Tree-level S-matrix of superstring field theory

H. Kunitomo (YITP, Kyoto U)

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Introduction

There are three main complementary approaches of superstring field theory.

- WZW-like approach: (Berkovits)
- Approach based on homotopy algebra: (Erler-Konopka-Sachs)
- Approach with an extra free field: (Sen)

In this talk, we consider the heterotic string field theory based on the second approach and show that it correctly reproduces the tree-level S-matrix calculated by the well-known first-quantized method. (cf. pioneering work by Konopka)

Heterotic string field theory with cyclic L_∞ structure

(Kunitomo-Sugimoto)

String fields: $\Phi = \Phi_{NS} + \Phi_R \in \mathcal{H}^{res} = \mathcal{H}^{NS(2,-1)}_{small} + \mathcal{H}^{R(2,-1/2)res}_{small}$.

Constraints: $b_0^-\Phi = L_0^-\Phi = 0\,, \quad XY\Phi_R = \Phi_R$ or $\mathcal{G}\mathcal{G}^{-1}\Phi = \Phi$,

Here,

$$\mathcal{G} = \pi^0 + \pi^1 X\,, \qquad \mathcal{G}^{-1} = \pi^0 + \pi^1 Y\,,$$

$$(\ \mathcal{G}\mathcal{G}^{-1} = \pi^0 + \pi^1 XY\ )$$

where π^0 (π^1) is the projection onto $\mathcal{H}^{NS}$ ($\mathcal{H}^R$). X and Y satisfy

$$XYX = X\,, \qquad [Q, \Xi] = X\,, \qquad (\ \Xi = \xi_0 + \cdots\ )\,.$$

Symplectic forms: $\omega_s(\Phi_1\,, \Phi_2) = \langle \Phi_1 | c_0^- | \Phi_2 \rangle$

$$\Omega(\Phi_1\,, \Phi_2) = \omega_s(\Phi_1\,, \mathcal{G}^{-1}\Phi_2)\,, \qquad \omega_l(\Phi_1, \Phi_2) = \omega_s(\xi_0\Phi_1, \Phi_2)\,,$$

for $\Phi_1, \Phi_2 \in \mathcal{H}^{res}$.

Bilinear rep.:

$$\begin{array}{ccccc} \langle\Omega| \,:\, \mathcal{H}^{res}\otimes\mathcal{H}^{res} & \longrightarrow & \mathbb{C} \\ \Cup & & \Cup \\ |\Phi_1\rangle\otimes|\Phi_2\rangle & \longmapsto & \Omega(\Phi_1\,,\Phi_2)\,, \end{array}$$

Heterotic string products is represented by

multi-linear map: $L_n \,:\, (\mathcal{H}^{res})^{\wedge n} \longrightarrow \mathcal{H}^{res}$

where $\mathcal{H}^{\wedge n}$ is the space of the symmetrized tensor product:

$$\mathcal{H}^{\wedge n} \ni \Phi_1\wedge\cdots\wedge\Phi_n \;=\; \sum_\sigma (-1)^{\epsilon(\sigma)}\Phi_{\sigma(1)}\otimes\cdots\otimes\Phi_{\sigma(n)}\,,$$

with $L_1\Phi \;=\; Q\Phi$ and $L_n(\Phi_1\,,\cdots,\Phi_n)\in\mathcal{H}^{res} \quad (n\geq 2)\,.$

If $\{L_n\}$ satisfy

$$\sum_{\sigma}\sum_{m=1}^{n}\frac{(-1)^{\epsilon(\sigma)}}{m!(n-m)!}L_{n-m+1}(L_m(\Phi_{\sigma(1)}\,,\cdots,\Phi_{\sigma(m)})\,,\Phi_{\sigma(m+1)}\,,\cdots,\Phi_{\sigma(n)})\;=\;0\,,$$

$$\Omega(\Phi_1\,,L_n(\Phi_2\,,\cdots,\Phi_{n+1}))\;=\;-(-1)^{|\Phi_1|}\Omega(L_n(\Phi_1\,,\cdots,\Phi_n)\,,\Phi_{n+1})\,,$$

it is called a cyclic L_∞ algebra $(\mathcal{H}^{res}\,,\,\Omega\,,\,\{L_n\})$.

