Workshop on String Field

Theory & Related Aspects

Towards bases of worldsheet integrals

for string amplitudes

Oliver Schlotterer (Uppsala University)

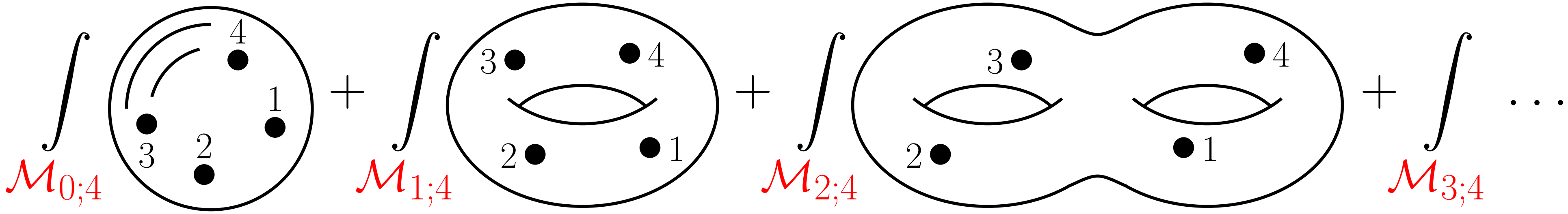
based on 1908.09848, 1908.10830 with C. Mafra

and 1911.03476, 2004.05156 with J. Gerken & A. Kleinschmidt

11.06.2020

Intro I : Towards linear algebra of string amplitudes

String perturbation theory: integrate (S)CFT correlators on surfaces Σ_g



$$\mathcal{A}_{\Sigma_g}(\{1,2,\ldots,n\}) \sim \int_{\mathcal{M}_{g;n}} \Big\langle \left(\begin{matrix} \text{PCOs and/or} \\ \text{b-ghosts \&} \\ \text{regulators} \end{matrix} \right) V_1(z_1)\, V_2(z_2)\, \ldots\, V_n(z_n) \Big\rangle_{\Sigma_g}$$

closed strings: moduli space $\{\tau_j\}$ of n-punctured (super-) Riemann surfaces at genus g

formalism dependent: e.g. RNS or (non-) minimal pure spinors

correlation function of n vertex operators V_j for ext. states on Riemann surface of genus g: depending on polarizations and momenta (kinematics)

Intro I : Towards linear algebra of string amplitudes

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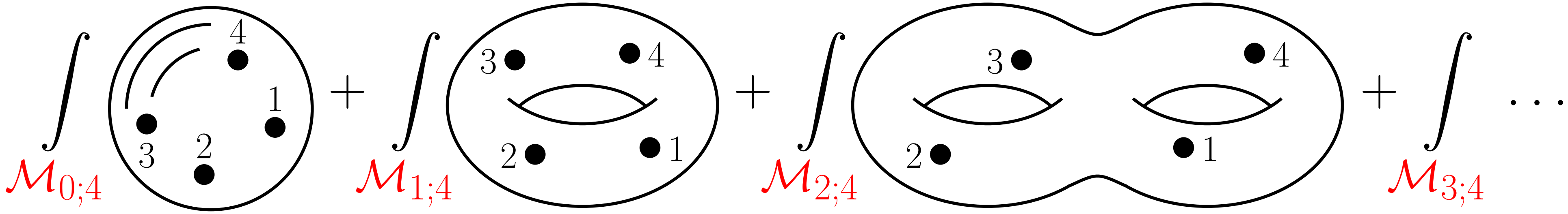

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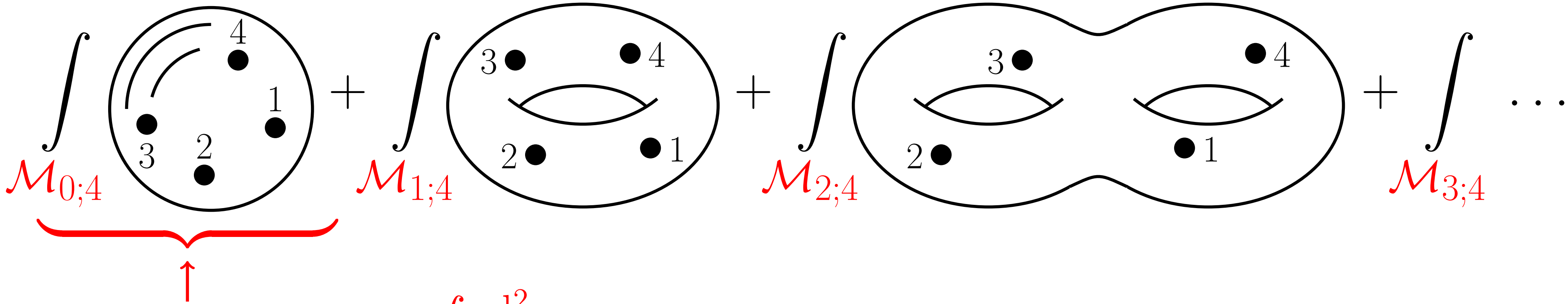
scalar function of $s_{ij} = \alpha'(k_i + k_j)^2$

This talk's mindset / questions / pipe dreams:

- pull polarization information out of the integral
- given n & Σ_g, how many independent fct.'s of z_i & τ_j to integrate?
- how to evaluate / α'-expand basis integrals?

