\tanh\left[\pi g(x+\tfrac{1}{x}) - ia\right]$$



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Projecting to the fundamental rep, $\int da\, e^{-ia}\langle Z(a)\mathcal{O}(0)\rangle$, we get

$$\int da\, e^{-ia}\langle Z(a)\mathcal{O}(0)\rangle = \oint \frac{dx(1-x^{-2})}{x^L} e^{2\pi g(x+\frac{1}{x})} = L\ \mathrm{I}_L(\sqrt{\lambda}),$$

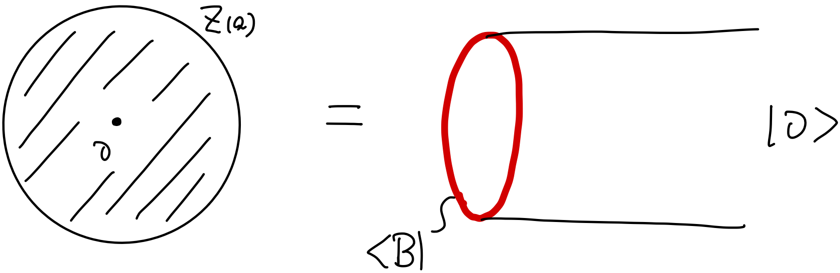
Agree with the result from localization [Pestun], [Giombi, Pestun].

Non-BPS $\mathcal{O}$ at finite coupling

Worldsheet Description of $\langle Z(a)\mathcal{O}\rangle$

As mentioned earlier, the AdS/CFT relates $\langle Z(a)\mathcal{O}\rangle$ to

$$\langle Z(a)\mathcal{O}\rangle \sim \text{Disk with 1 puncture}$$



$|\mathrm{MPS}\rangle \sim$ the weak-coupling limit of $|B\rangle$.

Goal: Determine $|B\rangle$ at finite coupling using integrability.

Bootstrap Program

- At weak coupling, $\langle Z(a)\mathcal{O}\rangle$ obeys some (unexpected) selection rule: Suggests the hidden symmetry and that $|B\rangle$ is an integrable boundary state [cf. Ghoshal, Zamolodchikov].

Bootstrap Program

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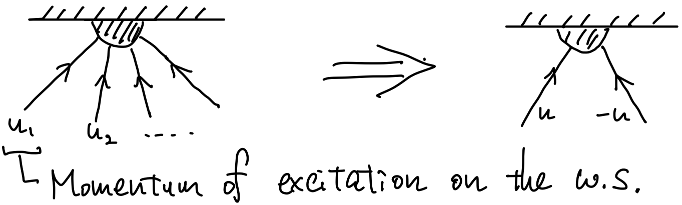
- For integrable boundary states, multiparticle overlaps $\langle B|u_1,\cdots\rangle$ can be reconstructed from two-particle overlaps $\langle B|u,-u\rangle$



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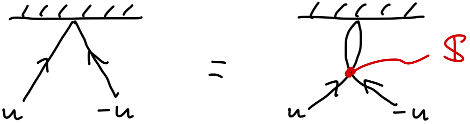
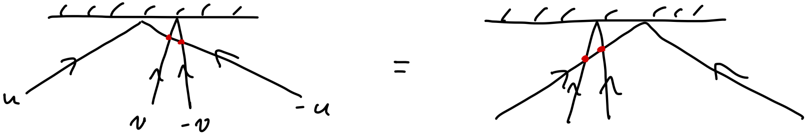
- For integrable boundary states, multiparticle overlaps $\langle B|u_1,\cdots\rangle$ can be reconstructed from two-particle overlaps $\langle B|u,-u\rangle$
- $\langle B|u,-u\rangle$ satisfies a set of axioms: Ward identity, Watson's equation, Boundary Yang-Baxter equation and Crossing symmetry.
- By solving them, one can determine $|B\rangle$ at finite coupling.