If we have such string products with proper ghost and picture numbers, the action

$$I\;=\;\sum_{n=0}^{\infty}\frac{1}{(n+2)!}\Omega(\Phi\,,\,L_{n+1}(\underbrace{\Phi\,,\,\cdots\,,\,\Phi}_{n+1}))\,,$$

is invariant under the gauge tf.

$$\delta\Phi=\sum_{n=0}^{\infty}\frac{1}{n!}L_{n+1}(\underbrace{\Phi\,,\,\cdots\,,\,\Phi}_{n}\,,\,\Lambda)\,.$$

$\clubsuit$ (Assume) bosonic products $\{L_n^{(0)}\}$ is known, which is an cyclic L_∞ algebra with proper ghost number but no picture number.

$\clubsuit$ Heterotic string products $\{L_n\}$ is constructed by inserting X and/or ξ_0 to $\{L_n^{(0)}\}$ (keeping cyclic L_∞ structure) so as to have proper picture number.

Coalgebra representation:

Symmetrized tensor algebra: $\mathcal{SH} = \mathcal{H}^{\wedge 0} \oplus \mathcal{H}^{\wedge 1} \oplus \mathcal{H}^{\wedge 2} \oplus \cdots$

Coderivation: $\boldsymbol{L}_n : \mathcal{SH} \longrightarrow \mathcal{SH}$

$$\begin{aligned}
\boldsymbol{L}_n\,\Phi_1 \wedge \cdots \wedge \Phi_m &= 0\,, && \text{for } m < n\,,\\
\boldsymbol{L}_n\,\Phi_1 \wedge \cdots \wedge \Phi_m &= L_n(\Phi_1 \wedge \cdots \wedge \Phi_m)\,, && \text{for } m = n\,,\\
\boldsymbol{L}_n\,\Phi_1 \wedge \cdots \wedge \Phi_m &= (L_n \wedge \mathbb{I}_{m-n})\Phi_1 \wedge \cdots \wedge \Phi_m\,, && \text{for } m < n\,.
\end{aligned}$$

Then we can consider coderivation $\boldsymbol{L} = \sum_{n=0}^{\infty} \boldsymbol{L}_{n+1}$. The L_∞ algebra is represented by a (degree odd) coderivation $\boldsymbol{L}$ satisfying

$$[\,\boldsymbol{L}\,,\boldsymbol{L}\,] \;=\; 0\,. \qquad \big(\,[\;,\;]: \text{graded commutator}\,\big)$$

Construction of $\boldsymbol{L}$:

Consider coderivations $\boldsymbol{B}(s,t)$ and $\boldsymbol{\lambda}(s,t)$:

$$\boldsymbol{B}(s,t) \;=\; \sum_{m,n,r=0}^{\infty} s^m t^n \boldsymbol{B}^{(n)}_{m+n+r+1}\big|^{2r}\,, \qquad \text{(degree odd)}\,,$$

$$\boldsymbol{\lambda}(s,t) \;=\; \sum_{m,n,r=0}^{\infty} s^m t^n \boldsymbol{\lambda}^{(n+1)}_{m+n+r+2}\big|^{2r}\,, \qquad \text{(degree even)}\,,$$

where m : 'picture no. deficit', (n) : picture no. and $2r$: cyclic Ramond no.

$\Big($cyclic Ramond no. $=$ no. of Ramond inputs $+$ no. of Ramond output.$\Big)$

Construct them by (cyclically) inserting X_0 and/or ξ_0 to $L_n^{(0)}$ so as to satisfy the differential eqs.

