Two-point boundary correlation functions of dense loop models

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Exactly Solvable Quantum Chains

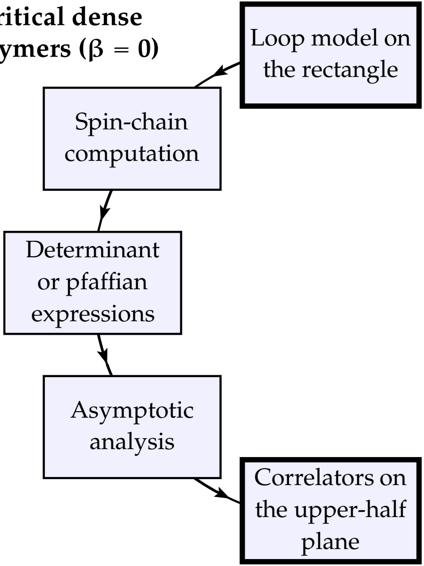
Natal, 22/06/2018

Joint work with Jesper Jacobsen

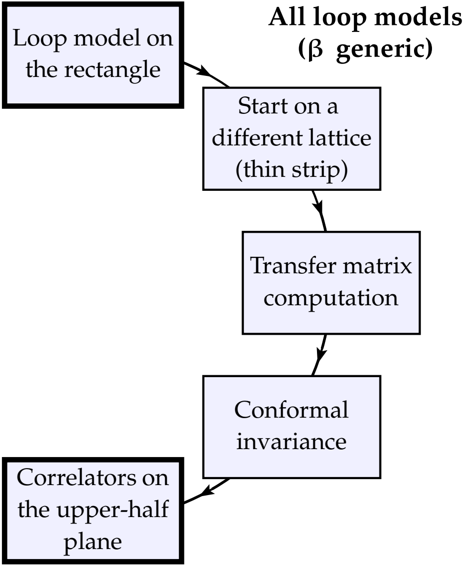
arXiv:1712.08657

Two approaches to correlation functions

Critical dense polymers ($\beta = 0$)



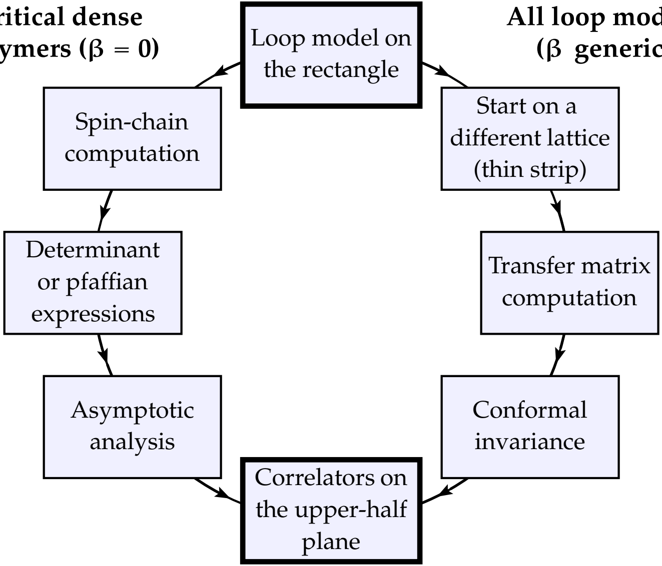
Two approaches to correlation functions



Two approaches to correlation functions

Critical dense polymers ($\beta = 0$)

All loop models (β generic)

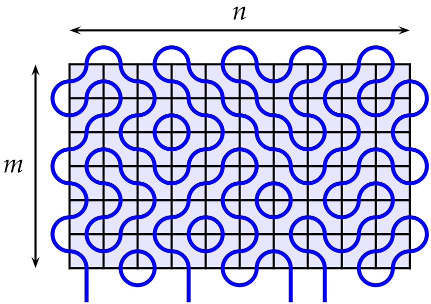


Dense loop model

- Loop model on the $m \times n$ rectangle
- Boundary conditions:

Dirichlet

Neumann



- Fugacity of the loops: $\begin{cases} \beta & \text{for bulk loops} \\ 1 & \text{for boundary loops} \end{cases}$
- Partition function: $Z = \sum_{\sigma} \beta^{n_\beta}$
- Model of **critical dense polymers:** $\beta = 0$

