Gauge field theory

Deformations, obstructions and higher structure

Jim Stasheff

U Penn and UNC-CH

Fundamental Aspects of String Theory
Sao Paulo, Brazil 8 June 2020

Introduction

In physics, a *field theory* refers to a set-up in which there are 'fields', often sections of some bundle.

A *gauge field theory* is a field theory presented as having a group $\mathfrak{G}$ of local symmetries.

In physics, ‘local’ means $\mathfrak{G}$ acts as fibre preserving morphisms of the bundle. Think of vertical vector fields as the analogous Lie algera $\mathfrak{g}$.

The designation *gauge transformation* refers to the infinitesimal action of $\mathfrak{g}$ on the fields.

History of L_∞ in physics

- 1982 D'Auria and Fré
- 1984 Berends, Burgers and van Dam
- 1983 Batalin, Fradkin, Vilkovisky and Batalin, Vilkovisky
- 1989 Zwiebach CSFT

Giovanni Felder remarked: *"the ∞-virus had a long incubation time and the outbreak came after I left the infection zone."*

Since then, it has expanded to epidemic proportions in the field theoretic physics community, being useful and popular !
in two major ways:

- Description of a physical problem leads to a structure, later recognized as an L_∞-algebra - BBvD's higher spin algebras, Zwiebach's CSFT, BFV and BV complexes
- Use of L_∞-algebras to better organize a generalized gauge field theory

It has been suggested that *every* classical perturbative gauge theory, including its dynamics and more, can be organized using an L_∞-structure that encodes the fields, the gauge structure, field equations, Noether identities and deformations of the theory; in short, everything one would like to know about a classical theory.

A significant part of the story is allowing ‘field dependent’ gauge transformations, which goes back to at least BBvD.

L_∞ structure

L_∞-algebras are a generalization of dg Lie algebras

- a dg vector space $(\mathfrak{X} = \bigoplus_m \mathfrak{X}_m, d_{\mathfrak{X}})$
- a coherent set of generalized brackets

$$\ell_n : \mathfrak{X} \otimes \mathfrak{X} \otimes \cdots \otimes \mathfrak{X} \to \mathfrak{X}$$

where $\ell_1 = d_{\mathfrak{X}}$.

Coherent means satisfying generalized Jacobi identities.

The CE_∞-algebra

For an L_∞algebra $\mathfrak{L}$, the relevant cochain algebra is the same as for a dg Lie algebra $\mathfrak{g}$, the CE_∞-algebra

$$Hom(\Lambda^\bullet \mathfrak{L}, \mathfrak{L})$$

but with the differential using all the brackets $\ell_n : \Lambda^n \mathfrak{L} \to \mathfrak{L}$.

$\Lambda^\bullet \mathfrak{L}$ denotes the free graded commutative COalgebra on $\mathfrak{L}$.

Before they were identified in the Chevalley-Eilenberg-algebra, generators were dubbed ‘ghosts’ in physics and the differential was called BRST.

All three, the higher bracket, the dg algebra and the coalgebra descriptions have their individual advantages; it is occasionally very useful to switch between them.

The CE_∞-coalgebra is particularly useful for defining L_∞-morphisms even of strict dg Lie algebras.
Example: Kontsevich’s proof of formality.

The Batalin and Vilkovisky complex $\mathcal{BV}$ deals with variational problems such as Euler-Lagrange equations
with symmetries/gauge transformations which form a group $\mathfrak{G}$ with infinitesimal Lie algebra $\mathfrak{g}$.

Start with a Lagrangian **L** in terms of fields regarded as sections of some bundle $p : E \to M$, its jet extension JE and use Loc(E), the algebra of Local functions on E.

A Lagrangian **L** determines a *solution surface* or *shell* $\Sigma \subset JE$ such that ϕ is a solution of the variational problem iff $j^\infty \phi$ has its image in Σ.

The corresponding algebra for Σ is the quotient $Loc(E)/\mathcal{I}$ where the *stationary ideal* $\mathcal{I}$ consists of local functions which vanish 'on shell', i.e. when restricted to the solution surface Σ.

Extend Loc E by adjoining generators of various degrees to form a dgca over Loc(E):

the Koszul-Tate resolution (KT, d_{KT}) for $Loc(E)/\mathcal{I}$.

For the symmetries, similarly adjoin generators' (called 'ghosts') to form the Chevalley-Eilenberg-complex for Loc(E) as a module over $\mathfrak{g}$ with differential d_{CE}.

Batalin and Vilkovisky 'walk on 2 feet', introducing at each stage a new variable u and an 'anti' u^*, giving rise to the odd Poisson bracket $\{u, u^*\} = 1$.