$$\begin{aligned}\partial_t \boldsymbol{B}_{n+2}(s,t) &= [\boldsymbol{Q}, \boldsymbol{\lambda}_{n+2}(s,t)] + \sum_{m=0}^{n-1} \Big(\boldsymbol{B}_{m+2}(s,t)\big(\pi(s)\pi_1 \boldsymbol{\lambda}_{n-m+1}(s,t) \wedge \mathbb{I}_{m+1}\big) \\ &\qquad - \boldsymbol{\lambda}_{m+2}(s,t)\big(\pi(s)\pi_1 \boldsymbol{B}_{n-m+1}(s,t) \wedge \mathbb{I}_{m+1}\big)\Big)\,, \qquad \text{(1a)} \\ \partial_s \boldsymbol{B}_{n+2}(s,t) &= [\boldsymbol{\eta}, \boldsymbol{\lambda}_{n+2}(s,t)] - \sum_{m=0}^{n-1} \Big(\boldsymbol{B}_{m+2}(s,t)\big(t\pi_1^1 \boldsymbol{\lambda}_{n-m+1}(s,t) \wedge \mathbb{I}_{m+1}\big) \\ &\qquad - \boldsymbol{\lambda}_{m+2}(s,t)\big(t\pi_1^1 \boldsymbol{B}_{n-m+1}(s,t) \wedge \mathbb{I}_{m+1}\big)\Big)\,, \qquad \text{(1b)}\end{aligned}$$

with the initial condition

$$\boldsymbol{B}(s,0) = \sum_{m,r=0}^{\infty} s^m \boldsymbol{L}_{m+r+1}^{(0)}\big|^{2r}\,.$$

Then $\boldsymbol{B}(s,t)$ satisfies

$$[\boldsymbol{Q}, \boldsymbol{B}_{n+2}(s,t)] \;=\; -\sum_{m=0}^{n-1} \boldsymbol{B}_{m+2}(s,t)\Big(\pi(s)\pi_1\boldsymbol{B}_{n-m+1}(s,t) \wedge \mathbb{I}_{m+1}\Big)\,, \quad \text{(2a)}$$

$$[\boldsymbol{\eta}, \boldsymbol{B}_{n+2}(s,t)] \;=\; \sum_{m=0}^{n-1} \boldsymbol{B}_{m+2}(s,t)\Big(t\pi_1^1\boldsymbol{B}_{n-m+1}(s,t) \wedge \mathbb{I}_{m+1}\Big)\,, \quad \text{(2b)}$$

where $\pi(s) = \pi^0 + s\pi^1$.

From $\boldsymbol{B}(s,t)$, we obtain a cyclic L_∞ algebra $(\mathcal{H}_{large}, \omega_l, \boldsymbol{D} - \boldsymbol{C})$ with

$$\boldsymbol{D} - \boldsymbol{C} \;=\; \boldsymbol{Q} - \boldsymbol{\eta} + \boldsymbol{B}\,, \quad \text{with} \quad \boldsymbol{B} \;=\; \boldsymbol{B}(0,1) \;=\; \boldsymbol{B}_2 + \boldsymbol{B}_3 \cdots .$$

It can be decomposed into two independent L_∞ algebras in $\mathcal{H}_{large}$

$$\pi_1\boldsymbol{D} \;=\; \pi_1\boldsymbol{Q} + \pi_1^0\boldsymbol{B}\,, \qquad \pi_1\boldsymbol{C} \;=\; \pi_1\boldsymbol{\eta} - \pi_1^1\boldsymbol{B}\,,$$

$$[\boldsymbol{D}, \boldsymbol{D}] \;=\; [\boldsymbol{C}, \boldsymbol{C}] \;=\; [\boldsymbol{D}, \boldsymbol{C}] \;=\; 0\,.$$

Then, we transform them by using the cohomomorphism

$$\pi_1 \hat{\boldsymbol{F}}^{-1} = \pi_1 \mathbb{I}_{\mathcal{SH}} - \Xi \pi_1^1 \boldsymbol{B}\,.$$

to another pair of the L_∞ algebras by the similarity tf.

$$\pi_1 \hat{\boldsymbol{F}}^{-1} \boldsymbol{D} \hat{\boldsymbol{F}} = \pi_1 \boldsymbol{Q} + \mathcal{G} \pi_1 \boldsymbol{B} \hat{\boldsymbol{F}} \equiv \pi_1 \boldsymbol{L}\,,$$
$$\pi_1 \hat{\boldsymbol{F}}^{-1} \boldsymbol{C} \hat{\boldsymbol{F}} = \pi_1 \boldsymbol{\eta}\,,$$

which satisfy

$$[\,\boldsymbol{\eta}\,,\boldsymbol{\eta}\,] \;=\; [\,\boldsymbol{L}\,,\boldsymbol{L}\,] \;=\; [\,\boldsymbol{\eta}\,,\boldsymbol{L}\,] = \;0\,.$$

We can show that $(\mathcal{H}^{res}\,,\,\Omega\,,\,\boldsymbol{L})$ is the cyclic L_∞ algebra representing the heterotic string products.

