Semigroup Quantum Spin Chains

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Based on work done with D. Texeira, D. Trancanelli ; F. Sugino and V. Korepin

Plan of the talk

1. Introduction to Semigroups and Inverse Semigroups

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1. Introduction to Semigroups and Inverse Semigroups
2. Integrable SUSY Spin Chains
3. Semigroup Motzkin and Fredkin Spin Chain

Inverse Semigroups

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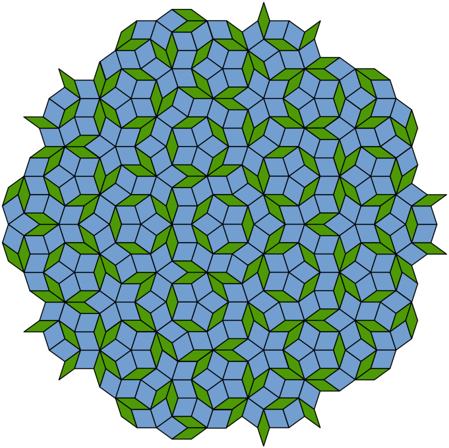
This structure is still not a group as there is no unique identity element. We now have partial identities.

Inverse Semigroups and Quasicrystals (M.V. Lawson *et. al* 00)



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As an example consider $\mathcal{S}_1^2$. There are four partial functions which we denote by $\{x_{1,1}, x_{1,2}, x_{2,1}, x_{2,2}\}$.

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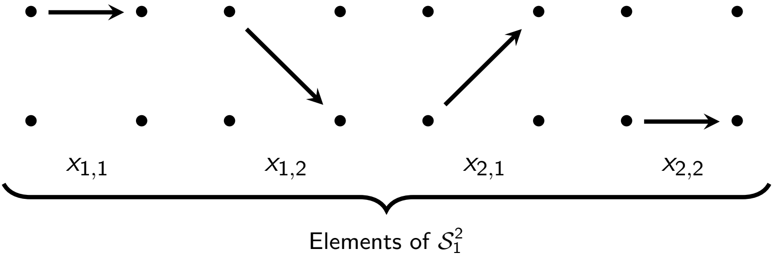
As an example consider $\mathcal{S}_1^2$. There are four partial functions which we denote by $\{x_{1,1}, x_{1,2}, x_{2,1}, x_{2,2}\}$.
They obey the following composition rule

$$x_{i,j} * x_{k,l} = \delta_{jk} x_{i,l}.$$

Diagrammatica for SISs



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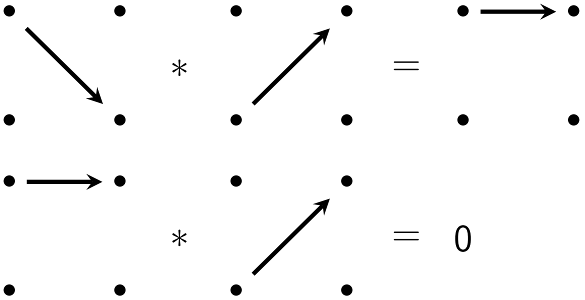
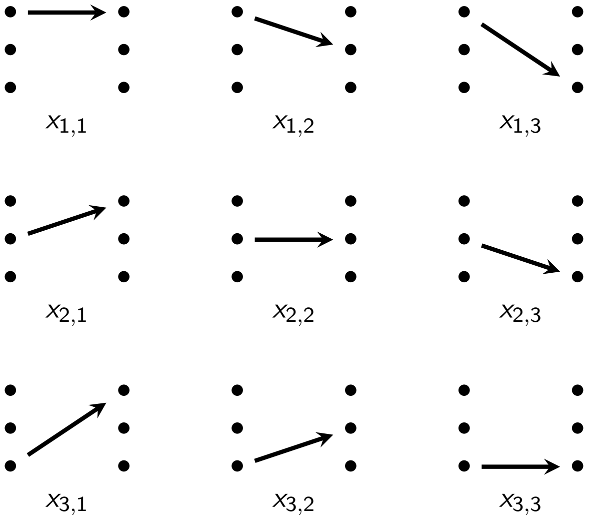


Diagrammatica for $\mathcal{S}_1^3$



A Matrix Representation

From the algebra of $\mathcal{S}_1^2$ and $\mathcal{S}_1^3$ it is easy to see that the elements are nothing but the $e_{i,j}$ matrices that span the space of 2 by 2 and 3 by 3 matrices respectively.