For example, for each field A, an 'anti-field' A^* of degree 1.

Unfortunately (or fortuitously?), the total $d_{CE} + d_{KT}$ does not square to 0, because the *'structure constants'* of $\mathfrak{g}$ are now *structure functions*.

The brilliance of Batalin and Vilkovisky and of Batalin, Fradkin and Vilkovisky was to add a sequence of terms of higher order $d_1, d_2, \ldots$ to achieve $D^2 = 0$ for $D = d_{CE} + d_{KT} + d_1 + d_2 + \cdots$ so as to kill the discrepancy step by step.

Organizing Principle

Perturbative gauge theory, including its dynamics, can be organized using an L_∞-structure .

A significant part of the story is allowing *field dependent* gauge transformations, going back to at least BBvD.

Hohm and Zwiebach began investigating the L_∞-structure of general gauge invariant perturbative field theories. They focused on the part of the theory dealing with gauge parameters, gauge fields and "equations of motion" organized as a dg graded vector space $\mathfrak{X}$:

$$\begin{array}{ccccccccc} \cdots & \longrightarrow & \mathfrak{X}_0 & \longrightarrow & \mathfrak{X}_1 & \longrightarrow & \mathfrak{X}_2 & \longrightarrow & \cdots . \\ & & \lambda & & A & & E & & \end{array}$$

The differential $\ell_1 = d_{\mathfrak{X}}$ and is theory dependent.

They emphasized the terms of higher order in A

the gauge transformations and gauge algebra:

$$\delta_\lambda(A) = \ell_1(\lambda) - \tfrac{1}{2!}\ell_2(\lambda, A) - \tfrac{1}{3!}\ell_2(\lambda, A, A) + \cdots$$

$$[\delta_\lambda, \delta_\mu](A) = \ell_2(\lambda, \mu) - \tfrac{1}{3!}\ell_3(\lambda, \mu, A) + \tfrac{1}{4!}\ell_4(\lambda, \mu, A, A) + \cdots$$

Exactly the BBvD pattern

and adding the field equations:

$$\ell_1(A) + \tfrac{1}{2!}\ell_2(A, A) - \tfrac{1}{3!}\ell_2(A, A, A) + \cdots = 0$$

Jurco, Raspollini, Sämann and Wolf and their team extend $\mathfrak{X}$ to

$$\cdots \underbrace{\begin{array}{c}\text{gauge}\\ \text{symmetries}\end{array}}_{\mathfrak{X}_0} \longrightarrow \underbrace{\begin{array}{c}\text{classical}\\ \text{fields}\end{array}}_{\mathfrak{X}_1} \longrightarrow \underbrace{\begin{array}{c}\text{equations}\\ \text{of}\\ \text{motion}\end{array}}_{\mathfrak{X}_2} \longrightarrow \underbrace{\begin{array}{c}\text{Noether}\\ \text{identi-}\\ \text{ties}\end{array}}_{\mathfrak{X}_3} \cdots$$

The classical BV complex $\mathcal{BV}$ provides precisely such a differential graded algebra .

Additional structures

Add a suitable inner product $\langle\ ,\ \rangle$ to define an action principle.

An L_∞-algebra is called *cyclic*
if $\quad < a_o,\ l_n(a_1, \cdots, a_n) >=< a_n,\ l_n(a_0, \cdots, a_{n-1}) >$

Čirić, Giotopaulos, Radonović and Szabohave added another 'twist',
a Drinfel'd-type *braiding*.

Bootstrapping is a process which starts with a known theory and then from initial data ℓ_1, ℓ_2 creates a deformation as an L_∞-algebra by inductively solving the L_∞-relations to determine the rest of the L_∞-structure.

For example, initiate a deformation using the star commutator = Poisson bracket. .

A key observation is the use of field dependent gauge transformations as well as an assumption that guarantees "closed implies exact".

Hierarchy refers to a structure much like a Postnikov tower or a Sullivan model.

Tensor hierarchies form a class of objects found in supergravity theories, arising from Leibniz algebras as gauge algebras.

Recall a Leibniz algebra $(V, \circ)$ is is a vector space V with a binary operation
$\circ : V \otimes V \to V$ satisfying the (left) derivation rule:

$$x \circ (y \circ z) = (x \circ y) \circ z + y \circ (x \circ z).$$

A Leibniz algebra gives rise to a tensor hierarchy and an L_∞-algebra

by lifting the skew-symmetric part of the Leibniz product,

sucessively adjoining additional variables to compensate for the failure of $\circ$ to satisfy Jacobi.