For later use, we denote

$$\boldsymbol{L} \;=\; \boldsymbol{Q} + \boldsymbol{L}_{int}\,, \qquad \pi_1 \boldsymbol{L}_{int} = \mathcal{G} \pi_1 \boldsymbol{B} \hat{\boldsymbol{F}} \equiv \mathcal{G} \pi_1 \boldsymbol{l}\,.$$

Tree-level S-matrix

Remove the ghost no. restriction and take Siegel-Ramond gauge:

$$b_0^+\Phi_{NS} \;=\; \beta_0\Phi_R \;=\; 0\,, \quad (\Phi_{NS}+\Phi_R \in \mathcal{H}_{SR}^{res})$$

Tree-level S-matrix (generating function): (Jevicki-Lee)

$$S[\Phi_0] \;=\; I[\Phi_{cl}(\Phi_0)]\,,$$

where $\Phi_{cl}(\Phi_0)$ is a classical sol. determined as a function of the homogeneous sol. Φ_0 obtained as follows.

Rewrite the EoM (diff. eq.) to the integral eq. (Note 1)

$$\Phi \;=\; \Phi_0 - Q^+\pi_1\boldsymbol{L}_{int}(e^{\wedge\Phi})\,, \qquad Q^+ \;=\; \frac{b_0^+}{L_0^+}(1-P_0)\,, \tag{3}$$

where P_0 is the proj. op. onto the on-shell sub-space $\mathcal{H}_0$ ($\ni \Phi_0$):

$$P_0\;:\;\mathcal{H}_{SR}^{res} \to \mathcal{H}_0 \;=\; \{\Phi\in\mathcal{H}_{SR}^{res}\,|\, L_0^+\Phi_{NS}=G\Phi_R=0\}\,.$$

Q^+ satisfyingn $QQ^+ + Q^+Q + P_0 = 1$ is called the contracting homotopy operator.

Eq. (3) can be solved in closed form using coalgebra rep. as function of Φ_0 :

$$\Phi_{cl}(\Phi_0) \; = \; \pi_1(\hat{\boldsymbol{I}} + \boldsymbol{H}\boldsymbol{L}_{int})^{-1}\hat{\boldsymbol{P}}(e^{\wedge\Phi_0}) \, ,$$

where $\boldsymbol{H}$, $\hat{\boldsymbol{P}}$ and $\hat{\boldsymbol{I}}$ are ops. acting on $\mathcal{SH}^{res}$ defined by

$$\boldsymbol{H} \; = \; \sum_{r,s=0}^{\infty} \frac{1}{(r+s+1)!} \, Q^+ \wedge (\mathbb{I}_1)^{\wedge r} \wedge (P_0)^{\wedge s}$$

$$\hat{\boldsymbol{P}} \; = \; \sum_{n=0}^{\infty} P_n \; = \; \sum_{n=0}^{\infty} \frac{1}{n!}(P_0)^{\wedge n} \, , \qquad \hat{\boldsymbol{I}} \; = \; \sum_{n=0}^{\infty} \mathbb{I}_n \; = \; \sum_{n=0}^{\infty} \frac{1}{n!}(\mathbb{I}_1)^{\wedge n} \, .$$

They satisfy

$$\boldsymbol{HQ} + \boldsymbol{QH} + \hat{\boldsymbol{P}} \; = \; \hat{\boldsymbol{I}} \, ,$$

$$\boldsymbol{H}\hat{\boldsymbol{P}} \; = \; \hat{\boldsymbol{P}}\boldsymbol{H} \; = \; \boldsymbol{HH} \; = \; 0 \, , \qquad [\boldsymbol{Q}, \hat{\boldsymbol{P}}] \; = \; 0 \, .$$

Putting $\Phi_{cl}(\Phi_0)$ to the action I , we obtain the tree-level S-matrix.

