Algebraic structures of effective string field theory

Harold Erbin

Università di Torino & Infn (Italy)

Workshop on Fundamental Aspects of String Theory
8th June 2020

In collaboration with: Carlo Maccaferri (Torino), Martin Schnabl, Jakub Vošmera (Prague) [1912.05463 + to appear]

Outline

1. Motivations
2. Perturbative description
3. Coalgebra description
4. Outlook

Effective actions

Goals:

- compute effective actions from string field theory
- study general structure (using A_∞ and L_∞ algebra)
- keep finite-momentum massless physical fields
- compute higher-derivative corrections
- include Ellwood invariant

Effective actions

Goals:

- compute effective actions from string field theory
- study general structure (using A_∞ and L_∞ algebra)
- keep finite-momentum massless physical fields
- compute higher-derivative corrections
- include Ellwood invariant

Two approaches:

- perturbative: intuitive, cumbersome beyond lowest orders
- coalgebra: all-order statements, natural field basis, simple deformations (homological perturbation)

Selected references

- algebraic aspects [hep-th/0107162, Lazaroiu; hep-th/0112228, Kajiura; math/0306332, Kajiura; 1609.00459, Sen; 1610.03251, Erler; 1901.08555, Matsunaga; 2003.05021, Masuda-Matsunaga]
- explicit computations [hep-th/0307019, Berkovits-Schnabl; 1801.07607, Maccaferri-Merlano; 1905.04958, Maccaferri-Merlano; 1912.05463, HE-Maccaferri-Vošmera]
- level-truncation [hep-th/0001201, Taylor; hep-th/0005085, David; hep-th/0306041, Coletti-Sigalov-Taylor; hep-th/0404102, Taylor; 1712.05935, Asada-Kishimoto]
- related topics [1902.10955, Mattiello-Sachs; 1910.00538, Vošmera; 2002.04043, Sen]

See also: Hiroaki's, Jakub's, Yuji's talks

Outline

1. Motivations
2. Perturbative description
3. Coalgebra description
4. Outlook

Classical string field theory

- $\mathcal{H}$ Hilbert space of string states, $\Psi \in \mathcal{H}$ string field
- action and equation of motion (A_∞ and L_∞)

$$S = \frac{1}{2}\langle \Psi, Q\Psi \rangle + \langle \Psi, V(\Psi) \rangle$$

$$\mathcal{E}(\Psi) := Q\Psi + V'(\Psi), \qquad V(\Psi) := \int_0^1 \mathrm{d}t\, V'(t\Psi)$$

- $\langle \cdot, \cdot \rangle$ inner-product on $\mathcal{H}$, $Q := \ell_1$ BRST charge

Classical string field theory

- $\mathcal{H}$ Hilbert space of string states, $\Psi \in \mathcal{H}$ string field
- action and equation of motion (A_∞ and L_∞)

$$S = \frac{1}{2}\langle \Psi, Q\Psi \rangle + \langle \Psi, V(\Psi) \rangle$$

$$\mathcal{E}(\Psi) := Q\Psi + V'(\Psi), \qquad V(\Psi) := \int_0^1 \mathrm{d}t \, V'(t\Psi)$$

- $\langle \cdot, \cdot \rangle$ inner-product on $\mathcal{H}$, $Q := \ell_1$ BRST charge
- L_∞ potential = interactions

$$V(\Psi) = \sum_{n \geq 2} \frac{1}{(n+1)!} \, \ell_n(\Psi^n), \qquad \ell_n : \mathcal{H}^{\otimes n} \to \mathcal{H}$$

- L_∞ relations $\Rightarrow$ gauge invariance

$$\delta_\Lambda \Psi = \sum_{n \geq 1} \frac{1}{n!} \, \ell_{n+1}(\Psi^n, \Lambda), \qquad 0 = \sum_{k+\ell=n} \ell_{k+1}\big(\cdots, \ell_\ell(\cdots)\big)$$

Definitions

- projector such that [1609.00459, Sen]

$$[L_0, P] = [Q, P] = [b_0, P] = 0$$
$$P^\dagger = P, \qquad \ker \widehat{L}_0 \in \operatorname{Im} P$$

- light states $P\mathcal{H}$
- heavy states $\bar{P}\mathcal{H} := (1 - P)\mathcal{H}$

Definitions

- projector such that [1609.00459, Sen]

$$[L_0, P] = [Q, P] = [b_0, P] = 0$$
$$P^\dagger = P, \qquad \ker \widehat{L}_0 \in \operatorname{Im} P$$

  - light states $P\mathcal{H}$
  - heavy states $\bar{P}\mathcal{H} := (1 - P)\mathcal{H}$

- example: projector on massless states (open string)

$$P_0 := \mathrm{e}^{-\infty \widehat{L}_0}, \qquad L_0 = \alpha' k^2 + \widehat{L}_0, \qquad \widehat{L}_0 =: \alpha' m^2$$

Definitions

- projector such that [1609.00459, Sen]

$$[L_0, P] = [Q, P] = [b_0, P] = 0$$
$$P^\dagger = P, \qquad \ker \widehat{L}_0 \in \operatorname{Im} P$$

  - light states $P\mathcal{H}$
  - heavy states $\bar{P}\mathcal{H} := (1 - P)\mathcal{H}$