L_∞-connections and twisted tensor products

Maurer-Cartan conditions:

- The integrability condition for deformation theory
- The master equation for field theories
- Flatness of a connection
- Definition of a *twisting cochain*

The L_∞ version (up to signs and rational fudge factors):

$$\sum \ell_n(a, \cdots, a) = 0.$$

The algebraic model of a fibration $F \to E \xrightarrow{p} B$ is a *twisted tensor product* $(B \otimes F, D)$ where the differential D is twisted:

$$d = d_B \otimes 1 + 1 \otimes d_F + \tau;$$

the "twisting term" τ is a differential and a perturbation with respect to filtration by the F degree i.e .

$$\tau : B^p \otimes F^q \subset \sum_{r<q} B \otimes F^r$$

Bonezzi and Holm have an example of such a twisted tensor product (though not so named):

$\Omega^\bullet(M) \otimes \mathfrak{L}$ with an L_∞-algebra $\mathfrak{L}$ as fibre.

A connection A or covariant derivative D_A with values in $\mathfrak{L}$.

If it is not flat, they then 'flatten the curvature' by sucessively using the higher brackets.

Deformation theory

Following the work of Gerstenhaber introducing algebraic deformation theory, it was applied to quantization by Bayen, Flato, Fronsdal, Lichnerowicz and Sternheimer.

A paradigm shift: Out with Hilbert space physics; in with cohomological physics.

Most recently, deformation theory appeared in the context of cohomological field theory, but there have been several other appearances in the meantime.

Gerstenhaber's Algebraic Deformation Theory:

An associative k-algebra is a pair (A, m) where A is a k-vector space and $m : A \otimes A \to A$ is a bilinear map satisfying the associativity condition.

A first-order (or *infinitesimal*) deformation of m is given by $m_1 : A \otimes A \to A$ such that $m + t\, m_1 : A[t] \otimes A[t] \to A[t]$ is associative modulo t^2.

The associativity implies that m_1 determines a class $[m_1] = \mathfrak{m}_1$ in what is known as the Hochschild cohomology $HH^2(A)$.

A *formal deformation* of m (over $k[[t]]$) is given by a formal power series

$$m_t = m + \sum t^n \, m_n$$

satisfying associativity.

The full deformation theory of associative algebras can be expressed conveniently in terms of 'higher structure' on the Hochschild cochain complex.

Obstructions

Given a choice of m_1, there may or may not exist a suitable m_2.

Gerstenhaber's eponymous bracket:

$$\{\ ,\ \} : HH^2 \otimes HH^2 \to HH^3$$

behaves like the Schouten-Nihenhuis bracket of multi-vector fields.

The *primary* obstruction to extending $m + t\ m_1$ to order t^2 is given by the Gerstenhaber bracket.

There exists m_2 such that $m + tm_1 + t^2 m_2$ is associative mod t^3 if and only if $\{\mathfrak{m}_1, \mathfrak{m}_1\} = 0$.

If the primary obstruction vanishes, there is a secondary obstruction which lies in a quotient of $HH^3(A)$.

In the case of complex structures, this was first worked out by Douady, who noticed the relation to a Massey product , Work in progress with Vladislav Kupriyanov

For a Poisson manifold, $\{\ ,\ \}$ can be constructed from a bi-vector Θ by the usual local coordinate formula or as a *derived bracket*:

$$\{f, g\} = [[\Theta, f]_{SN}, g]_{SN}.$$

The key requirement is that the Schouten-Nijenhuis bracket $[\Theta, \Theta]_{SN} = 0$.

Kupriyanov introduced the notion of *quasi-Poisson* bracket, given by an arbitrary bi-vector Θ.

In general, $[\Theta, \Theta]_{SN}$ will not vanish.

We seek to deform Θ to Θ_∞ so that $[\Theta_\infty, \Theta_\infty]_{SN} = 0$

Our approach is to follow the BFV procedure:

adjoin auxiliary variables and construct a symplectic embedding of the given quasi-Poisson structure in the extended space.

introduce Hamiltonian constraints and perform homological reduction to achieve the “gauge closure” condition:

the commutator of two field dependent gauge transformations is again such a field dependent gauge transformation .

George (Yuri) Rainich's advice

In 1923, Rainich wrote:

> *As to the method of the study it seemed to me better to avoid, as far as possible, the introduction of things which have no intrinsic meaning, such as coordinates, the g's, the three-indices symbols,...*

Rainich tried again in 1950

> *to introduce the idea of the tensor itself and to consider the components as something secondary .*