Here, we express the tree-level S-matrix by multilinear map:

$$\langle S| \;=\; \langle\Omega|P_0 \otimes \pi_1 \boldsymbol{S} \;:\; \mathcal{H}_0 \otimes \mathcal{S}\mathcal{H}_0 \to \mathbb{C}\,,$$
$$\boldsymbol{S} \;=\; \hat{\boldsymbol{P}}\boldsymbol{L}_{int}(\hat{\boldsymbol{I}} + \boldsymbol{H}\boldsymbol{L}_{int})^{-1}\hat{\boldsymbol{P}}\,,$$

which provides total S-matrix in the BRST formulation including unphysical states. (It is also obtained by HPT.)

The physical S-matrix is obtained by projecting it to the physical subspace:

$$\langle S^{phys}| \;=\; \langle S|P_{phys} \otimes \hat{\boldsymbol{P}}_{phys}\,,$$

where P_{phys} is the proj. op. onto the physical subspace,

$$P_{phys} \;:\; \mathcal{H}_0 \;\to\; \mathcal{H}_{phys} \;=\; \mathrm{Ker}(\widetilde{Q})/\mathrm{Im}(\widetilde{Q})\,,$$

and $\hat{\boldsymbol{P}}_{phys} = \sum_{n=0}^{\infty} \frac{1}{n!}(P_{phys})^{\wedge n}$. Unitarity of S^{phys} is guaranteed by

$$[\boldsymbol{Q}, \boldsymbol{S}] \;=\; 0\,.$$

The S-matrix element of $(n+3)$-string scattering is further expanded to those with different no. of external Ramond states ($=$ cyclic Ramond no.):

$$\langle S| \;=\; \sum_{n=0}^{\infty}\langle S_{n+3}| \;=\; \sum_{n=0}^{\infty}\sum_{r=0}^{[\frac{n+3}{2}]}\langle S_{n+3}^{(n-r+1)}|^{2r}\,.$$

For example, four-string scattering elements are

$$\langle S_4| \;=\; \langle S_4^{(2)}|^0 + \langle S_4^{(1)}|^2 + \langle S_4^{(0)}|^4\,.$$

The 1st, 2nd and 3rd terms represent the S-matrix elements of four-NS, two-NS-two-R and four-R scattering, respectively.

Evaluation of S-matrix

Since $\pi_1 \boldsymbol{L}_{int} = \mathcal{G}\pi_1 \boldsymbol{l}$, $\langle S|$ is also written as $\Big(\langle\Omega| = \langle\omega_l|(\xi_0 \otimes \mathcal{G}^{-1})\Big)$

$$\langle S| \;=\; \sum_{n=0}^{\infty}\langle S_{n+3}| \;=\; \sum_{n=0}^{\infty}\langle\omega_l|\xi_0 P_0 \otimes P_0\pi_1\boldsymbol{\Sigma}_{n+2}\,,$$

$$\pi_1\boldsymbol{\Sigma} \;=\; \pi_1\boldsymbol{l}(\hat{\boldsymbol{I}} + \boldsymbol{H}\boldsymbol{L}_{int})^{-1}\hat{\boldsymbol{P}}\,. \tag{4}$$

with $\boldsymbol{\Sigma}_{n+2} = \boldsymbol{\Sigma}\pi_{n+2}$. We can write (4) by using $\boldsymbol{B}$ in the form of the (classical) Dyson-Schwinger eq.

$$\pi_1\boldsymbol{\Sigma}_{n+2} \;=\; \sum_{m=0}^{n}\pi_1\boldsymbol{B}_{m+2}\left(\frac{1}{(m+2)!}\Big(P_0\pi_1 - (Q^+\mathcal{G} - \Xi\pi^1)\pi_1\boldsymbol{\Sigma}\Big)^{\wedge(m+2)}\right)\pi_{n+2}\,.$$

Since $\boldsymbol{\Sigma}_{l+2}$ with $l \geq n$ do not contribute in the r.h.s. we can recursively determine $\boldsymbol{\Sigma}_{n+2}$ from $\boldsymbol{\Sigma}_2 = \boldsymbol{B}_2 P_2$.