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$$\begin{aligned} x_{1,1} &= \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix},\ x_{1,2} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \\ x_{2,1} &= \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix},\ x_{2,2} = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix},. \end{aligned}$$

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This takes us one step closer to SUSY algebras !

Integrable SUSY Spin Chain

Supersymmetry Algebra in $0 + 1$ Dimensions

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It follows that the spectrum satisfies

$$E \geq 0.$$

Constructing Supercharges using SISs

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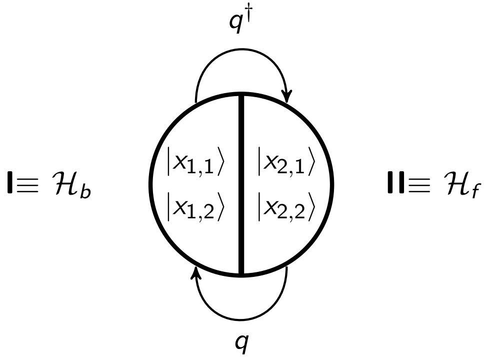
It introduces a grading of the Hilbert space



Supercharges out of SISs...

A more non-trivial supercharge built out of $\mathcal{S}_1^3$,

$$q = \frac{x_{1,2} + x_{1,3}}{\sqrt{2}} \ ; \ q^\dagger = \frac{x_{2,1} + x_{3,1}}{\sqrt{2}}.$$

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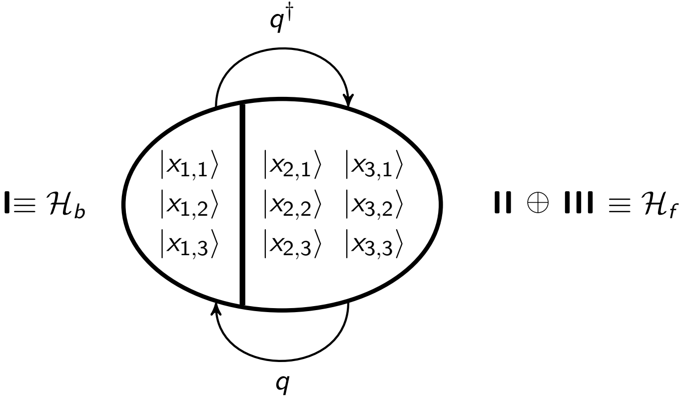

One-Particle SUSY Systems and Witten Index

For the $\mathcal{S}_1^2$ case the Hamiltonian is trivial.

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But for the $\mathcal{S}_1^3$ case we have

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Now the supercharges satisfy a centrally extended fermion algebra with

$$C = \frac{x_{2,3} + x_{3,2} - x_{2,2} - x_{3,3}}{2}$$

being the central extension.

Witten Index for $\mathcal{S}_1^3$ System

There are three unpaired "fermionic" zero modes making the Witten index 3 !

$$
\begin{aligned}
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The "bosons" and "fermions" are denoted by $\left|f^{1,2,3}\right\rangle$ and $\left|b^{1,2,3}\right\rangle$.

Building SUSY Chains - Non-Interacting

Associate local supercharges to sites, q_i.
A non-interacting SUSY chain is obtained from

$$Q = \sum_i a_i \theta_i \,, \qquad a_i \in \mathbb{C},$$

$$\theta_i = \prod_{1 \leq j < i} e^{i\pi F_j} q_i = \prod_{1 \leq j < i} (1 - 2F_j)\, q_i, \qquad i = 1, \ldots, N$$

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$$Q_{II} = \sum_{i=1}^{N-2}\ c_{i,i+1,i+2}\ M_i \theta_{i+1} M_{i+2}\,, \qquad c_{i,i+1,i+2} \in \mathbb{C}.$$

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The Local Integrals of Motion (LIOMs)

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Thus these models are integrable with N LIOMs.

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Thus these models are integrable with N LIOMs.
The states of the system are filled up by

$$\left| f_i^{1,2,3} \right\rangle, \ \left| b_i^{1,2,3} \right\rangle, \left| z_i^{1,2,3} \right\rangle$$

which are the local fermions, bosons and zero modes.

