Divergent to Complex Amplitudes in Two Dimensional String Theory

Ashoke Sen

Harish-Chandra Research Institute, Allahabad, India

Sao Paolo, June 2020

Plan:

1. Introduction to the problem

2. Solution

A.S.: arXiv:2003.12076 (arXiv:1607.06500)

The problem

In arXiv:1907.07688, Balthazar, Rodriguez and Yin (BRY) computed the D-instanton induced scattering amplitude in two dimensional string theory

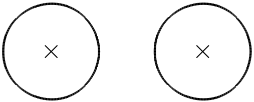
– time + c=25 Liouville theory as the world-sheet matter theory

In this theory closed string ‘tachyons’ are massless particles

They analyzed the scattering amplitude of closed string tachyons induced by a single D-instanton

– and compared this with the result obtained from a dual matrix model description

For 1→1 scattering the leading contribution comes from the product of two disk one point functions.



Result:

$$8\,\pi\,N\,e^{-1/g_s}\,\delta(\omega_1+\omega_2)\,\sinh(\pi|\omega_1|)\,\sinh(\pi|\omega_2|)$$

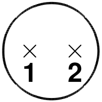
N: An overall normalization constant

g_s: string coupling constant

$-\omega_1, \omega_2$: energies of incoming / outgoing ‘tachyons’

At the next order, there are three contributions.

1. Two point function on the disk.

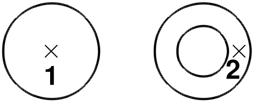


Result:

$$8\,\pi\,N\,e^{-1/g_s}\,g_s\,\delta(\omega_1+\omega_2)\,\sinh(\pi|\omega_1|)\,\sinh(\pi|\omega_2|)\,f(\omega_1,\omega_2)$$

$f(\omega_1, \omega_2)$ is a known function that will be described later.

2. Product of disk one point function and annulus one point function.



$+ (1 \Leftrightarrow 2)$

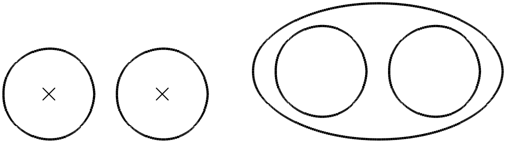
Result:

$$8\,\pi\,N\,e^{-1/g_s}\,g_s\,\delta(\omega_1+\omega_2)\,\sinh(\pi|\omega_1|)\,\sinh(\pi|\omega_2|)\,\{g(\omega_1)+g(\omega_2)\}$$

$g(\omega)$ is a known function that will be described below.

3. Product of two disk one point functions and the zero point function on a surface of Euler number -1

– disk with two holes or torus with one hole.



Result:

$$8\,\pi\, N\, e^{-1/g_s}\, g_s\, \delta(\omega_1+\omega_2)\, \sinh(\pi|\omega_1|)\, \sinh(\pi|\omega_2|)\, C$$

C: a real constant that can in principle be computed.

BRY compared the string theory result with the matrix model results for imaginary ω_1, ω_2 and found remarkable agreement.

Divergences in the open string channel forced them to express the result in terms of two unknown constants which had to be adjusted to fit the data.

These (and more) constants can be fixed from first principles using insights from string field theory

– discussed earlier and will not be discussed any further in this talk.

The same f,g,C can also be used to compute $1 \to n$ amplitude.

String theory result to first subleading order:

$$2^{n+2}\,\pi\,N\,e^{-1/g_s}\,g_s\,\delta(\omega_1+\cdots+\omega_{n+1})\prod_{i=1}^{n+1}\sinh(\pi|\omega_i|)$$

$$\left[1+g_s\sum_{j<k}f(\omega_j,\omega_k)+g_s\sum_j g(\omega_j)+C\,g_s\right]$$