Intro I : Towards linear algebra of string amplitudes

Warmup example: 4 gravitons (& superpartners) on the sphere



$$\mathcal{A}_{S^2}(\{1,2,3,4\}) = \int_{\mathbb{C}\setminus\{0,1,\infty\}} \frac{\mathrm{d}^2 z_2}{\pi} \Big\langle V_1(z_1{=}0)\, U_2(z_2)\, V_3(z_3{=}1)\, V_4(z_4{\to}\infty) \Big\rangle_{S^2}$$

$$= \mathcal{M}^{\rm SUGRA}_{\rm tree}(\{1,2,3,4\}) \times \frac{s_{12}}{\pi} \int_{\mathbb{C}\setminus\{0,1,\infty\}} \frac{\mathrm{d}^2 z_2\, |z_2|^{2s_{12}} |1-z_2|^{2s_{23}}}{|z_2|^2 (1-\bar{z}_2)}$$

$$= \mathcal{M}^{\rm SUGRA}_{\rm tree}(\{1,2,3,4\}) \times \frac{\Gamma(1+s_{12})\Gamma(1+s_{23})\Gamma(1+s_{13})}{\Gamma(1-s_{12})\Gamma(1-s_{23})\Gamma(1-s_{13})}$$

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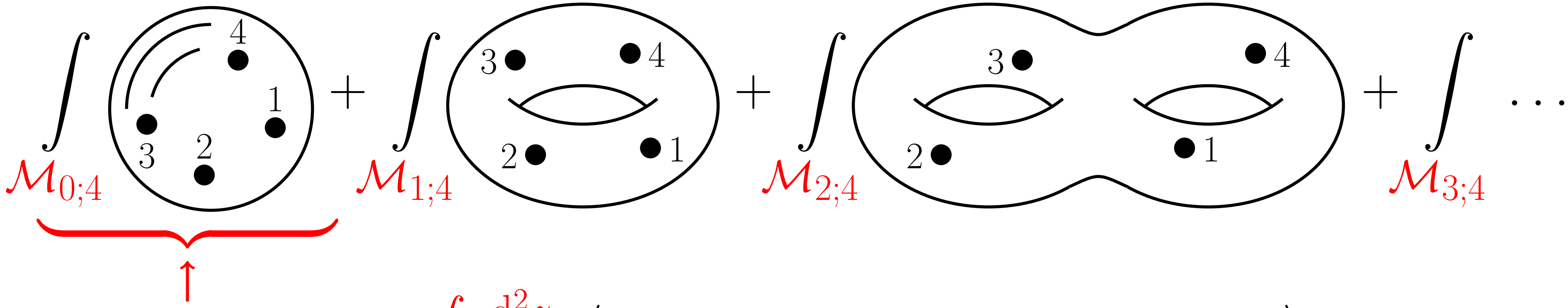

$$\mathcal{A}_{S^2}(\{1,2,3,4\}) = \int_{\mathbb{C}\setminus\{0,1,\infty\}} \frac{\mathrm{d}^2 z_2}{\pi} \Big\langle V_1(z_1{=}0)\, U_2(z_2)\, V_3(z_3{=}1)\, V_4(z_4{\to}\infty) \Big\rangle_{S^2}$$

$$= \mathcal{M}^{\rm SUGRA}_{\rm tree}(\{1,2,3,4\}) \times \frac{s_{12}}{\pi} \int_{\mathbb{C}\setminus\{0,1,\infty\}} \frac{\mathrm{d}^2 z_2\, |z_2|^{2s_{12}} |1{-}z_2|^{2s_{23}}}{|z_2|^2 (1{-}\bar z_2)}$$

$$= \mathcal{M}^{\rm SUGRA}_{\rm tree}(\{1,2,3,4\}) \times \frac{\Gamma(1+s_{12})\Gamma(1+s_{23})\Gamma(1+s_{13})}{\Gamma(1-s_{12})\Gamma(1-s_{23})\Gamma(1-s_{13})}$$

all polarizations: supergravity tensor structure $\forall$ orders in α'

single basis integral "Virasoro-Shapiro" can absorb different exponents for $z_2, \bar z_2$ $(1{-}z_2)$, $(1{-}\bar z_2)$ into $\Gamma(x+1) = x\Gamma(x)$

Spoiler: $[(n{-}3)!]^{\otimes 2}$ sphere integrals in n-point tree amplitudes (see later)

Intro II : Motivation & goals

Why expand string amplitudes in finite-dim. integral basis?

$$\mathcal{A}_{\Sigma_g}(\{1,2,\ldots,n\}) = \sum_{j=1}^{\dim(\text{basis})} K_g^{(j)}\left(\begin{array}{c}\text{polarizations}\\ \text{\& momenta}\end{array}\right) \times \mathcal{I}_g^{(j)}\left(s_{pq} = \alpha'(k_p+k_q)^2\right)$$

- $\alpha' \to 0$ limit: insights into hidden structure in field-theory amplitudes
- compare kinematic factors $K_g^{(j)}$ at different loop order g and fixed $\alpha'^{\#}$

  $\Rightarrow$ insights into string dualities (e.g. S-duality of Type II superstrings)
- basis of integrals $\mathcal{I}_g^{(j)}$ instrumental to perform α'-expansion, identify classes of coeff's (multiple zeta values, polylogs, modular forms, etc.)

  $\longrightarrow$ fruitful interplay with number theory & algebraic geometry

Intro III : Structure of integrands

Universal integrand: $V_j(z_j) \sim :e^{ik\cdot X(z)}: \implies$ "Koba-Nielsen factor"

$$\text{KN}^n_{\Sigma_g} = \exp\Big(\sum_{1\le i<j}^{n} s_{ij}\, G_{\Sigma_g}(z_i, z_j|\tau) \Big)$$

$s_{ij} = \alpha'(k_i+k_j)^2$

extra $\frac{1}{2}$ for open strings

Green function, e.g. $\log|z_{ij}|^2$ at tree level

punctures $z_{ij} = z_i - z_j$

moduli τ @ genus $g>0$

Additionally: OPE $\implies V_i(z_i)V_j(z_j) \sim z_{ij}^{-1} \times \left(\begin{matrix}\text{contractions of}\\ \text{polarizations}\end{matrix}\right)$

e.g. gluon/graviton polarization $\xi_\mu \partial_z X^\mu(z_i) \; :e^{ik_j\cdot X(z_j)}: \sim \; z_{ij}^{-1}(\xi\cdot k_j)$

Closed string: separate z_{ij}^{-1} and $\bar{z}_{ij}^{-1}$ for left- & right-movers

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punctures $z_{ij} = z_i - z_j$

moduli τ @ genus $g>0$

Not all combinations of "OPE contributions z_{ij}^{-1}" yield independent $\int_{\mathcal{M}_{g;n}}$:

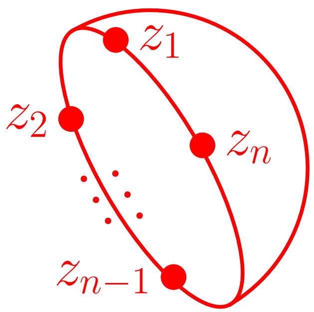
- integration by parts (Koba-Nielsen factor suppresses bdy terms in z_j)

$$0 = \int_{(\partial)\Sigma_g} \mathrm{d}^{(2)}z_i \frac{\partial}{\partial z_i}\Big(\mathrm{KN}^n_{\Sigma_g}\ldots\Big) = \int_{(\partial)\Sigma_g} \mathrm{d}^{(2)}z_i \Big(\mathrm{KN}^n_{\Sigma_g}\ldots\Big)\sum_{j\neq i} \underbrace{s_{ij}\partial_{z_i}G_{\Sigma_g}(z_i,z_j|\tau)}_{\text{locally } s_{ij}/z_{ij}}$$

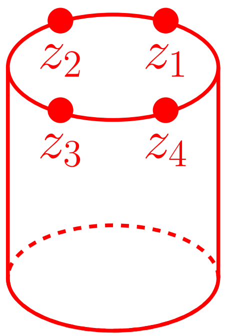
- partial fraction $\frac{1}{z_{12}z_{23}} + \mathrm{cyc}(z_1,z_2,z_3) = 0$ & Fay identities
- such relations define "twisted cohomology" **[e.g. Mizera's work; Casali's talk]**

Intro IV : Open strings = [cycles | cocycles⟩

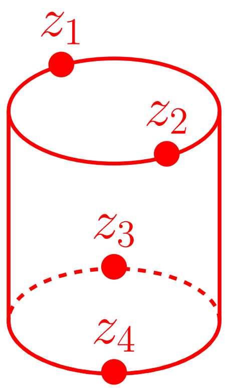
Open-string amplitudes: cyclic order @ boundary ⇒ integration cycle γ



$$\leftrightarrow \quad \gamma_{\rm disk}(1,2,\ldots,n) \;=\; \{-\infty<z_1<z_2<\ldots<z_n<\infty\}$$



$$\leftrightarrow \quad \gamma_{\rm cyl}(1,2,3,4)$$



$$\leftrightarrow \quad \gamma_{\rm cyl}\big(\begin{smallmatrix}1,2\\3,4\end{smallmatrix}\big)$$

Strip off Tr(Chan Paton factors) according to twisted cycle γ, **[Casali's talk]**

twisted cocycle: only z_j, no $\bar z_j$

$$A^{\rm open}_{\Sigma_g}(\gamma) \quad \leftrightarrow \quad \int_\gamma {\rm KN}^n_{\Sigma_g} \, \overbrace{\varphi(z_1,z_2,\ldots,z_n|\tau)} \quad = \quad [\gamma\,|\,\varphi\rangle_{\Sigma_g}$$

only defined up to $\partial_{z_j}({\rm KN}^n_{\Sigma_g}\ldots)$ ↗ ↖ will discuss $g=0,1$ bases

Intro V : Closed strings = $\langle$cocycles | cocycles$\rangle$

Given open-string integral basis with twisted cocycles φ_i

$$A^{\text{open}}_{\Sigma_g}(\gamma) \quad \leftrightarrow \quad \int_\gamma \text{KN}^n_{\Sigma_g}\, \varphi_i(z_1, z_2, \ldots, z_n|\tau) \quad = \quad [\gamma\,|\,\varphi_i\rangle_{\Sigma_g}$$

only defined up to $\partial_{z_j}(\text{KN}^n_{\Sigma_g}\ldots)$ ↗ ↖ will discuss $g = 0, 1$ bases

Obtain closed-string integral basis by $\int_\gamma \to \int_{\mathcal{M}_{g;n}}$

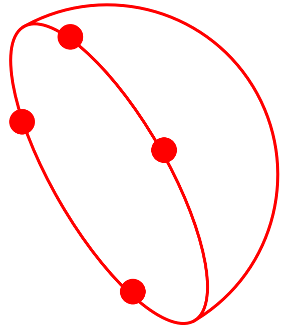
and adjoining complex conjugate cocycles $\overline{\varphi_j(\ldots)}$:

$$A^{\text{closed}}_{\Sigma_g}(\gamma) \quad \leftrightarrow \quad \int_{\mathcal{M}_{g;n}} \varphi_i(z_1, \ldots, z_n|\tau)\overline{\varphi_j(z_1, \ldots, z_n|\tau)} \quad = \quad \langle\varphi_j\,|\,\varphi_i\rangle_{\Sigma_g}$$

In both cases: bases of φ_i at given n and Σ_g should be universal to

- amplitudes in bosonic / heterotic / supersymmetric string theories
- massless & massive external states **hope for synergies with [Chakrabarti's talk]**

Tree level I : Basis of disk integrals



Open-string n-point tree amplitude $\rightarrow$ disk integrals Z_{tree}

$$Z_{\text{tree}}(\gamma \,|\, \rho(1,2,\ldots,n)) \;=\; \int_{\gamma\{-\infty<z_1<z_2<\ldots<z_n<\infty\}} \frac{\mathrm{d}z_1 \ldots \mathrm{d}z_n}{\mathrm{vol}\,\mathrm{SL}_2(\mathbb{R})}\, \mathrm{KN}^n_{\text{tree}}\, \mathrm{PT}(\rho(1,2,\ldots,n))$$

- Koba-Nielsen factor at genus zero $\mathrm{KN}^n_{\text{tree}} = \sum^n_{1\leq i<j} |z_{ij}|^{s_{ij}}$
- twisted cocycle basis $\in \big\{$Parke-Taylor factors $\varphi \;\rightarrow\; \mathrm{PT}(\ldots)\big\}$

$$\mathrm{PT}(\rho(1,2,\ldots,n)) \;=\; \frac{1}{\rho(z_{12}z_{23}\cdots z_{n-1,n}z_{n1})}\,, \quad \rho \;\cong\; \mathrm{perm}(1,2,\ldots,n)$$

- partial fraction & integration by parts $0 \;\cong\; \partial_{z_i}(\mathrm{KN}^n_{\text{tree}}\ldots)$

$$\Longrightarrow \quad (n-3)!\text{ basis } \big\{\mathrm{PT}(1,\rho(2,3,\ldots,n-2),n-1,n),\;\; \rho\in S_{n-3}\big\}$$