The Witten Index

The Witten Index for these systems is -3^N under the grading operator

$$W = \prod_{j=1}^{N} e^{i\pi F_j} = \prod_{j=1}^{N} (1 - 2F_j) \, , \qquad W^2 = \mathbb{I}.$$

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The index is stable under SUSY preserving perturbations

$$\Delta_k H = \sum_{i_1=1}^{N} \cdots \sum_{i_k=1}^{N} C(i_1, \cdots, i_k)(e^{\alpha_1} M_{i_1} + P_{i_1}) \cdots (e^{\alpha_k} M_{i_k} + P_{i_k}).$$

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It is also stable under deformed supercharges

$$q_d = \frac{1}{\sqrt{|a|^2 + |b|^2}} \left[a x_{1,2} + b x_{1,3}\right].$$

Related Work

- (H. Nicolai *et. al.* 77) has early works on Lattice SUSY and spin systems before Witten's SUSY QM.

$$Q = \sum_{i \in \mathbb{Z}} a_{2i-1} a_{2i}^{*} a_{2i+1}.$$

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$$Q = Q_{II} + \theta_1 M_2 + M_{N-1} \theta_N.$$

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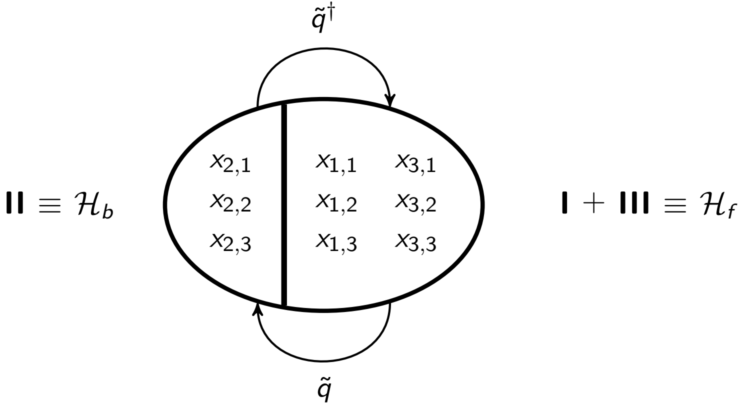
More recent works on Lattice SUSY spin systems including dynamical lattice SUSY systems.
- H.Moriya studies ergodicity and localization in the Nicolai SUSY many body system in arXiv:1610.09142.

Examples of Non-Integrable Many-Body SUSY Systems

Another possible grading of $\mathcal{S}_1^3$ is



Non-Integrable SUSY Systems.....

Choose the supercharge

$$Q' = F\tilde{Q}F^{-1},$$

with $\tilde{Q}$ is a global supercharge constructed using the new graded Hilbert space
and F is an invertible element made of the supercharge Q built out of the original grading.

$$F = e^{aQ} = 1 + aQ.$$

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Integrability is now broken as there are no longer LIOMs due to the loss of the unique grading of the local Hilbert spaces.

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Use the SIS, $\mathcal{S}_1^4$

$$\mathcal{H}_0 = \mathrm{I} + \mathrm{II}, \qquad \mathcal{H}_1 = \mathrm{III}, \qquad \mathcal{H}_2 = \mathrm{IV}.$$

Build parasupercharge

$$q = x_{1,3} + x_{2,3} + x_{3,4}, \qquad q^\dagger = x_{3,1} + x_{3,2} + x_{4,3}.$$

Semigroup Fredkin and Motzkin Spin Chains

Motzkin Spin Chain (P. Shor et. al. 2014)

- The local Hilbert space is given by $\{u^1, u^2, \cdots, u^s, 0, d^1, d^2, \cdots, d^s\}$, where u, d and 0 are dubbed "up", "down" and "flat" steps respectively.
- The system lives on a 1D chain and we can geometrically interpret the above steps as being along the $(1,1)$, $(1,-1)$ and $(1,0)$ directions respectively. s denotes the color of the step.
- For a $2n$-step/link chain the many body states are 2D paths. *Motzkin* walks are paths which start at $(0,0)$, end at $(2n,0)$, and always stays in the positive quadrant.
- The uniform superposition of such paths form the ground state of the Motzkin spin chain and has a half chain EE

$$S = 2\log_2(s)\sqrt{\frac{2\sigma n}{\pi}} + \frac{1}{2}\log_2(2\pi\sigma n) + O(1),$$

with $\sigma = \frac{\sqrt{s}}{2\sqrt{s}+1}$ and γ is Euler constant.