- example: projector on massless states (open string)

$$P_0 := \mathrm{e}^{-\infty \widehat{L}_0}, \qquad L_0 = \alpha' k^2 + \widehat{L}_0, \qquad \widehat{L}_0 =: \alpha' m^2$$

- procedure $\rightarrow$ effective action of light fields:
  1. Siegel gauge fixing heavy fields
  2. integrate out heavy fields
  3. check out-of-Siegel gauge constraints
  4. integrate out light auxiliary fields

Gauge fixing heavy fields (1)

- gauge fixing needed to invert kinetic term
- Siegel gauge projector

$$\Pi_s := b_0 c_0, \qquad \bar{\Pi}_s := c_0 b_0$$

- field decomposition

$$\Psi = \varphi + R_\downarrow + R_\uparrow$$

$$\varphi := P\Psi, \qquad R_\downarrow := \Pi_s \bar{P}\Psi, \qquad R_\uparrow := \bar{\Pi}_s \bar{P}\Psi$$

Gauge fixing heavy fields (1)

- gauge fixing needed to invert kinetic term
- Siegel gauge projector

$$\Pi_s := b_0 c_0, \qquad \bar{\Pi}_s := c_0 b_0$$

- field decomposition

$$\Psi = \varphi + R_\downarrow + R_\uparrow$$

$$\varphi := P\Psi, \qquad R_\downarrow := \Pi_s \bar{P}\Psi, \qquad R_\uparrow := \bar{\Pi}_s \bar{P}\Psi$$

- BRST charge decomposition

$$Q = c_0 L_0 - b_0 M_+ + \widehat{Q},$$

- equations of motion

$$\begin{aligned}
\mathcal{E}_\varphi(\Psi) := P\mathcal{E}(\Psi) \qquad &= Q\varphi + PV'(\Psi) \\
\mathcal{E}_{R_\downarrow}(\Psi) := \bar{P}\bar{\Pi}_s\mathcal{E}(\Psi) \qquad &= c_0 L_0 R_\downarrow + \widehat{Q}R_\uparrow + \bar{P}\bar{\Pi}_s V'(\Psi) \\
\mathcal{E}_{R_\uparrow}(\Psi) := \bar{P}\Pi_s\mathcal{E}(\Psi) \qquad &= \widehat{Q}R_\downarrow - b_0 M_+ R_\uparrow + \bar{P}\Pi_s V'(\Psi)
\end{aligned}$$

Gauge fixing heavy fields (2)

- Siegel gauge for heavy field

$$b_0(\bar{P}\Psi) = 0 \quad \Longrightarrow \quad R_\uparrow = 0$$

- gauge fixed equations of motion

$$\mathcal{E}_{\mathrm{gf},\varphi}(\Psi_{\mathrm{gf}}) = Q\varphi + PV'(\Psi_{\mathrm{gf}}),$$
$$\mathcal{E}_{\mathrm{gf},R_\downarrow}(\Psi_{\mathrm{gf}}) = c_0 L_0 R_\downarrow + \bar{P}\bar{\Pi}_s V'(\Psi_{\mathrm{gf}})$$

Gauge fixing heavy fields (2)

- Siegel gauge for heavy field

$$b_0(\bar{P}\Psi) = 0 \quad \Longrightarrow \quad R_\uparrow = 0$$

- gauge fixed equations of motion

$$\mathcal{E}_{\mathrm{gf},\varphi}(\Psi_{\mathrm{gf}}) = Q\varphi + PV'(\Psi_{\mathrm{gf}}),$$
$$\mathcal{E}_{\mathrm{gf},R_\downarrow}(\Psi_{\mathrm{gf}}) = c_0 L_0 R_\downarrow + \bar{P}\bar{\Pi}_s V'(\Psi_{\mathrm{gf}})$$

- Siegel gauge propagator (contracting homotopy operator)

$$\Delta := \frac{b_0}{L_0}\,\bar{P}_0, \qquad \{Q, \Delta\} = \bar{P}_0$$

- eom for $R_\downarrow$

$$\mathcal{E}_{\mathrm{gf},R_\downarrow}(\Psi_{\mathrm{gf}}) = 0 \quad \Longrightarrow \quad R_\downarrow = -\frac{b_0}{L_0}\,\bar{P}V'(\varphi + R_\downarrow)$$

L_0 can be singular in $\bar{P}\mathcal{H}$

Gauge fixing heavy fields (2)

- Siegel gauge for heavy field

$$b_0(\bar{P}\Psi) = 0 \quad \Longrightarrow \quad R_\uparrow = 0$$

- gauge fixed equations of motion

$$\mathcal{E}_{\mathrm{gf},\varphi}(\Psi_{\mathrm{gf}}) = Q\varphi + PV'(\Psi_{\mathrm{gf}}),$$
$$\mathcal{E}_{\mathrm{gf},R_\downarrow}(\Psi_{\mathrm{gf}}) = c_0 L_0 R_\downarrow + \bar{P}\bar{\Pi}_s V'(\Psi_{\mathrm{gf}})$$

- Siegel gauge propagator (contracting homotopy operator)

$$\Delta := \frac{b_0}{L_0}\,\bar{P}_0, \qquad \{Q, \Delta\} = \bar{P}_0$$

- eom for $R_\downarrow$

$$\mathcal{E}_{\mathrm{gf},R_\downarrow}(\Psi_{\mathrm{gf}}) = 0 \quad \Longrightarrow \quad R_\downarrow = -\frac{b_0}{L_0}\,\bar{P}V'(\varphi + R_\downarrow)$$