[Aomoto 1987; Broedel, Carrasco, Mafra, OS, Stieberger 2011 - 2016]

- at fixed disk ordering γ, $\exists$ universal $(n-3)!$ basis $\big\{Z_{\text{tree}}(\gamma|1,\rho,n-1,n)\big\}$

Tree level II : Open-string amplitudes in $Z_{\rm tree}$ basis

Open-string n-point tree amplitude $\rightarrow$ disk integrals $Z_{\rm tree}$

$$Z_{\rm tree}(\gamma \,|\, \rho(1,2,\ldots,n)) \; = \int_{\gamma\{-\infty<z_1<z_2<\ldots<z_n<\infty\}} \frac{{\rm d}z_1 \, \ldots \, {\rm d}z_n}{{\rm vol}\, {\rm SL}_2(\mathbb{R})} \, {\rm KN}^n_{\rm tree} \, {\rm PT}(\rho(1,2,\ldots,n))$$

- open superstring: coeff's of $Z_{\rm tree}$ $\rightarrow$ color ordered SYM tree amplitudes

$$A^{\rm super}_{\rm tree}(\gamma) \; = \sum_{\rho,\sigma \in S_{n-3}} Z_{\rm tree}(\gamma|1,\rho,n{-}1,n) \, S(\rho|\sigma)_1 \, A^{\rm SYM}_{\rm tree}(1,\sigma,n,n{-}1)$$

[Mafra, OS, Stieberger 1106.2645]

- $(n{-}3)! \times (n{-}3)!$ matrix $S(\rho|\sigma)_1$ from KLT relations for supergravity

$$M^{\rm SUGRA}_{\rm tree} \; = \sum_{\rho,\sigma \in S_{n-3}} \overline{A}^{\rm SYM}_{\rm tree}(1,\rho,n{-}1,n) \, S(\rho|\sigma)_1 \, A^{\rm SYM}_{\rm tree}(1,\sigma,n,n{-}1)$$

entries of KLT matrix $\sim s_{ij}^{n-3}$, e.g. $S(2|2)_1 = s_{12}$ at $n=4$ points

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[Mafra, OS, Stieberger 1106.2645]

- open bosonic string: α'-dependent coefficients $A^{\rm SYM}_{\rm tree} \rightarrow A^{(DF)^2+{\rm YM}}_{\rm tree}$

$$A^{\rm bos}_{\rm tree}(\gamma) \ = \sum_{\rho,\sigma \in S_{n-3}} Z_{\rm tree}(\gamma|1,\rho,n{-}1,n) \, S(\rho|\sigma)_1 \, A^{(DF)^2+{\rm YM}}_{\rm tree}(1,\sigma,n,n{-}1)$$

[Azevedo, Chiodaroli, Johansson, OS 1803.05452]

with $A^{(DF)^2+{\rm YM}}_{\rm tree}$ = amplitudes in gauge theory with dim 6 operators

[Johansson, Nohle 1707.02965; Jusinskas' talk]

Tree level III : Closed-string amplitudes in $W_{\rm tree}$ basis

Closed-string n-point tree amplitude $\rightarrow$ sphere integral $W_{\rm tree}$

$$W_{\rm tree}(\sigma(1,2,\ldots,n)\,|\,\rho(1,2,\ldots,n)) \;=\; \int_{\mathcal{M}_{0;n}} \frac{{\rm d}^2 z_1 \,\ldots\, {\rm d}^2 z_n}{{\rm vol}\,{\rm SL}_2(\mathbb{C})} \times {\rm KN}^n_{\rm tree}\, \overline{{\rm PT}(\sigma(1,2,\ldots,n))}\,{\rm PT}(\rho(1,2,\ldots,n))$$

antiholomorphic Parke-Taylor instead of disk ordering!

Closed-string trees from decorating $\sum_{\sigma,\rho} S(\beta|\sigma)_1 W_{\rm tree}(\sigma|\rho) S(\rho|\pi)_1$ with

$$A^{\rm SYM}_{\rm tree}(\beta) \times \overline{A}^{\rm SYM}_{\rm tree}(\pi) \;\Longrightarrow\; \text{type II superstrings}$$

$$A^{(DF)^2+{\rm YM}}_{\rm tree}(\beta) \times \overline{A}^{\rm SYM}_{\rm tree}(\pi) \;\Longrightarrow\; \text{heterotic strings (gravity)}$$

$$A^{(DF)^2+{\rm YM}}_{\rm tree}(\beta) \times \overline{A}^{(DF)^2+{\rm YM}}_{\rm tree}(\pi) \;\Longrightarrow\; \text{closed bosonic strings}$$

One loop I : Generalizing Parke-Taylor

What is one-loop analogue of the Parke-Taylor basis of genus-zero cocycles?

- tree level: established $(n-3)!$ basis of Parke-Taylor cocycles

$$Z_{\rm tree}(\gamma \,|\, \rho(1,2,\ldots,n)) \ = \ \int_\gamma \frac{{\rm d}z_1 \, \ldots \, {\rm d}z_n}{{\rm vol}\, {\rm SL}_2(\mathbb{R})} \, {\rm KN}^n_{\rm tree} \, {\rm PT}(\rho(1,2,\ldots,n))$$

$$W_{\rm tree}(\sigma \,|\, \rho) \ = \ \int_{\mathcal{M}_{0;n}} \frac{{\rm d}^2 z_1 \, \ldots \, {\rm d}^2 z_n}{{\rm vol}\, {\rm SL}_2(\mathbb{C})} \, {\rm KN}^n_{\rm tree} \, \overline{{\rm PT}(\sigma)} \, {\rm PT}(\rho)$$

- one loop: conjectural $(n-1)!$ basis of Kronecker-Eisenstein cocycles $\varphi^\tau_{\vec\eta}$

$$Z^\tau_{\vec\eta}(\gamma \,|\, 1,\rho(2,3,\ldots,n)) \ = \ \int_\gamma {\rm KN}^{\tau;n}_{g=1} \, \varphi^\tau_{\vec\eta}(1,\rho(2,3,\ldots,n))$$

$$W^\tau_{\vec\eta}(1,\sigma \,|\, 1,\rho) \ = \ \int_{\rm torus} {\rm KN}^{\tau;n}_{g=1} \, \overline{\varphi^\tau_{\vec\eta}(1,\sigma)} \, \varphi^\tau_{\vec\eta}(1,\rho)$$