Local Hilbert Space : Colored Motzkin

$|\uparrow\rangle \equiv$ 

$|\uparrow^k\rangle \equiv$ 

$|\downarrow\rangle \equiv$ 

$|\downarrow^k\rangle \equiv$ 

$|\rightarrow\rangle \equiv$ 

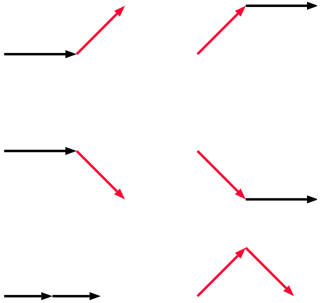
Motzkin Spin Chain Hamiltonian : $H_{Motzkin}$

- The local, frustration free Hamiltonian is built out of projectors to local equivalence moves

$$\begin{aligned} \left|D^k\right\rangle &= \frac{1}{\sqrt{2}}\left[\left|0d^k\right\rangle - \left|d^k0\right\rangle\right] \\ \left|U^k\right\rangle &= \frac{1}{\sqrt{2}}\left[\left|0u^k\right\rangle - \left|u^k0\right\rangle\right] \\ \left|F^k\right\rangle &= \frac{1}{\sqrt{2}}\left[\left|00\right\rangle - \left|u^kd^k\right\rangle\right] \end{aligned}$$

$$\Pi_{j,j+1} = \sum_{k=1}^{s}\left[\left|D^k\right\rangle_{j,j+1}\left\langle D^k\right| + \left|U^k\right\rangle_{j,j+1}\left\langle U^k\right| + \left|F^k\right\rangle_{j,j+1}\left\langle F^k\right|\right]$$

Local Equivalences : Colored Motzkin Chain



$H_{Motzkin}$.....

-The boundary term is

$$\Pi_{boundary} = \sum_{k=1}^{s} \left[\left| d^k \right\rangle_1 \left\langle d^k \right| + \left| u^k \right\rangle_{2n} \left\langle u^k \right| \right]$$

- A color balancing term

$$\Pi_{j,j+1}^{cross} = \sum_{k \neq i} \left| u^k d^i \right\rangle_{j,j+1} \left\langle u^k d^i \right|$$

- Finally

$$H_{Motzkin} = \Pi_{boundary} + \sum_{j=1}^{2n-1} \left[\Pi_{j,j+1} + \Pi_{j,j+1}^{cross} \right].$$

This is essentially a spin 1 chain. Model is gapless with gap scaling as n^{-c} with $c \geq 2$.

Fredkin Spin Chain (V. Korepin et. al. 2016)

- The local Hilbert space is spanned by $\{|\uparrow\rangle, |\downarrow\rangle\}$.
- Geometrically we have only "up" and "down" steps and no "flat" steps. The "up" step points along $(1, 1)$ and the "down" step points along $(1, -1)$.
- The states on the global Hilbert space are mapped to 2D *Dyck* walks which again start at $(0, 0)$ and end at $(2n, 0)$ without leaving the first quadrant.
- Notice that this is an uncolored local Hilbert space and the EE scales as

$$S = \frac{1}{2}\log(L) + O(1)$$

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$$|D_j\rangle = \frac{1}{\sqrt{2}}\left[|\uparrow_j, \downarrow_{j+1}, \downarrow_{j+2}\rangle - |\downarrow_j, \uparrow_{j+1}, \downarrow_{j+2}\rangle\right].$$

$$\Pi_{j,j+1,j+2} = |U_j\rangle\langle U_j| + |D_j\rangle\langle D_j|$$

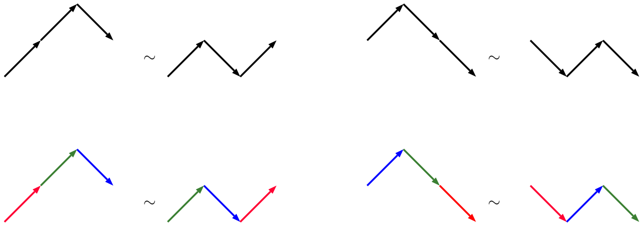
Boundary term is

$$H_{boundary} = \left[|\downarrow_1\rangle\langle\downarrow_1| + |\uparrow_{2n}\rangle\langle\uparrow_{2n}|\right]$$

$$H_{Fredkin} = H_{boundary} + \sum_{j=1}^{2n-2} \Pi_{j,j+1,j+2}.$$

- This is a spin $\frac{1}{2}$ chain. Has global $U(1)$ symmetry.