Matrix model result for the same amplitude:

$$2^{n+2}\,\pi\,N\,e^{-1/g_s}\,g_s\,\delta(\omega_1+\cdots+\omega_{n+1})\prod_{i=1}^{n+1}\sinh(\pi|\omega_i|)$$

$$\left[1-i\,g_s\sum_{j=1}^{n}\omega_j\left(1-\sum_{i=1}^{n}\pi\omega_i\coth(\pi\omega_i)\right)\right]$$

with $N=-1/(8\pi^2)$

Equality of string theory and matrix model results require:

$$\sum_{j<k=1}^{n+1} f(\omega_j,\omega_k) + \sum_{j=1}^{n+1} g(\omega_j) + C = -i\sum_{j=1}^{n}\omega_j \left(1 - \sum_{i=1}^{n}\pi\omega_i \coth(\pi\omega_i)\right)$$

for $\omega_1, \cdots, \omega_n > 0, \quad \omega_{n+1} = -\omega_1 - \cdots - \omega_n$

String theory results for $f(\omega_1,\omega_2)$ and $g(\omega)$:

$$\begin{aligned} f(\omega_1,\omega_2) &= 2^{-1/4}\pi^{1/2} 2^{(\omega_1^2+\omega_2^2)/2} \frac{1}{\sinh(\pi|\omega_1|)\sinh(\pi|\omega_2|)} \int_0^1 dy \\ &\quad y^{\omega_2^2/2}(1-y)^{1-\omega_1\omega_2}(1+y)^{1+\omega_1\omega_2} \langle V_{|\omega_1|/2}(i) V_{|\omega_2|/2}(i\,y)\rangle_{UHP}\,, \end{aligned}$$

$$g(\omega) = 2\,\pi^2 \frac{1}{\sinh(\pi|\omega|)} \int_0^\infty dt \int_0^{1/4} dx\, \eta(it) \left(\frac{2\pi}{\theta_1'(0|it)}\,\theta_1(2\,x|i\,t)\right)^{\omega^2/2} \langle V_{|\omega|/2}(2\pi x)\rangle_A\,.$$

$\theta_1(z|\tau)$ is the odd Jacobi theta function and $\theta_1'(z|\tau) \equiv \partial_z\theta_1(z|\tau)$.

$\langle V_{|\omega_1|/2}(i) V_{|\omega_2|/2}(i\,y)\rangle_D$: two point function on the upper half plane of a pair of primaries in the c=25 Liouville theory, carrying momenta $|\omega_1|/2$ and $|\omega_2|/2$, inserted at i and iy.

$\langle V_{|\omega|/2}(2\pi x)\rangle_A$: one point function of the Liouville primary of momentum $|\omega|/2$ on an annulus described by $0 \le \mathrm{Re}(w) \le \pi$, $w \equiv w + 2\pi\, i\, t$, with the vertex operator inserted at $\mathrm{Re}(w) = 2\pi x$.

$$\sum_{j<k=1}^{n+1} f(\omega_j,\omega_k) + \sum_{j=1}^{n+1} g(\omega_j) + C = -i\sum_{j=1}^{n}\omega_j \left(1 - \sum_{i=1}^{n}\pi\omega_i \coth(\pi\omega_i)\right)$$

The integrands of $f(\omega_1,\omega_2)$ and $g(\omega)$ are manifestly real.

How can the above relation hold?

This is a generic problem in the world-sheet description of string amplitudes.

World-sheet description always gives formally real amplitudes.

Whenever an amplitude is supposed to develop imaginary part, the world-sheet expression diverges.

Sundborg; Amano, Tsuchiya; D'Hoker, Phong; Berera; Witten; $\cdots$

f and g also diverge from $y \to 1$ and $t \to 0$ regions respectively

– divergences in the closed string channel.

The solution

Strategy:

1. Understand the origin of the divergences using (closed) string field theory.

2. Rectify them.

The world-sheet divergences always arise from the Schwinger parameter representation of the propagator:

$$(k^2+m^2)^{-1} = \int_0^\infty e^{-s(k^2+m^2)} ds = \int_0^1 dq\, q^{k^2+m^2-1}, \quad q \equiv e^{-s}$$

q's become world-sheet variables like y etc. after appropriate redefinition and the integral diverges for $k^2+m^2 \le 0$.