- leave integration over modular parameter τ (cylinder / torus) for later

- next slides: $(n-1)$ bookkeeping variables $\vec\eta = (\eta_2,\eta_3,\ldots,\eta_n)$

One loop II : The Kronecker-Eisenstein integrands

Parke–Taylor factors are related by partial fraction ($z_{ij} = z_i - z_j$)

$$\frac{1}{z_{12}z_{13}} = \frac{1}{z_{12}z_{23}} + \frac{1}{z_{13}z_{32}} \quad \Longrightarrow \quad \text{KK relations among PT}(\ldots)$$

Naive genus-1 generalization of z_{ij}^{-1}: odd Jacobi theta function

$$\partial_z \log \theta_1(z_{ij}|\tau) \; = \; \frac{1}{z_{ij}} + \left(\begin{array}{c} \text{quasi-periodic completion} \\ \text{w.r.t. } z \to z+1 \;\&\; z \to z+\tau \end{array} \right) ,$$

... more specifically:

$$\theta_1(z|\tau) \; = \; 2e^{i\pi\tau/4} \sin(\pi z) \prod_{n=1}^{\infty} \big(1-e^{2\pi i n\tau}\big)\big(1-e^{2\pi i(n\tau+z)}\big)\big(1-e^{2\pi i(n\tau-z)}\big)$$

Problem: quasi-periodic completion spoils partial fraction:

$$\partial_z \log \theta_1(z_{12}|\tau)\partial_z \log \theta_1(z_{13}|\tau) \; \neq \; \partial_z \log \theta_1(z_{12}|\tau)\partial_z \log \theta_1(z_{23}|\tau) \; + \; \partial_z \log \theta_1(z_{13}|\tau)\partial_z \log \theta_1(z_{32}|\tau)$$

One loop II : The Kronecker-Eisenstein integrands

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$$\frac{1}{z_{12}z_{13}} = \frac{1}{z_{12}z_{23}} + \frac{1}{z_{13}z_{32}} \quad \Longrightarrow \quad \text{KK relations among PT}(\ldots)$$

genus-1 generalization of z_{ij}^{-1}: doubly-periodic Kronecker–Eisenstein series

$$\Omega(z,\eta,\tau) \;=\; \exp\left(2\pi i\eta\,\frac{\mathrm{Im}\,z}{\mathrm{Im}\,\tau}\right)\frac{\theta_1'(0|\tau)\theta_1(z+\eta|\tau)}{\theta_1(z|\tau)\theta_1(\eta|\tau)}$$

Partial fraction generalizes to Fay identity involving auxiliary var's η

$$\Omega(z_{12},\eta_2,\tau)\,\Omega(z_{13},\eta_3,\tau) \;=\; \Omega(z_{12},\eta_2{+}\eta_3,\tau)\,\Omega(z_{23},\eta_3,\tau) + \Omega(z_{13},\eta_2{+}\eta_3,\tau)\,\Omega(z_{32},\eta_2,\tau)$$

Kronecker–Eisenstein integrand at n pt: $n{-}1$ auxiliary var's $\eta_2,\eta_3,\ldots,\eta_n$

$$\varphi^{\tau}_{\vec\eta}(1,2,\ldots,n) \;=\; \prod_{j=2}^{n}\Omega(z_{j-1,j},\eta_j{+}\eta_{j+1}{+}\ldots{+}\eta_n,\tau)$$

One loop II : The Kronecker-Eisenstein integrands

Fay identity among doubly-periodic Kronecker–Eisenstein series ...

$$\Omega(z,\eta,\tau) \;=\; \exp\left(2\pi i\eta\,\frac{\mathrm{Im}\, z}{\mathrm{Im}\,\tau}\right)\frac{\theta_1'(0|\tau)\theta_1(z+\eta|\tau)}{\theta_1(z|\tau)\theta_1(\eta|\tau)}$$

... propagates to Kronecker–Eisenstein integrands

$$\varphi^{\tau}_{\vec\eta}(1,2,\ldots,n) \;=\; \prod_{j=2}^{n}\Omega(z_{j-1,j},\eta_j{+}\eta_{j+1}{+}\ldots{+}\eta_n,\tau)$$

- iterated Fay id's $\Longrightarrow$ only $(n{-}1)!$ independent permutations of $1,2,\ldots,n$

- same counting for conjectural one-loop basis integrals: $\rho\in S_{n-1}$ basis

$$Z^{\tau}_{\vec\eta}(\gamma\,|\,1,\rho(2,3,\ldots,n)) \;=\; \int_\gamma \mathrm{KN}^{\tau;n}_{g=1}\,\varphi^{\tau}_{\vec\eta}(1,\rho(2,3,\ldots,n))$$

[Mafra, OS 1908.09848, 1908.10830]

- with auxiliary variables $\vec\eta$: generating fct. for one-loop string integrals

One loop II : The Kronecker-Eisenstein integrands

Example at four points: 6 permutations $\rho \in S_3$ of $(z_j, \eta_j) \in$ integrand

$$Z^{\tau}_{\eta_2,\eta_3,\eta_4}(\gamma\,|\,1, \rho(2,3,4)) \;=\; \int_{\gamma} \mathrm{KN}^{\tau;4}_{g=1}$$

$$\times\; \rho\Big\{\Omega(z_{12}, \eta_2{+}\eta_3{+}\eta_4, \tau)\Omega(z_{23}, \eta_3{+}\eta_4, \tau)\Omega(z_{34}, \eta_4, \tau)\Big\}$$