Local Equivalences : Colored Fredkin Chain



Colored Fredkin Spin Chain : $H_{colored,\,Fredkin}$

- Include s colors to each of the local basis states. The local equivalence moves now become

$$\begin{aligned}\left|U_j^{c_1,\,c_2,\,c_3}\right\rangle &= \frac{1}{\sqrt{2}}\left[\left|\uparrow_j^{c_1},\,\uparrow_{j+1}^{c_2},\,\downarrow_{j+2}^{c_3}\right\rangle - \left|\uparrow_j^{c_2},\,\downarrow_{j+1}^{c_3},\,\uparrow_{j+2}^{c_1}\right\rangle\right],\\ \left|D_j^{c_1,\,c_2,\,c_3}\right\rangle &= \frac{1}{\sqrt{2}}\left[\left|\uparrow_j^{c_2},\,\downarrow_{j+1}^{c_3},\,\downarrow_{j+2}^{c_1}\right\rangle - \left|\downarrow_j^{c_1},\,\uparrow_{j+1}^{c_2},\,\downarrow_{j+2}^{c_3}\right\rangle\right].\end{aligned}$$

$$B_{j,j+1} = \left|\uparrow_j^{c_1},\,\downarrow_{j+1}^{c_2}\right\rangle\left\langle\uparrow_j^{c_1},\,\downarrow_{j+1}^{c_2}\right|$$

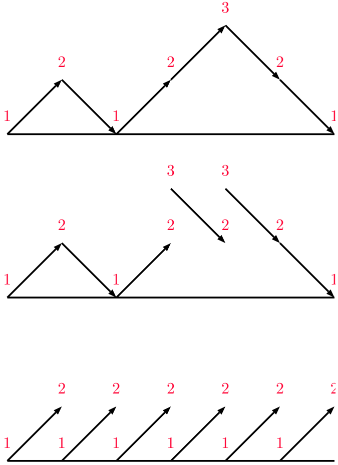
$$C_{j,j+1} = \Pi^{\frac{1}{\sqrt{2}}\left[\left|\uparrow_j^{c_1},\downarrow_{j+1}^{c_1}\right\rangle - \left|\uparrow_j^{c_2},\downarrow_{j+1}^{c_2}\right\rangle\right]}.$$

$$S \sim \frac{2}{\sqrt{\pi}}\log(s)\sqrt{\frac{(n+r)(n-r)}{n}} + +\frac{1}{2}\ln\frac{(n+r)(n-r)}{n} + O(1).$$

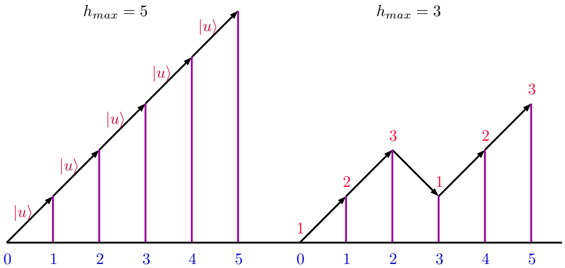
A Modification of the Motzkin Spin Chain (F.Sugino, PP, 2017)

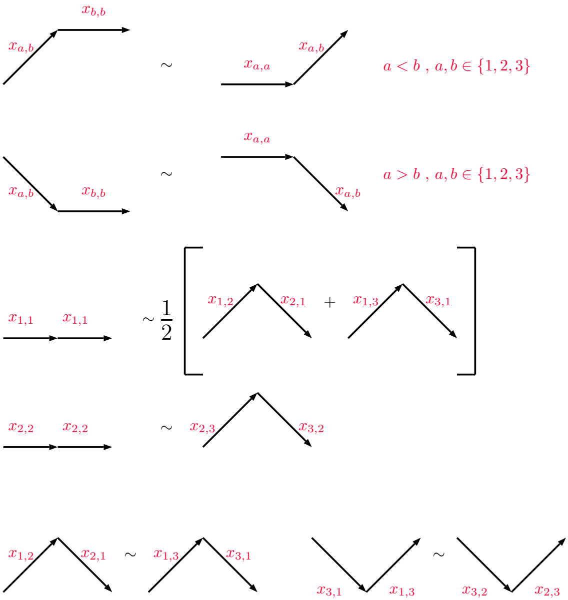
- Change the local Hilbert space to $\{|x_{a,b}\rangle;\ a, b \in \{1, 2, 3\}\}$. The "up" steps pointing along $(1, 1)$ occur when $a < b$, "down" steps pointing along $(1, -1)$ occur when $a > b$ and the "flat" steps pointing along $(1, 0)$ occur when $a = b$. These new indices can be thought of as arrow indices or more mathematically they are known as semigroup indices.
- This introduces different kinds of paths, *fully connected*, *partially connected* and *disconnected* paths.
- The maximum heights reached in a path is now smaller.