Therefore we use the replacement rule:

$$\int_0^1 dq\, q^{\alpha-1} \to (\alpha - i\epsilon)^{-1}$$

Note: The $-i\epsilon$ is included since the propagator is $(k^2+m^2-i\epsilon)^{-1}$

$$\int_0^1 \mathbf{dq}\,\mathbf{q}^{\alpha-1} \to (\alpha - \mathbf{i}\epsilon)^{-1}$$

For generic α, this rule is not sensitive to parameter redefinition.

If we define new variable

$$\widetilde{\mathbf{q}} = \mathbf{h}(\mathbf{q}) = \mathbf{a}\,\mathbf{q} + \mathbf{b}\,\mathbf{q}^2 + \cdots \quad \Leftrightarrow \quad \mathbf{q} = \widetilde{\mathbf{h}}(\widetilde{\mathbf{q}}),$$

express the integral as

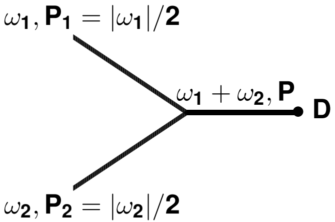
$$\int_0^{\mathbf{h}(1)} \mathbf{d}\widetilde{\mathbf{q}}\,\widetilde{\mathbf{h}}'(\widetilde{\mathbf{q}})\,(\widetilde{\mathbf{h}}(\widetilde{\mathbf{q}}))^{\alpha-1},$$

and then apply the replacement rule on $\widetilde{\mathbf{q}}$ integral after expanding the integrand in powers of $\widetilde{\mathbf{q}}$, we get the same result.

Divergent part of f(ω_1, ω_2) (from closed string channel):

$$\frac{1}{\sinh(\pi|\omega_1|)\sinh(\pi|\omega_2|)} \int_0^\infty dP \int_0^1 dy\,(1-y)^{-1+2P^2-(\omega_1+\omega_2)^2/2}\, 2^{-2P^2+(\omega_1+\omega_2)^2/2}\, C(|\omega_1|/2, |\omega_2|/2, P)\,\sinh(2\pi P)\,,$$

$C(P_1, P_2, P_3)$: the three point functions of Liouville primaries, carrying momenta P_1, P_2 and P_3.



Divergent part of $f(\omega_1, \omega_2)$ (from closed string channel):

$$\frac{1}{\sinh(\pi|\omega_1|)\sinh(\pi|\omega_2|)} \int_0^\infty dP \int_0^1 dy\,(1-y)^{-1+2P^2-(\omega_1+\omega_2)^2/2}\, 2^{-2P^2+(\omega_1+\omega_2)^2/2}$$

$$C(|\omega_1|/2, |\omega_2|/2, P)\, \sinh(2\pi P)\,,$$

Call q=1-y and replace:

$$\int_0^1 dy\,(1-y)^{-1+2P^2-(\omega_1+\omega_2)^2/2} = \int_0^1 dq\, q^{-1+2P^2-(\omega_1+\omega_2)^2/2}$$

$$\to \frac{1}{2P^2-(\omega_1+\omega_2)^2/2-i\epsilon}$$

This gives

$$\frac{1}{\sinh(\pi|\omega_1|)\sinh(\pi|\omega_2|)} \int_0^\infty dP\, \frac{1}{2P^2-(\omega_1+\omega_2)^2/2-i\epsilon}\, 2^{-2P^2+(\omega_1+\omega_2)^2/2}$$

$$C(|\omega_1|/2, |\omega_2|/2, P)\, \sinh(2\pi P)\,,$$

– finite, but has both real and imaginary parts!