Bootstrap Program

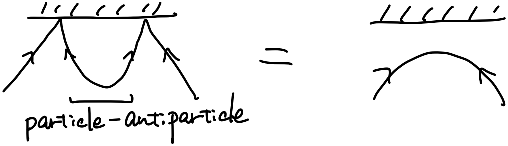
1. Watson's equation, $\langle B|u,-u\rangle = \langle B|\mathbb{S}|u,-u\rangle$:



2. Boundary YB, $\langle B|\mathbb{S}_{24}\mathbb{S}_{34}|u,v,-v,-u\rangle = \langle B|\mathbb{S}_{13}\mathbb{S}_{24}|u,v,-v,-u\rangle$:



3. Crossing equation:



Solution

We find a one-parameter family of solutions to the axioms:

$$\langle B(a)|u,-u\rangle \sim \frac{(u^2+\frac{1}{4})^2}{(u^2-(a+\frac{i}{2})^2)(u^2-(a-\frac{i}{2})^2)} \times \text{(complicated)}$$

which contains poles at $u = \pm(a \pm \frac{i}{2})$.

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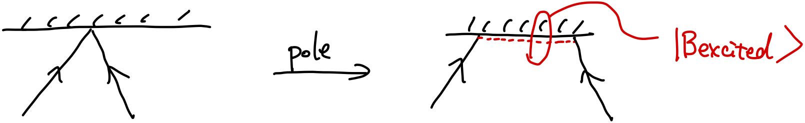
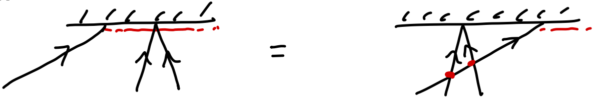
which contains poles at $u = \pm(a \pm \frac{i}{2})$.

Poles signal the existence of excited boundary states.



The overlap for the excited boundary state $|B_{\text{excited}}\rangle$ can be determined by the bootstrap axiom:



Excited Boundary States and MPS

$\langle B_{\rm excited}|u,-u\rangle$ turns out to have new poles at $u=\pm(a\pm\frac{3i}{2})$. This means there will be more excited states.

Repeating this procedure, we find infinitely many states

$$|B^{(n)}(a)\rangle \qquad (n=-\infty,\ldots,\infty)$$

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We thus have

$$\langle Z(a)\mathcal{O}_{u_1,\ldots,u_M}\rangle \sim \sum_{n=-\infty}^{\infty}\langle B^{(n)}(a)|u_1,\ldots,u_M\rangle$$

which, after the Sommerfeld-Watson, leads to

$$\oint \frac{dx(1-x^{-2})}{x^L}\tanh\left[\pi g(x+\tfrac{1}{x})-ia\right]\frac{Q(\frac{i}{2})Q(-\frac{i}{2})}{Q(v+\frac{i}{2})Q(v-\frac{i}{2})}\times\cdots$$

$$\left(v=\frac{\sqrt{\lambda}}{4\pi}(x+\tfrac{1}{x}), \qquad Q(u)\equiv\prod_{k=1}^{M}(u-u_k)\right)$$

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$$\left(v = \frac{\sqrt{\lambda}}{4\pi}(x+\tfrac{1}{x}), \qquad Q(u) \equiv \prod_{k=1}^{M}(u-u_k)\right)$$

- For non-BPS op, it reproduces the weak-coupling result computed by spin-chain methods.
- For BPS op, it reproduces the localization results at finite coupling.
- (# of excited boundary states)=(bond dim of MPS)
  Also true in other setups [SK, Wang]
- Auxiliary Hilbert space of MPS acquires a physical meaning at finite coupling as the DOF living on the boundary.

Conclusion

- Systematic approach to analyze $\langle \mathcal{W}\mathcal{O} \rangle$: MPS at weak coupling, Bootstrap at finite coupling.
- Key is to consider the generating function $Z(a)$ and project it to $\mathcal{W}_f$ only at the end by $\int da\, e^{-ia} Z(a)$.

Conclusion

- Systematic approach to analyze $\langle \mathcal{W}\mathcal{O} \rangle$: MPS at weak coupling, Bootstrap at finite coupling.
- Key is to consider the generating function $Z(a)$ and project it to $\mathcal{W}_f$ only at the end by $\int da\, e^{-ia} Z(a)$.

- Other set-ups? Instantons in $\mathcal{N} = 4$ SYM from ADHM?
- Can we connect the weak- and finite-coupling descriptions? Perhaps in a simpler model like c=1 string?
- Is the trick $\int da\, e^{-ia} Z(a)$ useful in other contexts? Flat-space analog?
- Classify all integrable $|B\rangle$ by the bootstrap axioms?