Different Kinds of Paths



Maximum Heights





$x_{b,b}$
$x_{a,b}$
$\sim$
$x_{a,a}$
$x_{a,b}$
$a < b \ , \ a,b \in \{1,2,3\}$
$x_{a,a}$
$x_{a,b}$
$x_{b,b}$
$\sim$
$x_{a,b}$
$a > b \ , \ a,b \in \{1,2,3\}$
$x_{1,1}$
$x_{1,1}$
$\sim \frac{1}{2}$
$x_{1,2}$
$x_{2,1}$
$+$
$x_{1,3}$
$x_{3,1}$
$x_{2,2}$
$x_{2,2}$
$\sim$
$x_{2,3}$
$x_{3,2}$
$x_{1,2}$
$x_{2,1}$
$\sim$
$x_{1,3}$
$x_{3,1}$
$x_{3,1}$
$x_{1,3}$
$\sim$
$x_{3,2}$
$x_{2,3}$

Projectors to the Modified Local Equivalence Moves

$$
\begin{aligned}
U_{j,j+1} &= \sum_{a,b=1;a<b}^{3} \Pi^{\frac{1}{\sqrt{2}}\left[\left|\left(x_{a,b}\right)_j,\left(x_{b,b}\right)_{j+1}\right\rangle - \left|(x_{a,a})_j,\left(x_{a,b}\right)_{j+1}\right\rangle\right]}, \\
D_{j,j+1} &= \sum_{a,b=1;a>b}^{3} \Pi^{\frac{1}{\sqrt{2}}\left[\left|\left(x_{a,b}\right)_j,\left(x_{b,b}\right)_{j+1}\right\rangle - \left|(x_{a,a})_j,\left(x_{a,b}\right)_{j+1}\right\rangle\right]}, \\
F_{j,j+1} &= \Pi^{\sqrt{\frac{2}{3}}\left[\left|(x_{1,1})_j,(x_{1,1})_{j+1}\right\rangle - \frac{1}{2}\left(\left|(x_{1,2})_j,(x_{2,1})_{j+1}\right\rangle + \left|(x_{1,3})_j,(x_{3,1})_{j+1}\right\rangle\right)\right]} \\
&\quad + \Pi^{\frac{1}{\sqrt{2}}\left[\left|(x_{2,2})_j,(x_{2,2})_{j+1}\right\rangle - \left|(x_{2,3})_j,(x_{3,2})_{j+1}\right\rangle\right]}, \\
W_{j,j+1} &= \Pi^{\frac{1}{\sqrt{2}}\left[\left|(x_{1,2})_j,(x_{2,1})_{j+1}\right\rangle - \left|(x_{1,3})_j,(x_{3,1})_{j+1}\right\rangle\right]} \\
&\quad + \mu \Pi^{\frac{1}{\sqrt{2}}\left[\left|(x_{3,1})_j,(x_{1,3})_{j+1}\right\rangle - \left|(x_{3,2})_j,(x_{2,3})_{j+1}\right\rangle\right]}.
\end{aligned}
$$

Boundary, Balancing and Bulk, Disconnected Terms

$$
\begin{aligned}
H_{left} &= \Pi^{\left|(x_{2,1})_1\right\rangle} + \Pi^{\left|(x_{3,1})_1\right\rangle} + \Pi^{\left|(x_{3,2})_1\right\rangle}, \\
H_{right} &= \Pi^{\left|(x_{1,2})_n\right\rangle} + \Pi^{\left|(x_{1,3})_n\right\rangle} + \Pi^{\left|(x_{2,3})_n\right\rangle}.
\end{aligned}
$$