Dense loop model

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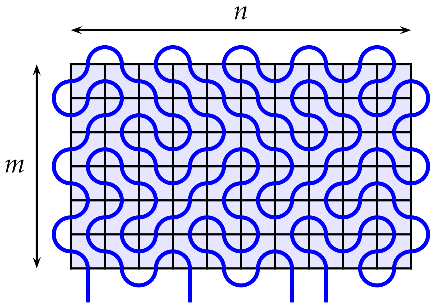

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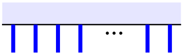

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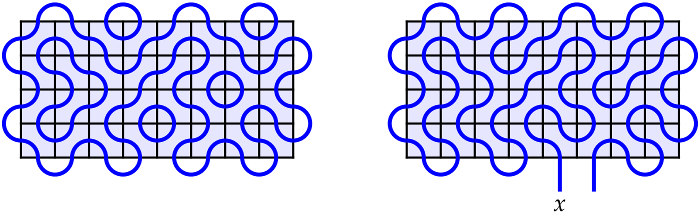
Boundary correlation functions

- Correlation function for two points x and y on the boundary:

$$C(x,y) = \frac{Z(x,y)}{Z^0}$$

- Z^0 is the **reference partition function**
- Boundary conditions for Z^0:

(β generic) (β = 0)



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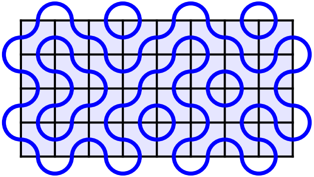

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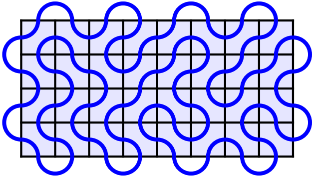

x

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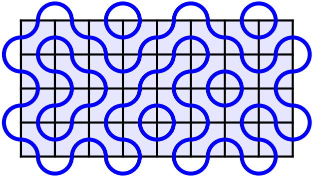

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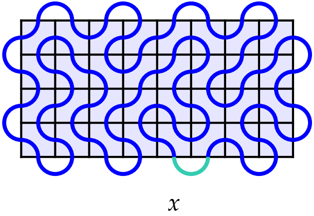

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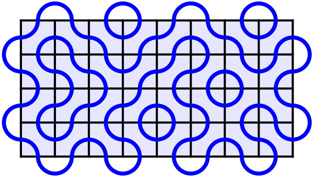

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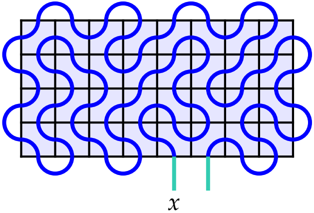
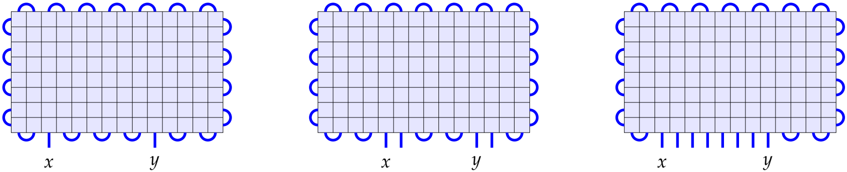


Six types of correlators: a, b and c

- Boundary conditions for $Z^{\mathrm{a,b,c}}(x,y)$:

(a) (b) (c)

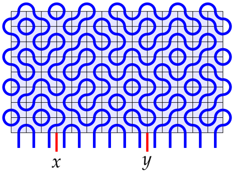


$$C^{\mathrm{a}}(x,y) = \frac{Z^{\mathrm{a}}(x,y)}{Z^0} \qquad C^{\mathrm{b}}(x,y) = \frac{Z^{\mathrm{b}}(x,y)}{Z^0} \qquad C^{\mathrm{c}}(x,y) = \frac{Z^{\mathrm{c}}(x,y)}{Z^0}$$

Six types of correlators: types d and e

- Boundary conditions and constraints for $Z^{\mathrm{d,e}}(x,y)$:

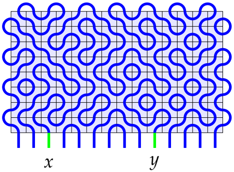
(d)



$$Z^{\mathrm{d}}(x,y) = \sum_{\sigma} \beta^{n_\beta}\, \delta_{c_x,c_y}$$

$$C^{\mathrm{d}}(x,y) = \frac{Z^{\mathrm{d}}(x,y)}{Z^0}$$

(e)



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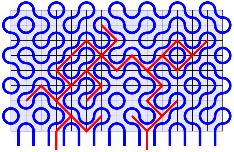

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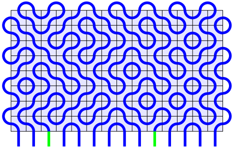

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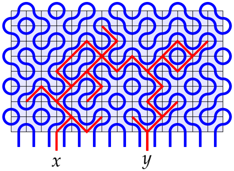

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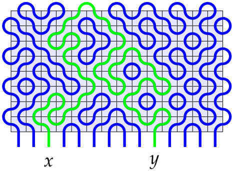

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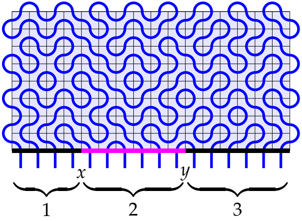
Six types of correlators: type f

- Boundary divided in segments 1, 2 and 3
- n_{ij}: number of loops tying segments i and j



$$n_{12} = 1$$
$$n_{23} = 3$$
$$n_{13} = 1$$

- Partition function and correlation function:

$$Z^{\mathrm{f}}(x,y) = \sum_{\sigma} \beta^{n_\beta} \tau^{n_{12}+n_{23}} \qquad C^{\mathrm{f}}(x,y) = \frac{Z^{\mathrm{f}}(x,y)}{Z^0}$$

- Related to the **valence bond entanglement entropy**: (Alet et al. '07)

$$\langle n_{12} + n_{23} \rangle = \frac{\sum_\sigma (n_{12}+n_{23})\beta^{n_\beta}}{\sum_\sigma \beta^{n_\beta}} = \frac{\mathrm{d}(\ln C^{\mathrm{f}}(x,y))}{\mathrm{d}\tau}\Big|_{\tau=1}$$

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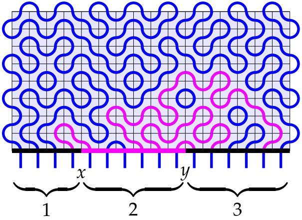

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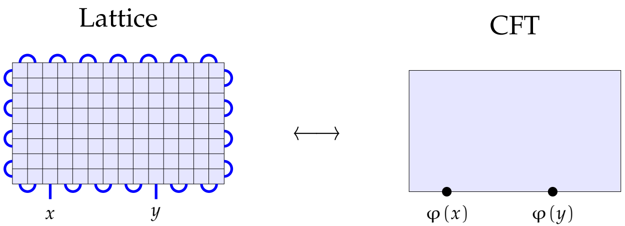
Critical behavior



- Two-point function of **primary fields** on the upper-half plane $\mathbb{H}$:

$$C_{\mathbb{H}}(x,y) \simeq \langle \varphi(x)\varphi(y)\rangle_{\mathbb{H}} = \frac{K}{|x-y|^{2\Delta}}$$

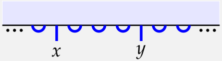
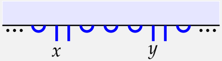
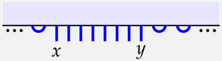
- Δ: conformal weight of the **boundary condition changing field**
- Partition function on the rectangle (Cardy-Peschel formula):

$$\ln Z = -mnf_b - (m+n)f_s - \ln(mn) \sum_{\text{corners}} \left(2\Delta - \frac{c}{16}\right) + \dots$$

where c is the **central charge**

Results for types a,b,c

- **Table of results** (with $r = |x-y|$)

	Polymers ($\beta = 0$)	Generic β
(a) 	$C^{\mathrm{a}}(x,y) = K\, r^{1/4}$	$C^{\mathrm{a}}(x,y) = \dfrac{K}{r^{2\Delta_{1,2}}}$
(b) 	$C^{\mathrm{b}}(x,y) = K_0 + K_1 \log r$	$C^{\mathrm{b}}(x,y) = \dfrac{K}{r^{2\Delta_{1,3}}}$
(c) 	$C^{\mathrm{c}}(x,y) = K\, r^{3/16}$	$C^{\mathrm{c}}(x,y) = \dfrac{K}{r^{2\Delta_{0,1/2}}}$