  L_0 can be singular in $\bar{P}\mathcal{H}$
- effective theory $\Rightarrow$ momentum cut-off $\alpha' k^2 \ll \min \widehat{L}_0$ in $\bar{P}\mathcal{H}$

Out-of-Siegel gauge constraints

- out-of-Siegel gauge constraints ($\sim$ Gauss constraints)

$$\mathcal{E}_{\mathrm{gf},R_\uparrow}(\Psi_{\mathrm{eff}}) = \widehat{Q}R_\downarrow + \bar{P}\Pi_s V'(\Psi_{\mathrm{eff}})$$

$\Psi_{\mathrm{eff}} := \varphi + R_\downarrow(\varphi)$

- cannot be derived from effective action

$$S_{\mathrm{eff}} = \frac{1}{2}\langle\varphi, Q\varphi\rangle + \langle\varphi, PV(\Psi_{\mathrm{eff}})\rangle + \left\langle \bar{\Pi}_s V'(\Psi_{\mathrm{eff}}), \frac{b_0}{L_0}\,\bar{P}\left(\frac{V'(\Psi_{\mathrm{eff}})}{2} - V(\Psi_{\mathrm{eff}})\right)\right\rangle$$

$\rightarrow$ must impose $\mathcal{E}_{\mathrm{gf},R_\uparrow} = 0$ on the side

Out-of-Siegel gauge constraints

- out-of-Siegel gauge constraints ($\sim$ Gauss constraints)

$$\mathcal{E}_{\mathrm{gf},R_\uparrow}(\Psi_{\mathrm{eff}}) = \widehat{Q} R_\downarrow + \bar{P}\Pi_s V'(\Psi_{\mathrm{eff}})$$

$\Psi_{\mathrm{eff}} := \varphi + R_\downarrow(\varphi)$

- cannot be derived from effective action

$$S_{\mathrm{eff}} = \frac{1}{2}\langle\varphi, Q\varphi\rangle + \langle\varphi, PV(\Psi_{\mathrm{eff}})\rangle + \left\langle \bar{\Pi}_s V'(\Psi_{\mathrm{eff}}), \frac{b_0}{L_0}\bar{P}\left(\frac{V'(\Psi_{\mathrm{eff}})}{2} - V(\Psi_{\mathrm{eff}})\right)\right\rangle$$

$\rightarrow$ must impose $\mathcal{E}_{\mathrm{gf},R_\uparrow} = 0$ on the side

- result: automatic if light eom holds

$$\mathcal{E}_{\mathrm{gf},\varphi} = 0 \quad \Longrightarrow \quad \mathcal{E}_{\mathrm{gf},R_\uparrow} = 0$$

Perturbative solution

- expand all fields and potential with $\mu \ll 1$

$$\varphi = \sum_{n\geq 1} \mu^n \varphi_n, \qquad R_\downarrow = \sum_{n\geq 1} \mu^n R_n, \qquad V' = \sum_{n\geq 2} \mu^n V'_n$$

- solve order by order in μ and resum

$$\begin{aligned} R_\downarrow &= -\frac{1}{2}\,\frac{b_0}{L_0}\bar{P}\ell_2(\varphi^2) + \frac{1}{2}\,\frac{b_0}{L_0}\bar{P}\ell_2\left(\varphi, \frac{b_0}{L_0}\bar{P}\ell_2(\varphi^2)\right) \\ &\quad - \frac{1}{3!}\,\frac{b_0}{L_0}\bar{P}\ell_3(\varphi^3) + O(\varphi^4) \\ \varphi &= \mu\varphi_1 + \mu^2\varphi_2 + O(\mu^3) \end{aligned}$$

- effective action

$$\begin{aligned} S_{\text{eff}} = \frac{1}{2}\,\langle\varphi, Q\varphi\rangle + \frac{1}{3!}\,\langle\varphi, P\ell_2(\varphi^2)\rangle + \frac{1}{4!}\,\langle\varphi, P\ell_3(\varphi^3)\rangle \\ - \frac{1}{8}\,\left\langle \bar{\Pi}_s\ell_2(\varphi^2), \frac{b_0}{L_0}\bar{P}\ell_2(\varphi^2)\right\rangle + O(\varphi^5) \end{aligned}$$

many interesting cases: compute with localization [Jakub's talk]

Effective gauge invariance

- equation of motion

$$\mathcal{E}_{\mathrm{gf},\varphi} = Q\varphi + \sum_{n\geq 2} \frac{1}{n!}\, \tilde{\ell}_n(\varphi^n)$$

- effective L_∞ structure

$$\tilde{\ell}_1(A_1) = PQA_1, \qquad \tilde{\ell}_2(A_1, A_2) = P\ell_2(A_1, A_2),$$

$$\begin{aligned}\tilde{\ell}_3(A_1, A_2, A_3) = P\ell_3(A_1, A_2, A_3) &- P\ell_2\left(A_1, \frac{b_0}{L_0}\bar{P}\ell_2(A_2, A_3)\right)\\ &- (-1)^{A_1(A_2+A_3)}\, P\ell_2\left(A_2, \frac{b_0}{L_0}\bar{P}\ell_2(A_3, A_1)\right)\\ &- (-1)^{A_3(A_1+A_2)}\, P\ell_2\left(A_3, \frac{b_0}{L_0}\bar{P}\ell_2(A_1, A_2)\right)\end{aligned}$$