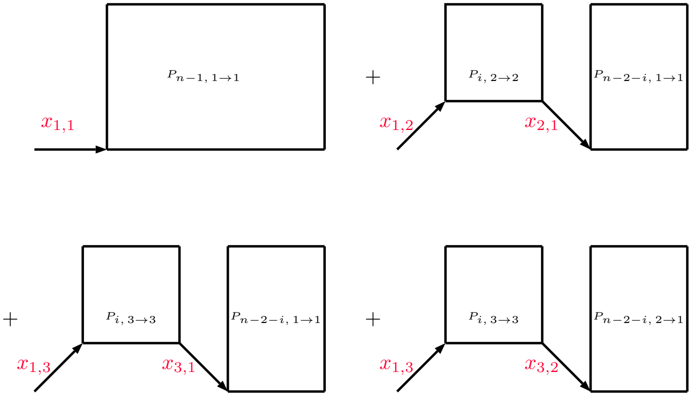
$$
B_{j,j+1} = \Pi^{\left|(x_{1,3})_j,(x_{3,2})_{j+1}\right\rangle} + \Pi^{\left|(x_{2,3})_j,(x_{3,1})_{j+1}\right\rangle}.
$$

$$
H_{bulk,\,disconnected} = \sum_{j=1}^{n-1} \sum_{a,b,c,d=1;b\neq c}^{3} \Pi^{\left|(x_{a,b})_j,(x_{c,d})_{j+1}\right\rangle}.
$$

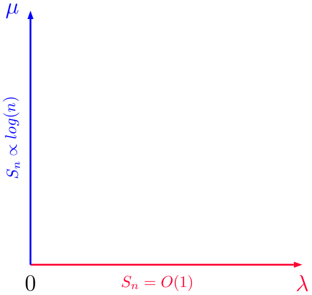
$$
H_{\mathcal{S}_1^3,\,Motzkin} = H_{left} + H_{right} + H_{bulk} + \lambda \sum_{j=1}^{2n-1} B_{j,j+1} + H_{bulk,\,disconnected}.
$$

Ground States

-This system has a ground state degeneracy (GSD) of 5 given by the equivalence classes, $\{11\}$, $\{12\}$, $\{21\}$, $\{22\}$ and $\{33\}$.
- We can use techniques from enumerative combinatorics to compute the normalization of these states.



Quantum Phase Transition



Colored $\mathcal{S}_1^3$ Motzkin Chain

- We introduce a color degree of freedom to each of the basis states, $\left|x_{a,b}^k\right\rangle,\ k \in \{1, 2\}$.

$$H^{balanced} = \mu \sum_{i=1}^{n} C_j + \sum_{j=1}^{n-1} \Big[U_{j,j+1} + D_{j,j+1} + F_{j,j+1}^{balanced} + W_{j,j+1}^{balanced} + R_{j,j+1}^{balanced} + H_{left} + H_{right} \Big]$$

with new equivalence moves

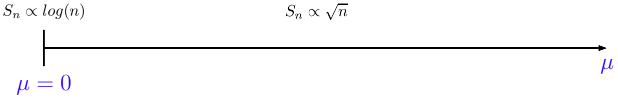
$$C_j = \sum_{a=1}^{3} \Pi^{\frac{1}{\sqrt{2}}\left[\left|(x_{a,a}^1)_j\right\rangle - \left|(x_{a,a}^2)_j\right\rangle\right]},$$

$$R_{j,j+1}^{balanced} = \sum_{a,b,c=1;\ b>a,c}^{3} \left[\Pi^{\left|(x_{a,b}^1)_j,(x_{b,c}^2)_{j+1}\right\rangle} + \Pi^{\left|(x_{a,b}^2)_j,(x_{b,c}^1)_{j+1}\right\rangle} \right].$$

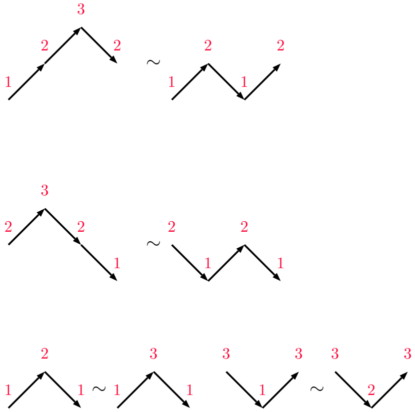
Quantum Phase Transition

$$H_{\mathcal{S}_1^3,\, colored\ Motzkin} = H^{balanced} + H_{bulk,disconnected}.$$

$$\begin{aligned} S_{A,\,1\to 1} \quad &= \quad (2\ln 2)\sqrt{\frac{2\sigma n}{\pi}} + \frac{1}{2}\ln n + \frac{1}{2}\ln(2\pi\sigma) + \gamma - \frac{1}{2} + \ln\frac{3}{2^{1/3}} \\ &\quad +(\text{terms vanishing as } n \to \infty) \end{aligned}$$