Divergent part of $g(\omega)$:

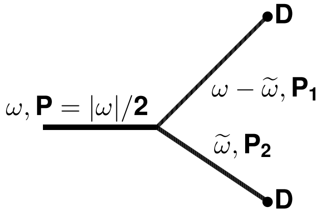
$$\frac{1}{\sinh(\pi|\omega|)}\,\pi^{-1/2}\int_0^{\omega} dP_1 \int_0^{\omega} dP_2\, C(|\omega|/2, P_1, P_2)\sinh(2\pi P_1)\sinh(2\pi P_2)$$

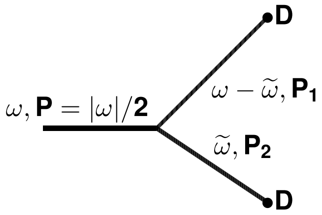
$$\int_0^{\infty} dt_1 \int_0^{\infty} dt_2\,(t_1+t_2)^{-1/2}\exp\left[-t_1 P_1^2 - t_2 P_2^2 + \frac{t_1 t_2}{t_1+t_2}\frac{\omega^2}{4}\right].$$

t_1, t_2 are related to (t,x) via:

$$t_1 = 2\pi(1-x)/t, \quad t_2 = 4\pi x/t$$

Integration over t_1, t_2 diverges at large t_1, t_2 for $\omega > 2(P_1+P_2)$





Consider the integral:

$$\mathbf{I} = \int_{-\infty}^{\infty} \mathbf{d}\widetilde{\omega} \, \frac{1}{-\frac{1}{4}(\omega - \widetilde{\omega})^2 + \mathbf{P_1^2} - \mathbf{i}\epsilon} \, \frac{1}{-\frac{1}{4}\widetilde{\omega}^2 + \mathbf{P_2^2} - \mathbf{i}\epsilon}$$

– has poles at $\widetilde{\omega} = \pm(\mathbf{2P_2} - \mathbf{i}\epsilon), \ \omega \pm (\mathbf{2P_1} - \mathbf{i}\epsilon)$

– can be evaluated to give a finite integral

$$\mathbf{I} = \int_{-\infty}^{\infty} d\widetilde{\omega} \, \frac{1}{-\frac{1}{4}(\omega - \widetilde{\omega})^2 + P_1^2 - i\epsilon} \, \frac{1}{-\frac{1}{4}\widetilde{\omega}^2 + P_2^2 - i\epsilon}$$

Schwinger parameter representation:

$$\mathbf{I} = \int_0^{\infty} dt_1 \int_0^{\infty} dt_2 \int_{-\infty}^{\infty} d\widetilde{\omega} \, \exp[-t_1\{-\frac{1}{4}(\omega - \widetilde{\omega})^2 + P_1^2\} - t_2\{-\frac{1}{4}\widetilde{\omega}^2 + P_2^2\}]$$

Wick rotate $\widetilde{\omega} \to i\,u$ and carry out the u-integral using Gaussian integration

$$\mathbf{I} = 2\,i\,\sqrt{\pi} \int_0^{\infty} dt_1 \int_0^{\infty} dt_2 (t_1 + t_2)^{-1/2} \exp\left[\frac{t_1 t_2}{4(t_1 + t_2)}\omega^2 - t_1 P_1^2 - t_2 P_2^2\right]$$

– diverges for $\omega > 2(P_1 + P_2)$

Replace world-sheet expressions of this kind by the top expression which has no divergence!

This procedure is robust, i.e. does not change under reparametrization of the t_i's.