- effective (non-canonical) gauge invariance

$$\delta_\lambda\varphi = Q\lambda + \tilde{\ell}_2(\varphi, \lambda) + \frac{1}{2}\,\tilde{\ell}_3(\varphi^2, \lambda) + O(\varphi^3), \qquad \bar{P}\lambda = 0$$

Integrating out auxiliary massless fields (1)

- consider massless gauge field (open bosonic string)
- $\widehat{L}_0 = 0$ field at $N_{\text{gh}} = 1$

$$\varphi_A := \frac{\sqrt{2}}{\alpha'} A_\mu(k)\, c\, \mathrm{i}\partial X^\mu \mathrm{e}^{\mathrm{i}k\cdot X}, \qquad \varphi_B := \frac{B(k)}{\sqrt{2}}\, \partial c\, \mathrm{e}^{\mathrm{i}k\cdot X}$$

- $A_\mu(k)$ gauge field: φ_A primary if $k \cdot A = 0$, on-shell for $k^2 = 0$
- $B(k)$ Nakanishi–Lautrup (NL) auxiliary field, φ_B not primary
  note: massless ghost zero-mode from [2002.04043, Sen]

Integrating out auxiliary massless fields (1)

- consider massless gauge field (open bosonic string)
- $\widehat{L}_0 = 0$ field at $N_{\text{gh}} = 1$

$$\varphi_A := \frac{\sqrt{2}}{\alpha'} A_\mu(k)\, c\, \mathrm{i}\partial X^\mu \mathrm{e}^{\mathrm{i}k\cdot X}, \qquad \varphi_B := \frac{B(k)}{\sqrt{2}}\, \partial c\, \mathrm{e}^{\mathrm{i}k\cdot X}$$

- $A_\mu(k)$ gauge field: φ_A primary if $k \cdot A = 0$, on-shell for $k^2 = 0$
- $B(k)$ Nakanishi–Lautrup (NL) auxiliary field, φ_B not primary
  note: massless ghost zero-mode from [2002.04043, Sen]
- Siegel gauge condition + constraint:

$$\begin{cases} b_0 \varphi_B = 0 \\ \widehat{Q}\varphi_A = 0 \end{cases} \implies \begin{cases} B(k) = 0 \\ k \cdot A(k) = 0 \end{cases}$$

Integrating out auxiliary massless fields (1)

- consider massless gauge field (open bosonic string)
- $\widehat{L}_0 = 0$ field at $N_{\text{gh}} = 1$

$$\varphi_A := \frac{\sqrt{2}}{\alpha'} A_\mu(k)\, c\, \mathrm{i}\partial X^\mu \mathrm{e}^{\mathrm{i}k\cdot X}, \qquad \varphi_B := \frac{B(k)}{\sqrt{2}}\, \partial c\, \mathrm{e}^{\mathrm{i}k\cdot X}$$

- $A_\mu(k)$ gauge field: φ_A primary if $k \cdot A = 0$, on-shell for $k^2 = 0$
- $B(k)$ Nakanishi–Lautrup (NL) auxiliary field, φ_B not primary
  note: massless ghost zero-mode from [2002.04043, Sen]
- Siegel gauge condition + constraint:

$$\begin{cases} b_0 \varphi_B = 0 \\ \widehat{Q}\varphi_A = 0 \end{cases} \implies \begin{cases} B(k) = 0 \\ k \cdot A(k) = 0 \end{cases}$$

- keep gauge invariance → integrate out NL field

Integrating out auxiliary massless fields (2)

Integrate out $B(k)$ field:

- solve equation

$$\Pi_s \mathcal{E}_{\mathrm{gh},\varphi} = 0 \quad \Longrightarrow \quad \varphi_B = c_0 M_- \Big(\widehat{Q}\varphi_A - PV'(\varphi_A + \varphi_B) \Big)$$

- use $\mathrm{SU}(1,1)$ algebra

$$[M_+, M_-] = \widehat{N}_{\mathrm{gh}}, \qquad [\widehat{N}_{\mathrm{gh}}, M_\pm] = \pm 2 M_\pm$$

$\widehat{N}_{\mathrm{gh}}$ ghost number without zero-mode

Integrating out auxiliary massless fields (2)

Integrate out $B(k)$ field:

- solve equation

$$\Pi_s \mathcal{E}_{\mathrm{gh},\varphi} = 0 \quad \Longrightarrow \quad \varphi_B = c_0 M_- \Big(\widehat{Q}\varphi_A - PV'(\varphi_A + \varphi_B) \Big)$$

- use $\mathrm{SU}(1,1)$ algebra

$$[M_+, M_-] = \widehat{N}_{\mathrm{gh}}, \qquad [\widehat{N}_{\mathrm{gh}}, M_\pm] = \pm 2 M_\pm$$

$\widehat{N}_{\mathrm{gh}}$ ghost number without zero-mode

- $\varphi_B = O(\varphi_A)$

- free action before integrating out

$$S = \frac{1}{2} \int \frac{\mathrm{d}^D k}{(2\pi)^D} \Big[A_\mu(k) k^2 A_\mu(-k) - B(k)B(-k) + 2k \cdot A(k)B(-k) \Big]$$

Integrating out auxiliary massless fields (3)

Better approach:

- field redefinition to make state with A_μ primary

$$\tilde{\varphi}_A := \frac{A_\mu(k)}{\sqrt{2}} \left(\frac{2}{\alpha'}\, c\, \mathrm{i}\partial X^\mu + k^\mu\, \partial c \right) \mathrm{e}^{\mathrm{i}k\cdot X}$$

$$\varphi_\beta := \frac{\beta(k)}{\sqrt{2}}\, \partial c\, \mathrm{e}^{\mathrm{i}k\cdot X}, \qquad \beta(k) := B(k) - k\cdot A(k)$$

Integrating out auxiliary massless fields (3)

Better approach:

- field redefinition to make state with A_μ primary

$$\tilde{\varphi}_A := \frac{A_\mu(k)}{\sqrt{2}} \left(\frac{2}{\alpha'} \, c \, \mathrm{i}\partial X^\mu + k^\mu \, \partial c \right) \mathrm{e}^{\mathrm{i}k\cdot X}$$

$$\varphi_\beta := \frac{\beta(k)}{\sqrt{2}} \, \partial c \, \mathrm{e}^{\mathrm{i}k\cdot X}, \qquad \beta(k) := B(k) - k \cdot A(k)$$

- can be implemented with modified projector Π

$$\tilde{\varphi}_A := \Pi(\varphi_A + \varphi_B), \qquad \varphi_\beta := \bar{\Pi}(\varphi_A + \varphi_B)$$

- free action

$$S = \frac{1}{2} \int \frac{\mathrm{d}^D k}{(2\pi)^D} \Big[A_\mu(k)(k^2 \, \eta^{\mu\nu} - k^\mu k^\nu) A_\nu(-k) - \beta(k)\beta(-k) \Big]$$

- $\varphi_\beta = O(\tilde{\varphi}_A^2)$

Outline

Motivations

Advantages of coalgebra description:

- no need for explicit field decomposition
- optimal projector clearly characterized
- package perturbative expansion and effective interactions
- read directly effective L_∞ structure
- perform both projections at the same time ("horizontal composition")
- deformation (e.g. Ellwood invariant) combined directly with projection ("vertical composition")

Applications (A_∞ SFT, but works also for L_∞):

- effective action for gauge and NL fields
- effective action for gauge field only
- effective action with Ellwood invariant

Coalgebra description (1)

- tensor product Hilbert space

$$T\mathcal{H} := \mathbb{C} \oplus \mathcal{H} \oplus \mathcal{H}^{\otimes 2} + \cdots \,, \qquad \pi_k : T\mathcal{H} \to \mathcal{H}^{\otimes k}$$

- coderivation → embed A_∞ products $m_n : \mathcal{H}^{\otimes n} \to \mathcal{H}$

$$\boldsymbol{m}_n : T\mathcal{H} \to T\mathcal{H}, \qquad \boldsymbol{m} := \sum_{n \geq 1} \boldsymbol{m}_n$$

$$\boldsymbol{m}_n \pi_N = \sum_{k=0}^{N-n} 1_{\mathcal{H}}^{\otimes(N-n-k)} \otimes m_n \otimes 1_{\mathcal{H}}^{\otimes k}$$

- A_∞ relation

$$[\boldsymbol{m}, \boldsymbol{m}] = 0$$

- group-like element

$$\frac{1}{1-A} = 1_{T\mathcal{H}} + A + A^{\otimes 2} + \cdots$$

- symplectic form

$$\omega : \mathcal{H}^{\otimes 2} \to \mathbb{C}$$

Coalgebra description

- action

$$S = \int_0^1 \mathrm{d}t\, \omega \left(\pi_1 \boldsymbol{\partial}_t \, \frac{1}{1 - \Psi(t)} \otimes \pi_1 \boldsymbol{m} \, \frac{1}{1 - \Psi(t)} \right)$$

$\Psi(1) = \Psi, \quad \Psi(0) = 0, \quad \pi_1 \boldsymbol{\partial}_t \pi_1 = \partial_t$

- equation of motion

$$\pi_1 \boldsymbol{m} \, \frac{1}{1 - \Psi(t)} = 0$$

- gauge transformation

$$\delta_\Lambda \, \frac{1}{1 - \Psi} = [\boldsymbol{m}, \boldsymbol{\Lambda}] \, \frac{1}{1 - \Psi}$$

$\pi_1 \boldsymbol{\Lambda} \pi_0 = \Lambda \in \mathcal{H}$

Homological perturbation lemma (1)

- encode free SFT as strong deformation retract:
  - vector space $T\mathcal{H}$
  - differential = BRST charge $\boldsymbol{Q} = \boldsymbol{m}_1$
  - contracting operator = free propagator $\boldsymbol{\Delta}$
  - projector $\boldsymbol{P}$
- describe interactions as perturbation
  - perturbation $\delta\boldsymbol{m}$ = interactions $\boldsymbol{m}_n$ for $n \geq 2$
  - full differential $\boldsymbol{m} = \boldsymbol{m}_1 + \delta\boldsymbol{m}$
  - full contracting operator $\boldsymbol{\eta}$
  - full projector $\boldsymbol{\Pi}$

Homological perturbation lemma (1)