- Loop weight: $\beta = 2\cos\lambda \qquad \lambda = \pi(1-t) \qquad t \in (0,1)$
- Central charge and conformal weights:

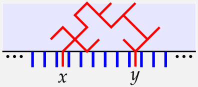
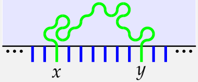
$$c = 13 - 6(t + t^{-1}) \qquad \Delta_{r,s} = \frac{1-rs}{2} + \frac{r^2-1}{4t} + \frac{(s^2-1)t}{4}$$

- For polymers:

$$\lambda = \frac{\pi}{2} \qquad t = 2 \qquad c = -2 \qquad \Delta_{r,s} = \frac{(2r-s)^2-1}{8}$$

Results for types d,e

- **Table of results** (with $r = |x - y|$)

		Polymers ($\beta = 0$)	Generic β
(d)		$C^{\mathrm{d}}(x,y) = \frac{K}{r^{3/4}}$	$C^{\mathrm{d}}(x,y) = \frac{K}{r^{2\Delta_{1,0}}}$
(e)		$C^{\mathrm{e}}(x,y) = \frac{K}{r^{2}}$	$C^{\mathrm{e}}(x,y) = \frac{K}{r^{2\Delta_{1,-1}}}$

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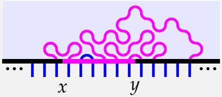
Results for types f

- **Table of results** (with $r = |x - y|$)

		Polymers ($\beta = 0$)	Generic β
(f)		$C^{\mathrm{f}}(x,y) = \dfrac{K}{r^{\frac{2\theta}{\pi}(1+\frac{2\theta}{\pi})}}$	$C^{\mathrm{f}}(x,y) = \dfrac{K}{r^{2\Delta_{1+\frac{2\theta}{\lambda},1+\frac{2\theta}{\lambda}}}}$

- Loop weights:
  - bulk loops: $\beta = 2\cos\lambda \qquad \lambda = \pi(1-t) \qquad t \in (0,1)$
  - boundary loops: $\tau = \dfrac{\cos(\frac{\lambda}{2}+\theta)}{\cos(\frac{\lambda}{2})}$
- Central charge and conformal weights:

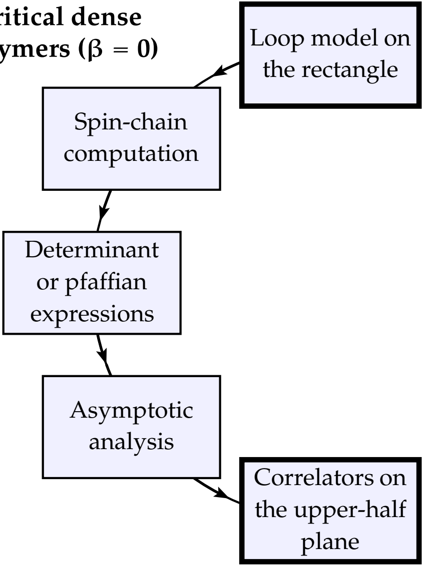
$$c = 13 - 6(t + t^{-1}) \qquad \Delta_{r,s} = \frac{1-rs}{2} + \frac{r^2-1}{4t} + \frac{(s^2-1)t}{4}$$

- For polymers:

$$\lambda = \frac{\pi}{2} \qquad t = 2 \qquad c = -2 \qquad \Delta_{r,s} = \frac{(2r-s)^2-1}{8}$$

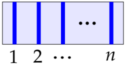
Approach 1: Exact derivations

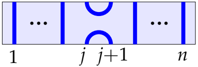
Critical dense polymers ($\beta = 0$)



Temperley-Lieb algebra

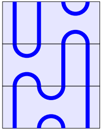
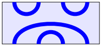
- The Temperley-Lieb algebra $\mathsf{TL}_n(\beta) = \langle I, e_j \rangle$:

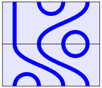
$I =$ 

$e_j =$ 

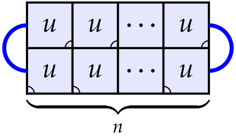
$j = 1, \dots, n-1$

- Examples of Temperley-Lieb products for $n = 4$:

$e_2 e_1 e_3 =$  $=$ 

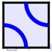
 $= \beta$ 

- Transfer tangle: (recall $\beta = 2\cos\lambda$)

$\boldsymbol{D}(u) =$ 

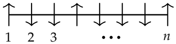
$= \dfrac{\sin(\lambda - u)}{\sin\frac{\lambda}{2}}$  $+ \dfrac{\sin u}{\sin\frac{\lambda}{2}}$ 

- Isotropic point: $u = \frac{\lambda}{2}$

$=$  $+$ 

XXZ spin chain

- XXZ representation of $\mathsf{TL}_n(\beta)$ for $\beta = q + q^{-1}$:



$$\mathsf{X}_n(e_j) = \underbrace{\mathbb{I}_2 \otimes \cdots \otimes \mathbb{I}_2}_{j-1} \otimes \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & q & 1 & 0 \\ 0 & 1 & q^{-1} & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \otimes \underbrace{\mathbb{I}_2 \otimes \cdots \otimes \mathbb{I}_2}_{n-j-1}$$

- Spin-chain Hamiltonian and transfer matrix:

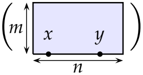
$$H = -\sum_{j=1}^{n-1} \mathsf{X}_n(e_j) \qquad\qquad D(u) = \mathsf{X}_n\big(\boldsymbol{D}(u)\big)$$

$$= -\frac{1}{2}\Big(\sum_{j=1}^{n-1} \sigma_j^x \sigma_{j+1}^x + \sigma_j^y \sigma_{j+1}^y - \frac{q+q^{-1}}{2}(\sigma_j^z \sigma_{j+1}^z - \mathbb{I})\Big) - \frac{q-q^{-1}}{4}(\sigma_1^z - \sigma_n^z)$$

- Special values:

Polymers $(\beta = 0)$	$\longleftrightarrow$	**XX spin-chain** $(q = \mathrm{i})$

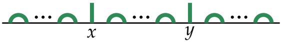
Correlators from the spin chain

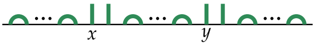
- Partition functions on the rectangle: 

$$\boxed{Z^{\mathrm{a,b,c}}(x,y) = \langle v^{\mathrm{a,b,c}} | D(\tfrac{\lambda}{2})^{m/2} | v^0 \rangle}$$

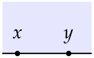
with

$|v^0\rangle = |$  $\rangle$

$|v^{\mathrm{a}}\rangle = |$  $\rangle$

$|v^{\mathrm{b}}\rangle = |$  $\rangle$

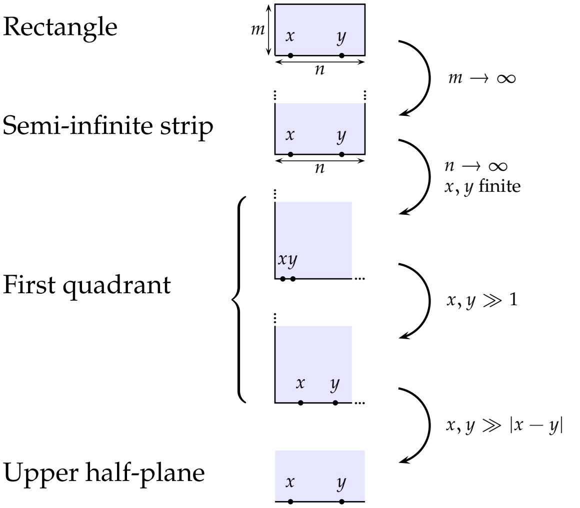
$|v^{\mathrm{c}}\rangle = |$  $\rangle$

- Correlation function on the upper-half plane: 