Modified Fredkin Chain (F.Sugino, PP, V.Korepin, 2018)



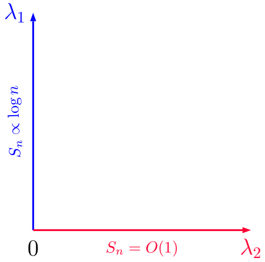
Modified Fredkin Chain Hamiltonian

$$
\begin{aligned}
U_{j,j+1,j+2} &= \Pi_{\frac{1}{\sqrt{2}}\left[\left|(x_{1,2})_j,(x_{2,3})_{j+1},(x_{3,2})_{j+2}\right\rangle - \left|(x_{1,2})_j,(x_{2,1})_{j+1},(x_{1,2})_{j+2}\right\rangle\right]} \\
D_{j,j+1,j+2} &= \Pi_{\frac{1}{\sqrt{2}}\left[\left|(x_{2,3})_j,(x_{3,2})_{j+1},(x_{2,1})_{j+2}\right\rangle - \left|(x_{2,1})_j,(x_{1,2})_{j+1},(x_{2,1})_{j+2}\right\rangle\right]} \\
W_{j,j+1} &= \Pi_{\frac{1}{\sqrt{2}}\left[\left|(x_{1,2})_j,(x_{2,1})_{j+1}\right\rangle - \left|(x_{1,3})_j,(x_{3,1})_{j+1}\right\rangle\right]} \\
&\quad + \lambda_1 \Pi_{\frac{1}{\sqrt{2}}\left[\left|(x_{3,1})_j,(x_{1,3})_{j+1}\right\rangle - \left|(x_{3,2})_j,(x_{2,3})_{j+1}\right\rangle\right]},
\end{aligned}
$$

$$
H_F = H_{left} + H_{bulk,\,connected} + H_{right} + \lambda_2 \sum_{j=1}^{n-1} B_{j,j+1} + H_{bulk,\,disconnected}.
$$

Quantum Phase Transition

-The GSD is 4, we no longer have the $\{33\}$ equivalence class.
$\lambda_1 = \lambda_2 = 0$ is a special phase where there is an extensive GSD in each equivalence class.
- When $\lambda_1, \lambda_2 > 0$ the Hamiltonian is no longer frustration free and is not shown in the figure.



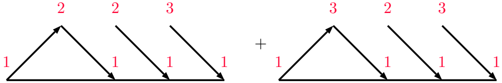
Excitations

- There are three kinds of excitations in these systems, fully connected, partially connected and disconnected excitations.
- The partially connected excitations are localized both in the low energy and high energy sector.

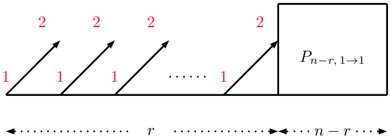
$$|x_{2,3}\rangle_i \langle x_{1,2}| \triangleright |P_{n,\,1\to 1}\rangle = \sum_{h=0}^{h_{max,i}} \left[\left| P^{(0\to h)}_{i-1,\,1\to 1} \right\rangle \otimes |x_{2,3}\rangle_i \otimes \left| P^{(h+1\to 0)}_{n-i,\,2\to 1} \right\rangle \right].$$

Partially Connected Excitations

A low energy example



A high energy example



Localization

- The partially connected excitations are localized as can be seen by computing connected 2-point correlation functions.

$$\langle pce|\theta_i(t)\theta_j(0)|pce\rangle - \langle pce|\theta_i(t)|pce\rangle\langle pce|\theta_j(0)|pce\rangle = 0,$$

$$\theta_i(0) = |x_{a_1,b_1}\rangle_i\langle x_{a_2,b_2}|,\ \ a_1 \neq a_2 \text{ and } b_1 \neq b_2,$$

$$\theta_i(0) = \sum_{a,b} k_{a,b}|x_{a,b}\rangle_i\langle x_{a,b}|,\ \ a, b \in \{1, 2, 3\}.$$

Thank you !