- open superstring: 4pt integrand is 1 in place of $\Omega^3 \;\Rightarrow$ pick η_j^{-3} order

from product of $$\Omega(z,\eta,\tau) \;=\; \frac{1}{\eta} + \underbrace{\partial_z \log\theta_1(z|\tau) \;+\; 2\pi i\frac{\mathrm{Im}\, z}{\mathrm{Im}\,\tau}}_{\partial_z G_{\mathrm{torus}}(z|\tau)} + \mathcal{O}(\eta)$$

$$A^{\mathrm{super}}_{\text{1-loop}}(\gamma(1,2,3,4)) \;\sim\; \int_0^{i\infty} \mathrm{d}\tau \int_{\gamma(1,2,3,4)} \mathrm{d}z_2\, \mathrm{d}z_3\, \mathrm{d}z_4\, \mathrm{KN}^{\tau;4}_{g=1} \times 1$$

$$= \int_0^{i\infty} \mathrm{d}\tau\;\, Z^{\tau}_{\eta_2,\eta_3,\eta_4}(\gamma\,|\,1,2,3,4)\,\Big|_{\eta_j^{-3}} \leftrightarrow \text{ set } \Omega(\ldots)\to 1$$

[Brink, Green, Schwarz 1982]

One loop II : The Kronecker-Eisenstein integrands

Example at four points: 6 permutations $\rho \in S_3$ of $(z_j, \eta_j) \in$ integrand

$$Z^{\tau}_{\eta_2,\eta_3,\eta_4}(\gamma \,|\, 1, \rho(2,3,4)) \;=\; \int_{\gamma} \mathrm{KN}^{\tau;4}_{g=1}$$

$$\times\; \rho\Big\{\Omega(z_{12}, \eta_2{+}\eta_3{+}\eta_4, \tau)\Omega(z_{23}, \eta_3{+}\eta_4, \tau)\Omega(z_{34}, \eta_4, \tau)\Big\}$$

- open superstring: 4pt integrand is 1 in place of Ω^3 $\Rightarrow$ pick η_j^{-3} order

from product of $$\Omega(z,\eta,\tau) \;=\; \frac{1}{\eta} + \underbrace{\partial_z \log\theta_1(z|\tau) \,+\, 2\pi i \frac{\operatorname{Im} z}{\operatorname{Im}\tau}}_{\partial_z G_{\text{torus}}(z|\tau)} + \mathcal{O}(\eta)$$

- open bos. string: 4pt integrand $\sim \partial^4_{z_i}$ of $(\log\theta_1)$'s $\Rightarrow$ pick η_j^{+1} order

In fact, $Z^{\tau}_{\vec{\eta}}$ are generating series of genus-one integrals in string amplitudes:

different orders in η_j $\longleftrightarrow$ different string theories / amounts of SUSY

One loop III : Open-string differential equations

Another benefit of η_j-dependent $\Omega(z,\eta,\tau)$ in the integrand of

$$Z^{\tau}_{\vec\eta}(\gamma\,|\,1,\rho(2,3,\ldots,n)) \ = \ \int_{\gamma} \mathrm{KN}^{\tau;n}_{g=1} \times \rho\Big\{ \prod_{j=2}^{n} \Omega(z_{j-1,j},\eta_j{+}\eta_{j+1}{+}\ldots{+}\eta_n,\tau)\Big\} \, .$$

$\Longrightarrow$ $(n{-}1)!$-family $\rho\in S_{n-1}$ closes under τ-derivative

$$2\pi i\,\frac{\partial}{\partial\tau}\,Z^{\tau}_{\vec\eta}(\gamma\,|\,1,\rho(2,3,\ldots,n)) \ = \sum_{\alpha\in S_{n-1}} D^{\tau}_{\vec\eta}(\rho|\sigma)\,Z^{\tau}_{\vec\eta}(\gamma\,|\,1,\sigma(2,3,\ldots,n))$$

with $(n{-}1)!\times(n{-}1)!$ matrix $D^{\tau}_{\vec\eta}(\rho|\sigma)$ linear in α' (i.e. $s_{ij}=\alpha'(k_i{+}k_j)^2$).

[Mafra, OS 1908.09848, 1908.10830]

- in comparison to Feynman integrals, α' takes role of the dim-reg ϵ
- closure under ∂_τ $\Rightarrow$ evidence the $Z^{\tau}_{\vec\eta}(\gamma\,|\,1,\rho)$ furnish a basis at fixed γ

One loop III : Open-string differential equations

Two-point example: "1×1 matrix" $D^{\tau}_{\eta_2}(2|2)$

$$2\pi i \frac{\partial}{\partial \tau} Z^{\tau}_{\eta_2}(\gamma|1,2) = \underbrace{s_{12}\Big(\frac{1}{2}\frac{\partial^2}{\partial \eta_2^2} - \wp(\eta_2,\tau) - 2\zeta_2\Big)}_{D^{\tau}_{\eta_2}(2|2)} Z^{\tau}_{\eta_2}(\gamma|1,2)$$

with Weierstraß function generating holomorphic Eisenstein series

$$\wp(\eta,\tau) = \frac{1}{\eta^2} + \sum_{k=4}^{\infty}(k-1)\,\eta^{k-2}\,\mathrm{G}_k(\tau)\,, \qquad \mathrm{G}_k(\tau) = \sum_{\substack{(m,n)\in\mathbb{Z}^2 \\ (m,n)\neq(0,0)}} \frac{1}{(m\tau+n)^k}$$

All-order α'-expansion from path-ordered exponential $\mathbb{P}\exp(\ldots)$

$$Z^{\tau}_{\eta_2}(\gamma|1,2) = \underbrace{\mathbb{P}\exp\Big\{\int_{i\infty}^{\tau}\frac{\mathrm{d}\tau'}{2\pi i}\,D^{\tau'}_{\eta_2}(2|2)\Big\}}_{\text{iterated Eisenstein integrals}} Z^{i\infty}_{\eta_2}(\gamma|1,2)$$

"tree-lv. data"

$$Z^{i\infty}_{\eta_2}(\gamma|1,2) = \pi\cot(\pi\eta_2)\,\frac{\Gamma(1-s_{12})}{\big[\Gamma(1-\frac{s_{12}}{2})\big]^2}$$