We extend it to the generating function with two parameters as

$$\pi_1 \boldsymbol{\Sigma}_{n+2}(s,t) \;=\; \sum_{m=0}^{n} \pi_1 \boldsymbol{B}_{m+2}(s,t) \left(\frac{1}{(m+2)!} \Big(P_0 \pi_1 - \Delta(s,t) \pi_1 \boldsymbol{\Sigma}(s,t) \Big)^{\wedge (m+2)} \right),$$

with $\boldsymbol{B}_{m+2}(s,t) = \boldsymbol{B}(s,t)\pi_{m+2}$ and

$$\Delta(s,t) = Q^{+}(\pi^0 + (tX + s)\pi^1) - t\Xi\pi^1 \,.$$

Here, $\Delta(s,t)$ is determined so that $[\boldsymbol{\eta}, \boldsymbol{\Sigma}] = [\boldsymbol{Q}, \boldsymbol{\Sigma}] = 0$ are preserved:

$$[\boldsymbol{\eta}, \boldsymbol{\Sigma}(s,t)] \;=\; [\boldsymbol{Q}, \boldsymbol{\Sigma}(s,t)] \;=\; 0 \,,$$

and $\boldsymbol{\Sigma}(0,1) = \boldsymbol{\Sigma}$. $\Delta(s,t)$ satisfies

$$\partial_s \Delta(s,t) \;=\; -\{\eta, Q^{+}\Xi\} \,, \qquad \partial_t \Delta(s,t) \;=\; -\{Q, Q^{+}\Xi\} \,.$$

Then, using Eqs.(1) and (2), we can show that

$$\partial_s \boldsymbol{\Sigma}(s,t) \;=\; [\boldsymbol{\eta}, \boldsymbol{\rho}(s,t)]\,, \qquad \partial_t \boldsymbol{\Sigma}(s,t) \;=\; [\boldsymbol{Q}, \boldsymbol{\rho}(s,t)]\,,$$

with a fixed $\boldsymbol{\rho}(s,t)$ (Note 2). Further, if we introduce an operation $\mathcal{O}\circ$ ($\mathcal{O} = \xi_0$ or X_0) on coderivation $\boldsymbol{D}_n$ defined by

$$\pi_1 \mathcal{O} \circ \boldsymbol{D}_n \;=\; \frac{1}{n+1}\Big(\mathcal{O} D_n + (-1)^{|\mathcal{O}||D|} D_n (\mathcal{O}\pi_1 \wedge \mathbb{I}_{n-1})\Big)\,.$$

we find the relation

$$\partial_t \boldsymbol{\Sigma}(s,t) - X_0 \circ \partial_s \boldsymbol{\Sigma}(s,t) \;=\; [\boldsymbol{Q}, [\boldsymbol{\eta}, \boldsymbol{T}(s,t)]]\,, \qquad (5)$$

with $\boldsymbol{T}(s,t) = \xi_0 \circ \boldsymbol{\rho}(s,t)$ holds. From (5) we have

$$\partial_t \langle S_{n+3}(s,t)| \;=\; \partial_s \langle S_{n+3}(s,t)| X_0 + \cdots\,,$$

for $\langle S(s,t)|$, where dots represents the terms vanishing on $\mathcal{H}_{phys}$ and

$$\langle S_{n+3}(s,t)| X_0 \;=\; \langle S_{n+3}(s,t)| (X_0 \otimes \mathbb{I}_{n+2} + \mathbb{I}_1 \otimes X_0 \wedge \mathbb{I}_{n+1})\,.$$

Repeatedly using (6), we can find that

$$\langle S_{n+3}^{phys}| \;=\; \sum_{p=0}^{n+1} \langle S_{n+3}^{phys(0)}|^{2(n-p+1)} (X_0)^p \,. \tag{6}$$

Here, $\langle S_{n+3}^{phys(0)}|$ is the part of physical S-matrix element integrated over the *whole moduli space* (or added up all the possible Feynman diagrams) without inserting the PCOs. (Note 3)

Hence, the r.h.s. of (6) is independent of the position X_0 inserted, and nothing but the tree-level physical S-matrix obtained in the 1st-quantized method.