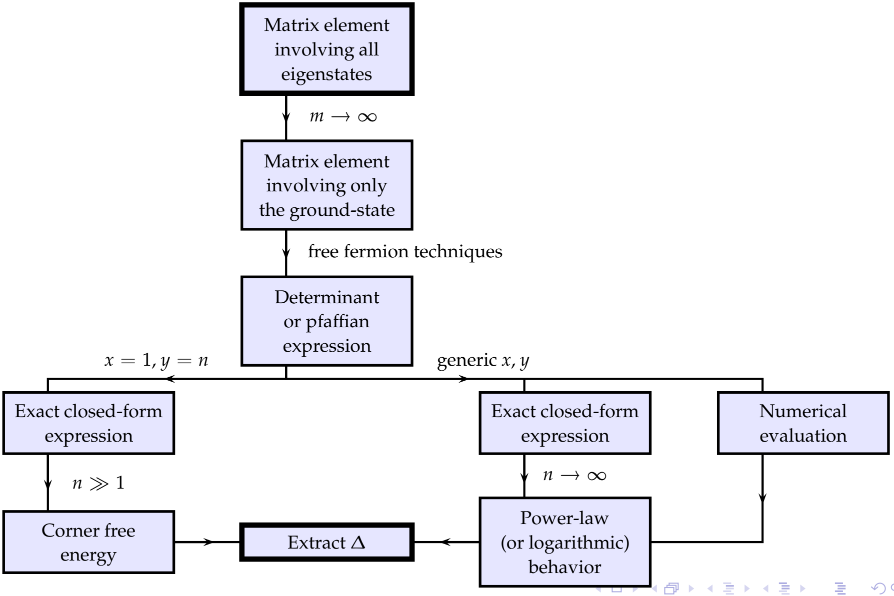
$$\boxed{\frac{Z^{\mathrm{a,b,c}}(x,y)}{Z^0} \xrightarrow{?} \frac{K}{|x-y|^{2\Delta^{\mathrm{a,b,c}}}}}$$

From the rectangle to the upper-half plane

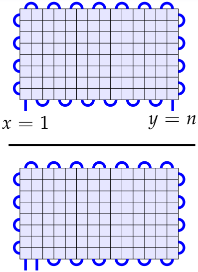
- Sequence of limits in the lattice calculation:



Strategy to compute Δ for $\beta = 0$



Correlators of type a



- Corner free energy analysis:

$$\ln C^{\mathrm{a}}(1,n) = \ln\left(\frac{Z^{\mathrm{a}}(1,n)}{Z^0}\right) = \tfrac{1}{2}\ln n - \tfrac{1}{2}\ln 2$$

- Expected from CFT:

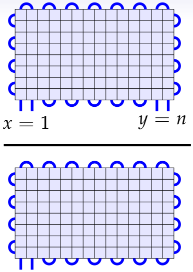
$$\ln Z/Z' = -n(f_s - f_s') - 2\ln n \sum_{\text{corners}} (\Delta - \Delta') + \dots$$

- Result of the exact calculation for x, y generic:

$$C^{\mathrm{a}}(x,y) = K(y-x)^{1/4} \qquad K = \frac{\overset{\text{Barnes' } G(z) \text{ function}}{G^2(\frac{3}{2})}}{2^{1/4}}$$

- **Conformal weights:** $\Delta^{\mathrm{a}} = -\dfrac{1}{8} \qquad \Delta^0 = 0$

Correlators of type b



- Corner free energy analysis:

$$C^{\mathrm{b}}(1,n) = \tfrac{4}{\pi}\ln n + 1 + \tfrac{4}{\pi}(\overset{\text{Euler-Mascheroni constant}}{\gamma} + 2\ln 2 - \ln \pi - 1) + \dots$$

$$\ln C^{\mathrm{b}}(1,n) = \ln(\ln n) + \ln(4/\pi) + \dots$$

- Result of the exact calculation for x, y generic:

$$C^{\mathrm{b}}(x,y) = K_0 \ln(y-x) + K_1 \qquad K_0 = \frac{2}{\pi} \qquad K_1 = 1 + \frac{2}{\pi}(\gamma + \ln 2)$$

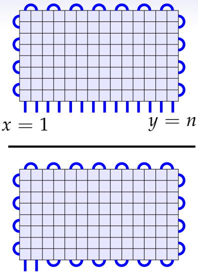
- **Logarithmic field of conformal weight:** $\Delta^{\mathrm{b}} = 0$

- CFT: two-point function of a log. field in a rank 2 Jordan cell:

$$\langle \omega(z_0)\omega(z_1)\rangle = \frac{K_0 \ln|z_0 - z_1| + K_1}{|z_0 - z_1|^{2\Delta}}$$