One loop III : Open-string differential equations

Two-point example: "1×1 matrix" $D^{\tau}_{\eta_2}(2|2)$

$$2\pi i \, \frac{\partial}{\partial \tau} \, Z^{\tau}_{\eta_2}(\gamma|1,2) \ = \ \underbrace{s_{12}\Big(\frac{1}{2}\,\frac{\partial^2}{\partial \eta_2^2} - \wp(\eta_2,\tau) - 2\zeta_2\Big)}_{D^{\tau}_{\eta_2}(2|2)} \, Z^{\tau}_{\eta_2}(\gamma|1,2)$$

Three-point example: 2×2 differential operator $D^{\tau}_{\eta_2,\eta_3}(2,3|\alpha(2,3))$

$$\begin{aligned} 2\pi i \partial_\tau Z^{\tau}_{\eta_2,\eta_3}(\gamma|1,2,3) \ &= \ \Big(s_{12}\big[\tfrac{1}{2}\partial^2_{\eta_2} - \wp(\eta_2{+}\eta_3,\tau)\big] + s_{23}\big[\tfrac{1}{2}(\partial_{\eta_2}{-}\partial_{\eta_3})^2 - \wp(\eta_3,\tau)\big] \\ &\qquad + s_{13}\big[\tfrac{1}{2}\partial^2_{\eta_3} - \wp(\eta_3,\tau)\big] - 2\zeta_2 s_{123}\Big) Z^{\tau}_{\eta_2,\eta_3}(\gamma|1,2,3) \\ &\qquad + s_{13}\big[\wp(\eta_2{+}\eta_3,\tau) - \wp(\eta_3,\tau)\big] Z^{\tau}_{\eta_2,\eta_3}(\gamma|1,3,2) \\ &= \sum_{\alpha \in S_2} D^{\tau}_{\eta_2,\eta_3}(2,3|\alpha(2,3)) \, Z^{\tau}_{\eta_2,\eta_3}(\gamma|1,\alpha(2,3)) \, . \end{aligned}$$

In general, all τ-dependence of $D^{\tau}_{\vec{\eta}}$ occurs via $\wp(\eta,\tau)$ & hence $\mathrm{G}_k(\tau)$.

One loop IV : Closed-string basis / differential equations

Conjectural basis of closed-string integrals in bos / het / SUSY theories

$$W^{\tau}_{\vec{\eta}}(1,\sigma(2,\ldots,n)\,|\,1,\rho(2,\ldots,n)) \;=\; \int_{\text{torus}} \Bigg(\prod_{j=1}^{n} \frac{\mathrm{d}^2 z_j}{\operatorname{Im}\tau}\Bigg)\,\mathrm{KN}^{\tau;n}_{g=1}$$

$$\times \prod_{j=2}^{n} \sigma\Big\{\overline{\Omega(z_{j-1,j},\eta_j{+}\eta_{j+1}{+}\ldots{+}\eta_n,\tau)}\Big\}\,\rho\Big\{\Omega(z_{j-1,j},\eta_j{+}\eta_{j+1}{+}\ldots{+}\eta_n,\tau)\Big\}$$

[Gerken, Kleinschmidt, OS 1911.03476, 2004.05156]

Expansion in $\eta_j, \bar{\eta}_j$ and α' generates modular graph forms

[D'Hoker, Green, Gürdogan, Vanhove 1512.06779; D'Hoker, Green 1603.00839]

Differential eq. involves modular version of $\frac{\partial}{\partial\tau}$ "Maaß operators"

$$\nabla_{\vec{\eta}} \;=\; (\tau{-}\bar{\tau})\,\frac{\partial}{\partial\tau} + n - 1 + \sum_{j=2}^{n} \eta_j\,\frac{\partial}{\partial\eta_j}$$

$$\overline{\nabla}_{\vec{\eta}} \;=\; (\bar{\tau}{-}\tau)\,\frac{\partial}{\partial\bar{\tau}} + n - 1 + \sum_{j=2}^{n} \bar{\eta}_j\,\frac{\partial}{\partial\bar{\eta}_j}$$

One loop IV : Closed-string basis / differential equations

Largely recycle differential operators $D^{\tau}_{\vec{\eta}}$ from open string ...

$$2\pi i \frac{\partial}{\partial \tau} Z^{\tau}_{\vec{\eta}}(\gamma|1,\rho(2,3,\ldots,n)) = \sum_{\alpha \in S_{n-1}} D^{\tau}_{\vec{\eta}}(\rho|\alpha)\, Z^{\tau}_{\vec{\eta}}(\gamma|1,\alpha(2,3,\ldots,n))$$

... but drop the ζ_2 term (indicated by "sv" notation)

e.g. $$\text{sv}\, D_{\eta_2}(2|2) = s_{12}\Big(\frac{1}{2}\partial^2_{\eta_2} - \wp(\eta_2,\tau) \cancel{-2\zeta_2}\Big) \qquad @ \text{ 2pt}$$

$$\text{sv}\, D_{\eta_2,\eta_3}(2,3|2,3) = s_{12}\big[\tfrac{1}{2}\partial^2_{\eta_2} - \wp(\eta_2{+}\eta_3,\tau)\big] + s_{23}\big[\tfrac{1}{2}(\partial_{\eta_2}{-}\partial_{\eta_3})^2 - \wp(\eta_3,\tau)\big]$$

$$+ s_{13}\big[\tfrac{1}{2}\partial^2_{\eta_3} - \wp(\eta_3,\tau)\big] \cancel{-2\zeta_2 s_{123}} \qquad @ \text{ 3pt}$$

One loop IV : Closed-string basis / differential equations

Largely recycle differential operators $D^\tau_{\vec{\eta}}$ from open string ...

$$2\pi i \frac{\partial}{\partial \tau} Z^\tau_{\vec{\eta}}(\gamma|1,\rho(2,3,\ldots,n)) = \sum_{\alpha\in S_{n-1}} D^\tau_{\vec{\eta}}(\rho|\alpha)\, Z^\tau_{\vec{\eta}}(\gamma|1,\alpha(2,3,\ldots,n))$$

... but drop the ζ_2 term (indicated by "sv" notation)

$$\text{e.g.} \quad \text{sv}\, D_{\eta_2}(2|2) = s_{12}\Big(\frac{1}{2}\partial^2_{\eta_2} - \wp(\eta_2,\tau) \cancel{- 2\zeta_2}\Big) \qquad @ \text{ 2pt}$$