Summary

◇ We have shown that the tree-level physical S-matrix of the heterotic string field theory agrees with that in the 1-st quantized formulation.

♣ Similarly, we can show that the tree-level physical S-matrices of the type II and the open superstring field theories also agree with those in the 1-st quantized formulation.

♠ Extension to the loop-level: Loop L_∞ (BV) algebra, LSZ reduction formula, unitarity, Heisenberg rep., asymptotic fields, and so on.

Appendix

Note 1: Expanding the ghost zero-mode, the sates in $\mathcal{H}^{res}$ has the form

$$\mathcal{H}^{res} \ \ni \ \Phi \ = \ (\phi_{NS} - c_0^+ \psi_{NS}) + \left(\phi_R - \frac{1}{2}(\gamma_0 + 2c_0^+ G)\psi_R\right).$$

EoMs are obtained by projecting $\pi_1 \boldsymbol{L}(e^{\wedge\Phi}) = 0$ onto ψ-component.

$$NS \ : \ L_0^+ (\Phi_{cl})_{NS} + b_0^+ \pi_1^0 \boldsymbol{L}_{int}(e^{\wedge(\Phi_{cl})_{NS}}) \ = \ 0\,,$$

$$R \ : \ G(\Phi_{cl})_R + \frac{b_0^+}{2G} \pi_1^1 \boldsymbol{L}_{int}(e^{\wedge(\Phi_{cl})_R}) \ = \ 0\,.$$

They can be rewritten as the form of integral eq. ($(1 - P_0)$ is ommitted.)

$$\Phi \ = \ \Phi_0 - \frac{b_0^+}{L_0^+} \pi_1 \boldsymbol{L}(e^{\wedge\Phi})\,.$$

Note 2: $\boldsymbol{\rho}(s,t)$ is determined by solving the recursion relation

$$\begin{aligned}
&\pi_1\boldsymbol{\rho}_{n+2}(s,t) \\
&= \sum_{m=0}^{n} \pi_1\boldsymbol{\lambda}_{m+2}(s,t)\big(D_{m+2}(s,t)\big)P_{n+2}\pi_{n+2} \\
&- \sum_{m=0}^{n-1} \pi_1\boldsymbol{B}_{m+2}(s,t)\Big(D_{m+1}(s,t)\wedge\Big(\Delta(s,t)\pi_1\boldsymbol{\rho}(s,t)+Q^{+}\Xi\pi_1^1\boldsymbol{\Sigma}(s,t)\Big)\Big)P_{n+2}\pi_{n+2}\,,
\end{aligned}$$

with $\boldsymbol{\rho}_2(s,t)=\boldsymbol{\lambda}_2(s,t)P_2\pi_2$, where

$$D_M(s,t) \;=\; \frac{1}{M!}\Big(P_0\pi_1-\Delta(s,t)\pi_1\boldsymbol{\Sigma}(s,t)\Big)^{\wedge M}\,.$$

Note 3:

For example, (one of the) two-NS-two-Ramond scattering S-matrix element $\langle S_4^{phys(0)}|_0^2$ is written as

$$\begin{aligned}
&\langle S_4^{phys(0)}|_0^2 \\
&= \langle\omega_s|\Big(P_{phys}\pi^1 \\
&\qquad \otimes \pi_1^1\big(\boldsymbol{L}_2^{(0)}|_0^2 - \boldsymbol{L}_2^{(0)}|_0^2\frac{b_0^+}{L_0^+}\boldsymbol{L}_2^{(0)}|_0^0 - \boldsymbol{L}_2^{(0)}|_0^2\frac{b_0^+}{L_0^+}\boldsymbol{L}_2^{(0)}|_0^2\ \big)\Big)P_{phys}\pi_3 \\
&\qquad\qquad \times\Big(X_0\otimes\mathbb{I}_3+\mathbb{I}_1\otimes X_0\wedge\mathbb{I}_2\Big)^2 .
\end{aligned}$$