- encode free SFT as strong deformation retract:
  - vector space $T\mathcal{H}$
  - differential = BRST charge $\boldsymbol{Q} = \boldsymbol{m}_1$
  - contracting operator = free propagator $\boldsymbol{\Delta}$
  - projector $\boldsymbol{P}$
- describe interactions as perturbation
  - perturbation $\delta\boldsymbol{m}$ = interactions $\boldsymbol{m}_n$ for $n \geq 2$
  - full differential $\boldsymbol{m} = \boldsymbol{m}_1 + \delta\boldsymbol{m}$
  - full contracting operator $\boldsymbol{\eta}$
  - full projector $\boldsymbol{\Pi}$
- conditions on $\boldsymbol{Q}$ and $\boldsymbol{m}$

$$[\boldsymbol{P}, \boldsymbol{Q}] = 0, \qquad \boldsymbol{Q}^2 = 0, \qquad [\boldsymbol{\Pi}, \boldsymbol{m}] = 0, \qquad \boldsymbol{m}^2 = 0$$

- gauge-fixing and Hodge–Kodaira decomposition

$$\boldsymbol{\Delta}\Psi = 0, \qquad [\boldsymbol{Q}, \boldsymbol{\Delta}] = 1 - \boldsymbol{P}, \qquad \boldsymbol{P}\boldsymbol{\Delta} = \boldsymbol{\Delta}\boldsymbol{P} = \boldsymbol{\Delta}^2 = 0$$

$$\boldsymbol{\eta}\Psi = 0, \qquad [\boldsymbol{m}, \boldsymbol{\eta}] = 1 - \boldsymbol{\Pi}, \qquad \boldsymbol{\Pi}\boldsymbol{\eta} = \boldsymbol{\eta}\boldsymbol{\Pi} = \boldsymbol{\eta}^2 = 0$$

Homological perturbation lemma (2)

- theory diagram: projector $\boldsymbol{P}$ and perturbation $\delta\boldsymbol{m}$

$$
\begin{array}{ccc}
\boldsymbol{\Delta} \circlearrowright (T\mathcal{H}, \boldsymbol{Q}) & \xrightarrow{\boldsymbol{P}} & (TP\mathcal{H}, \boldsymbol{Q}) \\
\Big\downarrow \delta\boldsymbol{m} & & \Big\downarrow \delta\tilde{\boldsymbol{m}} \\
\boldsymbol{\eta} \circlearrowright (T\mathcal{H}, \boldsymbol{m}) & \xrightarrow{\boldsymbol{\Pi}} & (T\Pi\mathcal{H}, \tilde{\boldsymbol{m}})
\end{array}
$$

- homological perturbation lemma

$$
\delta\tilde{\boldsymbol{m}} = \boldsymbol{P}\delta\boldsymbol{m}\,\frac{1}{1+\boldsymbol{\Delta}\delta\boldsymbol{m}}, \qquad \boldsymbol{\eta} = \boldsymbol{\Delta} - \boldsymbol{\Delta}\delta\tilde{\boldsymbol{m}}\boldsymbol{\Delta},
$$

$$
\boldsymbol{\Pi} = \frac{1}{1+\boldsymbol{\Delta}\delta\boldsymbol{m}}\,\boldsymbol{P}\,\frac{1}{1+\delta\boldsymbol{m}\boldsymbol{\Delta}}
$$

Application: integrate out heavy fields

- $\boldsymbol{P}$ integrates out heavy fields, $\delta \boldsymbol{m}$ adds interactions
- theory diagram

$$\begin{array}{ccc} \text{free theory} & \xrightarrow{\boldsymbol{P}} & \text{free effective theory} \\ \Big\downarrow{\scriptstyle \delta \boldsymbol{m}} & & \Big\downarrow{\scriptstyle \delta \tilde{\boldsymbol{m}}} \\ \text{interacting theory} & \xrightarrow{\boldsymbol{\Pi}} & \text{interacting effective theory} \end{array}$$

- read effective interactions by expanding

$$\tilde{m}_2(\varphi^2) = Pm_2(\varphi^2),$$
$$\tilde{m}_3(\varphi^3) = Pm_3(\varphi^3) - 2\, Pm_2(\Delta m_2(\varphi^2), \varphi)$$

Horizontal composition

- theory diagram: two successive projections $\boldsymbol{P}_1$ and $\boldsymbol{P}_2$

$$\begin{array}{ccccc}
\boldsymbol{\Delta}_1 \circlearrowright (T\mathcal{H}, \boldsymbol{Q}) & \xrightarrow{\boldsymbol{P}_1} & \boldsymbol{\Delta}_2 \circlearrowright (TP_1\mathcal{H}, \boldsymbol{Q}) & \xrightarrow{\boldsymbol{P}_2} & (TP_2P_1\mathcal{H}, \boldsymbol{Q}) \\
\downarrow{\scriptstyle \delta \boldsymbol{m}} & & \downarrow{\scriptstyle \delta \tilde{\boldsymbol{m}}} & & \downarrow{\scriptstyle \delta \boldsymbol{m}'} \\
\boldsymbol{\eta}_1 \circlearrowright (T\mathcal{H}, \boldsymbol{m}) & \xrightarrow{\boldsymbol{\Pi}_1} & \boldsymbol{\eta}_2 \circlearrowright (T\Pi_1\mathcal{H}, \tilde{\boldsymbol{m}}) & \xrightarrow{\boldsymbol{\Pi}_2} & (T\Pi_2\Pi_1\mathcal{H}, \boldsymbol{m}')
\end{array}$$