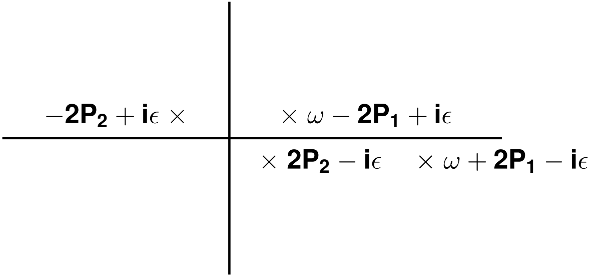
Divergent part of $g(\omega)$:

$$\frac{1}{\sinh(\pi|\omega|)}\,\pi^{-1/2}\int_0^{\omega} dP_1\int_0^{\omega} dP_2\; C(|\omega|/2, P_1, P_2)\,\sinh(2\pi P_1)\,\sinh(2\pi P_2)$$
$$\int_0^{\infty} dt_1\int_0^{\infty} dt_2\,(t_1+t_2)^{-1/2}\exp\left[-t_1\,P_1^2 - t_2\,P_2^2 + \frac{t_1 t_2}{t_1+t_2}\,\frac{\omega^2}{4}\right]$$

$$\to \frac{1}{\sinh(\pi|\omega|)}\,\pi^{-1/2}\int_0^{\omega} dP_1\int_0^{\omega} dP_2\; C(|\omega|/2, P_1, P_2)\,\sinh(2\pi P_1)\,\sinh(2\pi P_2)$$
$$\frac{1}{2\,i\,\sqrt{\pi}}\int d\widetilde{\omega}\,\frac{1}{-\frac{1}{4}(\omega-\widetilde{\omega})^2 + P_1^2 - i\epsilon}\,\frac{1}{-\frac{1}{4}\widetilde{\omega}^2 + P_2^2 - i\epsilon}$$

– has finite real and imaginary parts!

Pole positions in the $\widetilde{\omega}$ plane:



We can evaluate this by rotating the contour to be along the imaginary axis, picking up the residue at $\omega - 2P_1 + i\epsilon$.

$$
\begin{aligned}
&\frac{1}{\sinh(\pi|\omega|)}\,\pi^{-1/2}\int_0^{\omega} dP_1 \int_0^{\omega} dP_2\, C(|\omega|/2, P_1, P_2)\sinh(2\pi P_1)\sinh(2\pi P_2)\\
&\qquad \frac{1}{2\sqrt{\pi}}\int_{-\infty}^{\infty} du\, \frac{1}{\frac{1}{4}(u+i\omega)^2 + P_1^2}\,\frac{1}{\frac{1}{4}u^2 + P_2^2} \qquad\qquad \rightarrow \quad \text{real}\\
&+\frac{1}{\sinh(\pi|\omega|)}\int_0^{\omega/2} dP_1 \int_0^{\omega} dP_2\, C(|\omega|/2, P_1, P_2)\sinh(2\pi P_1)\sinh(2\pi P_2)\,\frac{1}{P_1}\,\frac{4}{(2P_1 + 2P_2 - \omega - i\epsilon)(2P_2 - 2P_1 + \omega)}
\end{aligned}
$$

This procedure gives manifestly finite $f(\omega_1, \omega_2)$ and $g(\omega)$.

Furthermore, from these expressions one can get analytic expressions for the imaginary parts of f and g.

$$f_{imaginary}(\omega_1, \omega_2) = \frac{1}{2}\, i\, \pi\, \omega_1\, \omega_2\, \{\coth(\pi\omega_1) + \coth(\pi\omega_2)\}\ \mathrm{sign}(\omega_1 + \omega_2)$$

$$g_{imaginary}(\omega) = \frac{i\,\pi}{2}\, |\omega| \left\{ \omega \coth(\pi\omega) - \frac{1}{\pi} \right\}$$

These exactly reproduce the matrix model result for $1 \to n$ tachyon scattering amplitude which is purely imaginary!

Note: If we had just wanted the imaginary parts, they could be found using Cutkosky rules of string field theory. Pius, A.S.

With the imaginary parts out of the way, the agreement between the matrix model and string theory results would require

$$\sum_{j<k=1}^{n+1} f_{real}(\omega_j, \omega_k) + \sum_{j=1}^{n+1} g_{real}(\omega_j) + C = 0$$

for $\omega_1, \cdots, \omega_n > 0$, $\quad \omega_{n+1} = -\omega_1 - \cdots - \omega_n$

– can be verified numerically in principle for real ω_i's.