- The constant K_0 is **universal** (Vasseur, Jacobsen '14)

Correlators of type c



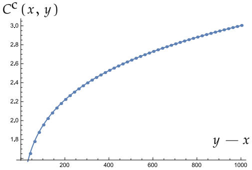
- Corner free energy analysis:

$$\ln C^{c}(1,n) = \ln\left(\frac{Z^{c}(1,n)}{Z^{0}}\right) = n\underbrace{\frac{G}{\pi}}_{} + \frac{3}{8}\ln n + \dots$$

Catalan's constant

- Expected from CFT:

$$\ln Z/Z' = -n(f_s - f_s') - 2\ln n \sum_{\text{corners}} (\Delta - \Delta') + \dots$$



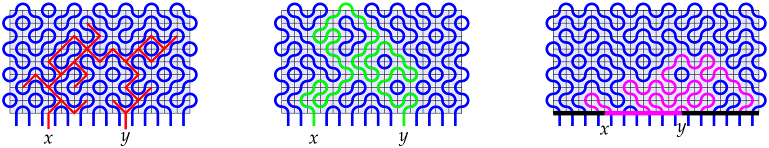
- Results for generic x, y:
  - Pfaffian formula for $C^{c}(x,y)$
  - No closed form obtained
  - Numerical evaluation: $\Delta^{c} \simeq -0.09405$

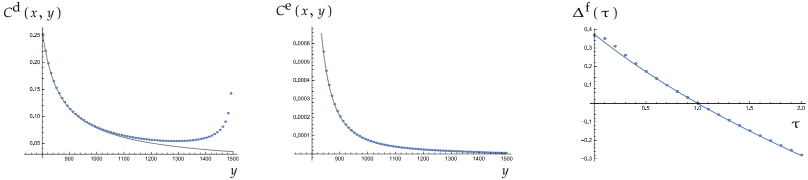
- **Conformal weight:** $\Delta^{c} = -\dfrac{3}{32} = -0.09375$

Correlators of type d, e and f

- Pfaffian formulas for $C^{\mathrm{d,e,f}}(x,y)$ $\qquad$ (recall $\tau = \dfrac{\cos(\frac{\lambda}{2}+\theta)}{\cos(\frac{\lambda}{2})}$)

(d) $\qquad$ (e) $\qquad$ (f)

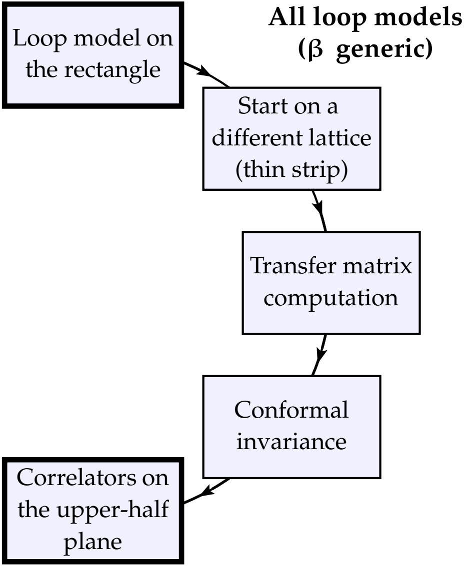




- Numerical evaluation: $\qquad \Delta^{\mathrm{d}} \simeq 0.37247 \qquad \Delta^{\mathrm{e}} \simeq 0.994845$

- **Conformal weights:** $\qquad \Delta^{\mathrm{d}} = \dfrac{3}{8} \qquad \Delta^{\mathrm{e}} = 1 \qquad \Delta^{\mathrm{f}} = \dfrac{\theta}{\pi}\left(1+\dfrac{2\theta}{\pi}\right)$