Horizontal composition

- theory diagram: two successive projections $\boldsymbol{P}_1$ and $\boldsymbol{P}_2$

$$
\begin{array}{ccccc}
\boldsymbol{\Delta}_1 \circlearrowright (T\mathcal{H},\boldsymbol{Q}) & \xrightarrow{\boldsymbol{P}_1} & \boldsymbol{\Delta}_2 \circlearrowright (TP_1\mathcal{H},\boldsymbol{Q}) & \xrightarrow{\boldsymbol{P}_2} & (TP_2P_1\mathcal{H},\boldsymbol{Q}) \\
\Big\downarrow \delta\boldsymbol{m} & & \Big\downarrow \delta\tilde{\boldsymbol{m}} & & \Big\downarrow \delta\boldsymbol{m}' \\
\boldsymbol{\eta}_1 \circlearrowright (T\mathcal{H},\boldsymbol{m}) & \xrightarrow{\boldsymbol{\Pi}_1} & \boldsymbol{\eta}_2 \circlearrowright (T\Pi_1\mathcal{H},\tilde{\boldsymbol{m}}) & \xrightarrow{\boldsymbol{\Pi}_2} & (T\Pi_2\Pi_1\mathcal{H},\boldsymbol{m}')
\end{array}
$$

- result: equivalent to

$$
\begin{array}{ccc}
\boldsymbol{\Delta}_{12} \circlearrowright (T\mathcal{H},\boldsymbol{Q}) & \xrightarrow{P_{12}} & (TP_{12}\mathcal{H},\boldsymbol{Q}) \\
\Big\downarrow \delta\boldsymbol{m} & & \Big\downarrow \delta\tilde{\boldsymbol{m}} \\
\boldsymbol{\eta}_{12} \circlearrowright (T\mathcal{H},\boldsymbol{m}) & \xrightarrow{\Pi_{12}} & (T\Pi_{12}\mathcal{H},\tilde{\boldsymbol{m}})
\end{array}
$$

$$
\begin{aligned}
P_{12} &= P_2P_1 \\
\Pi_{12} &= \Pi_2\Pi_1 \\
\Delta_{12} &= \Delta_1 + \Delta_2 P_1 \\
\eta_{12} &= \eta_1 + \eta_2 \Pi_1
\end{aligned}
$$

Application: integrate out NL field

- $P_1 = P_0$ (resp. $P_2 = \Pi$) integrates out heavy (resp. NL) fields
- Hodge–Kodaira decomposition for $P_0 Q$ fixes Π, Δ_2

$$\Delta_2 = c_0 M^- P_0 = \frac{1}{2}\, c_0 b_{-1} b_1 P_0$$

$$\Pi = \Pi_s - c_0 W, \qquad W := [M^-, \widehat{Q}]$$

Π corrected w.r.t. Siegel projector Π_s

Application: integrate out NL field

- $P_1 = P_0$ (resp. $P_2 = \Pi$) integrates out heavy (resp. NL) fields
- Hodge–Kodaira decomposition for $P_0 Q$ fixes Π, Δ_2

$$\Delta_2 = c_0 M^- P_0 = \frac{1}{2} c_0 b_{-1} b_1 P_0$$

$$\Pi = \Pi_s - c_0 W, \qquad W := [M^-, \widehat{Q}]$$

  Π corrected w.r.t. Siegel projector Π_s
- horizontal composition → vertices for physical field $\tilde{\varphi}_A$

$$\Delta_{\text{eff}} = \frac{b_0}{L_0} \bar{P}_0 + c_0 M^- P$$

$$m_1'(A_2) = \Pi P_0\, Q A_1, \qquad m_2'(A_1, A_2) = \Pi P_0\, m_2(A_1, A_2)$$

$$m_3'(A_1, A_2, A_3) = \Pi P_0\, m_2\Big(\Delta_{\text{eff}} m_2(A_1, A_2), A_3\Big) + \cdots$$

  additional algebraic propagator from NL field [2002.04043, Sen]
- $k = 0 \Rightarrow m_1' = 0$ ($\Pi P \to$ cohomology) → minimal model

Vertical composition

- theory diagram: two successive deformations $\delta \boldsymbol{m}_1$ and $\delta \boldsymbol{m}_2$

$$
\begin{array}{ccc}
\boldsymbol{\Delta} \circlearrowright (T\mathcal{H}, \boldsymbol{Q}) & \xrightarrow{\boldsymbol{P}} & (TP\mathcal{H}, \boldsymbol{Q}) \\
\Big\downarrow{\scriptstyle \delta \boldsymbol{m}_1} & & \Big\downarrow{\scriptstyle \delta \tilde{\boldsymbol{m}}_1} \\
\boldsymbol{\eta}_1 \circlearrowright (T\mathcal{H}, \boldsymbol{m}) & \xrightarrow{\boldsymbol{\Pi}_1} & (T\Pi_1\mathcal{H}, \tilde{\boldsymbol{m}}) \\
\Big\downarrow{\scriptstyle \delta \boldsymbol{m}_2} & & \Big\downarrow{\scriptstyle \delta \tilde{\boldsymbol{m}}_2} \\
\boldsymbol{\eta}_2 \circlearrowright (T\mathcal{H}, \boldsymbol{M}) & \xrightarrow{\boldsymbol{\Pi}_2} & (T\Pi_2\mathcal{H}, \tilde{\boldsymbol{M}})
\end{array}
$$