Holomorphic differential only acts on 2nd entry ρ:

$$2\pi i \nabla_{\vec{\eta}} W^\tau_{\vec{\eta}}(1,\sigma|1,\rho) = (\tau-\bar{\tau}) \sum_{\alpha\in S_{n-1}} \text{sv}\, D^\tau_{\vec{\eta}}(\rho|\alpha)\, W^\tau_{\vec{\eta}}(1,\sigma|1,\alpha)$$

$$+\, 2\pi i \sum_{j=2}^{n} \bar{\eta}_j \partial_{\eta_j} W^\tau_{\vec{\eta}}(1,\sigma|1,\rho)$$

no open-string analogue:
mild interaction between
left & right after loop integral

[Gerken, Kleinschmidt, OS 1911.03476, 2004.05156]

One loop IV : Closed-string basis / differential equations

Also, Laplace operator closes on $W^{\tau}_{\vec{\eta}}$: simply mixes component integrals

$$\Delta_{\vec{\eta}} = (\overline{\nabla}_{\vec{\eta}} - 1)\nabla_{\vec{\eta}} - \Big(n-1+\sum_{j=2}^{n} \eta_j \partial_{\eta_j}\Big)\Big(n-2+\sum_{j=2}^{n} \bar{\eta}_j \partial_{\bar{\eta}_j}\Big)$$

generate Laplace equations for all modular graph forms ($\partial_{\eta_1} = 0$):

$$\begin{aligned}
(2\pi i)^2 \Delta_{\vec{\eta}} W^{\tau}_{\vec{\eta}}(1,\sigma|1,\rho) = \sum_{\alpha,\beta \in S_{n-1}} \Big\{ \delta_{\sigma,\alpha}\delta_{\rho,\beta}\Big[(2\pi i)^2 (2-n)\Big(n-1+\sum_{i=2}^{n}(\eta_i \partial_{\eta_i} + \bar{\eta}_i \partial_{\bar{\eta}_i})\Big) \\
+ (2\pi i)^2 \sum_{2 \leq i < j}^{n} (\eta_i \bar{\eta}_j - \eta_j \bar{\eta}_i)(\partial_{\eta_j}\partial_{\bar{\eta}_i} - \partial_{\eta_i}\partial_{\bar{\eta}_j}) \\
+ 2\pi i (\tau - \bar{\tau}) \sum_{1 \leq i < j \leq n} s_{ij} (\partial_{\eta_j} - \partial_{\eta_i})(\partial_{\bar{\eta}_j} - \partial_{\bar{\eta}_i}) \Big] \\
+ 2\pi i (\tau - \bar{\tau}) \Big[\delta_{\sigma,\alpha} \sum_{i=2}^{n} \eta_i \partial_{\bar{\eta}_i} \mathrm{sv}\, D^{\tau}_{\vec{\eta}}(\rho|\beta) + \delta_{\rho,\beta} \sum_{i=2}^{n} \bar{\eta}_i \partial_{\eta_i} \overline{\mathrm{sv}\, D^{\tau}_{\vec{\eta}}}(\sigma|\alpha) \Big] \\
+ (\tau - \bar{\tau})^2 \mathrm{sv}\, D^{\tau}_{\vec{\eta}}(\sigma|\alpha)\, \overline{\mathrm{sv}\, D^{\tau}_{\vec{\eta}}}(\rho|\beta) \Big\} W^{\tau}_{\vec{\eta}}(1,\alpha|1,\beta) \,.
\end{aligned}$$

same sv $D^{\tau}_{\vec{\eta}}$

as in $\nabla_{\vec{\eta}} W^{\tau}_{\vec{\eta}}$

Number theory of string tree-level and one-loop amplitudes

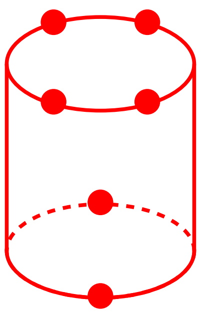
Expanding $Z_{\rm tree}, W_{\rm tree}$ in $s_{ij}{=}\alpha'(k_i{+}k_j)^2 \Rightarrow$ multiple zeta values (MZVs)

$$\zeta_{n_1,n_2,\ldots,n_r} \; = \sum_{0<k_1<k_2<\ldots<k_r}^{\infty} k_1^{-n_1} k_2^{-n_2} \ldots k_r^{-n_r} \,, \qquad n_r \geq 2$$

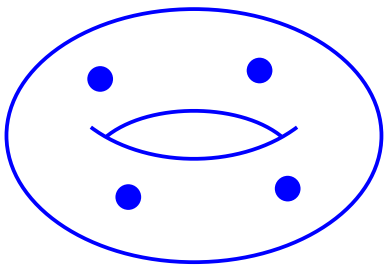
[e.g. Aomoto, Brown, Dupont, Terasoma, Zerbini (math)]

[e.g. Broedel, Mafra, OS, Stieberger, Vanhove, Taylor (physics)]

Analogous α'-expansion at genus one $\rightarrow$ functions of τ (integrate later)



$\longleftrightarrow \; Z^{\tau}_{\vec\eta}(\cdot|\cdot) \; \Longrightarrow$ { elliptic MZVs [Enriquez 1301.3042] [Brödel, Mafra, Matthes, OS 1412.5535]



$\longleftrightarrow \; W^{\tau}_{\vec\eta}(\cdot|\cdot) \; \Longrightarrow$ { modular (graph) forms [D'Hoker, Green, Gürdogan, Vanhove 1512.06779] [D'Hoker, Green 1603.00839]

Summary & Outlook

- advocated integration-by-parts reduction to bases of Σ_g-integrals

  tree lv: $(n-3)!$ Parke-Taylor's, 1 loop: $(n-1)!$ Kronecker-Eisenstein's

- at one-loop: generating fct's of string integrands with variables η_j

  $\longrightarrow$ applicable to massive-state correlators? **[correspondence with A. Schwarz]**

- higher genus: need to generalize Kronecker-Eisenstein series

  & beware of non-splitness of supermoduli space **[Donagi, Witten 1304.7798]**

- can export (meromorphic part of) Kronecker-Eisenstein integrands to

  ambitwistor strings **[talks of Geyer, Mason]**