Vertical composition

- theory diagram: two successive deformations $\delta \boldsymbol{m}_1$ and $\delta \boldsymbol{m}_2$

$$
\begin{array}{ccc}
\boldsymbol{\Delta} \circlearrowright (T\mathcal{H}, \boldsymbol{Q}) & \xrightarrow{\boldsymbol{P}} & (TP\mathcal{H}, \boldsymbol{Q}) \\
\downarrow{\scriptstyle \delta \boldsymbol{m}_1} & & \downarrow{\scriptstyle \delta \tilde{\boldsymbol{m}}_1} \\
\boldsymbol{\eta}_1 \circlearrowright (T\mathcal{H}, \boldsymbol{m}) & \xrightarrow{\boldsymbol{\Pi}_1} & (T\Pi_1\mathcal{H}, \tilde{\boldsymbol{m}}) \\
\downarrow{\scriptstyle \delta \boldsymbol{m}_2} & & \downarrow{\scriptstyle \delta \tilde{\boldsymbol{m}}_2} \\
\boldsymbol{\eta}_2 \circlearrowright (T\mathcal{H}, \boldsymbol{M}) & \xrightarrow{\boldsymbol{\Pi}_2} & (T\Pi_2\mathcal{H}, \tilde{\boldsymbol{M}})
\end{array}
$$

- result: equivalent to

$$
\begin{array}{ccc}
\boldsymbol{\Delta} \circlearrowright (T\mathcal{H}, \boldsymbol{Q}) & \xrightarrow{\boldsymbol{P}} & (TP\mathcal{H}, \boldsymbol{Q}) \\
\downarrow{\scriptstyle \delta \boldsymbol{m}_{12}} & & \downarrow{\scriptstyle \delta \tilde{\boldsymbol{m}}_{12}} \\
\boldsymbol{\eta}_2 \circlearrowright (T\mathcal{H}, \boldsymbol{M}) & \xrightarrow{\boldsymbol{\Pi}_2} & (T\Pi_2\mathcal{H}, \tilde{\boldsymbol{M}})
\end{array}
$$

$$\delta \boldsymbol{m}_{12} = \delta \boldsymbol{m}_1 + \delta \boldsymbol{m}_2$$
$$\delta \tilde{\boldsymbol{m}}_{12} = \delta \tilde{\boldsymbol{m}}_1 + \delta \tilde{\boldsymbol{m}}_2$$

Application: Ellwood invariant (1)

- Ellwood invariant: open string 1-point function [Yuji's talk]

$$E[\Psi] := \langle \mathcal{V}(i) f \circ \Psi \rangle_{\text{UHP}} := \langle \Psi, e_0 \rangle$$

$\mathcal{V}$ on-shell closed string state at mid-point $\to$ 0-product e_0

- result: effective invariant for massless $k = 0$ fields

$$\tilde{E}[\varphi] = \sum_{n \geq 0} \frac{1}{n+1} \omega(\varphi, \tilde{e}_n(\varphi^n)) = E\big[\Psi_{\text{eff}}(\varphi)\big]$$

$$\tilde{e}_n(\varphi^n) = -\tilde{m}_{n+1}(-\Delta e_0, \varphi^n) + \text{perms}$$

- off-shell gauge invariant $\to$ deform the action

$$S_{\text{ell}}[\Psi] = S[\Psi] + \lambda\, E[\Psi], \qquad S_{\text{eff,ell}}[\varphi] = S_{\text{eff}}[\varphi] + \lambda\, \tilde{E}[\varphi] + O(\lambda^2)$$

- 0-product = tadpole $\to$ vacuum shift [1404.6254, Pius-Rudra-Sen]
- obstruction to vacuum shift of full SFT = massless eom of effective SFT (simple cases: reduce to tadpole in $S_{\text{eff,ell}}$)

Application: Ellwood invariant (2)

- vertical composition$\to$ effective action [Jakub's, Yuji's talks]
  - $\delta \boldsymbol{m}_1 = \boldsymbol{m}_2 + \cdots$
  - $\delta \boldsymbol{m}_2 = \lambda \boldsymbol{e}$
- result: effective products

$$\tilde{\boldsymbol{M}} = \boldsymbol{P}(\boldsymbol{m} + \lambda \boldsymbol{e}) \frac{1}{1 + \boldsymbol{\Delta}(\boldsymbol{m} + \lambda \boldsymbol{e} - \boldsymbol{Q})} = \tilde{\boldsymbol{m}} + \lambda \, \boldsymbol{\Pi}_1 \boldsymbol{e} \, \frac{1}{1 + \lambda \boldsymbol{\eta}_1 \boldsymbol{e}}$$

$\to$ implements automatically vacuum shift of products

note: $\tilde{\boldsymbol{m}}$, $\boldsymbol{\Pi}_1$, $\boldsymbol{\eta}_1$ effective theory of light fields (with NL field)

- in components

$$\tilde{M}_n(A_1, \ldots, A_n) = \sum_{k \geq 0} \tilde{m}_{n+k}\Big((-\Delta e_0)^k, A_1, \ldots, A_n\Big) + \text{perms}$$

Outline

Conclusion and outlook

Achievements:

- efficient way to combine multiple projections and perturbations
- understand better the role of the NL field
- effective action with Ellwood's invariant

Conclusion and outlook

Achievements:

- efficient way to combine multiple projections and perturbations
- understand better the role of the NL field
- effective action with Ellwood's invariant

Outlook:

- compute quartic effective interaction with full α' corrections (Witten's open SFT)
- generalize to open-closed SFT
- compute ghost-dilaton contributions