Approach 2: CFT derivations

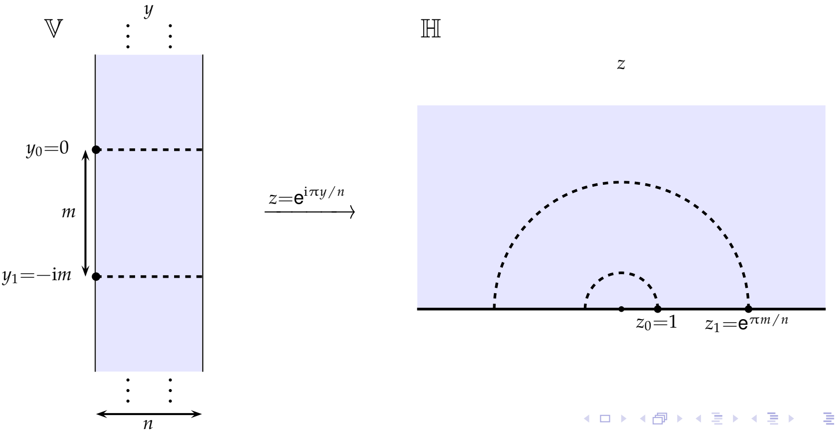


Covariance of two-point functions

- Transformation law for two-point functions of primary fields:

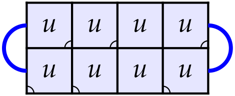
$$\langle \phi(y_0)\phi(y_1)\rangle_{\mathbb{V}} = \left|\frac{\mathrm{d}y}{\mathrm{d}z}\right|_{y=y_0}^{-\Delta} \left|\frac{\mathrm{d}y}{\mathrm{d}z}\right|_{y=y_1}^{-\Delta} \langle \phi(z_0)\phi(z_1)\rangle_{\mathbb{H}}$$

- The domains $\mathbb{V}$ and $\mathbb{H}$:



Finite-size corrections

- Reminder: the transfer matrix is defined as

$$D(u) = \mathsf{X}_n\big(\boldsymbol{D}(u)\big) = \mathsf{X}_n\Bigg($$ 

$$\Bigg)$$

- Behavior of the eigenvalues of the low-lying states of $D(\frac{\lambda}{2})$:

$$\Lambda_i = \exp\Big(-nf_b - f_s \underbrace{-\tfrac{2\pi}{n}\big(\Delta(i) - \tfrac{c}{24}\big) + \dots}_{\text{finite-size corrections}}\Big)$$

- f_b: bulk free energy
- f_s: surface free energy

$\Big\}$ Do not depend on i

- The finite-size corrections depend on the **central charge** and a **conformal weight**

Correlators of type a

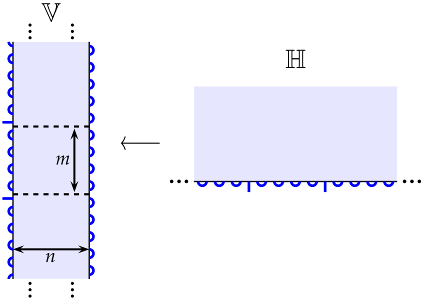
- Compute $C^{\mathrm{a}}_{\mathbb{V}}(y_0, y_1)$ in two ways:

  1) using **conformal invariance:**

$$C^{\mathrm{a}}_{\mathbb{H}}(z_0, z_1) = \frac{K}{|z_0 - z_1|^{2\Delta^{\mathrm{a}}}}$$

$$C^{\mathrm{a}}_{\mathbb{V}}(y_0, y_1) = \frac{K' \mathrm{e}^{\frac{\pi m}{n}\Delta^{\mathrm{a}}}}{(\mathrm{e}^{\frac{\pi m}{n}} - 1)^{2\Delta^{\mathrm{a}}}}$$

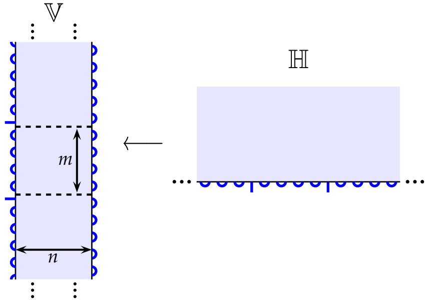
$$\xrightarrow{m \gg n} K' \mathrm{e}^{-\frac{\pi m}{n}\Delta^{\mathrm{a}}}$$



Correlators of type a

- Compute $C^{\mathrm{a}}_{\mathbb{V}}(y_0, y_1)$ in two ways:

1) using **conformal invariance:**



$$C^{\mathrm{a}}_{\mathbb{H}}(z_0, z_1) = \frac{K}{|z_0 - z_1|^{2\Delta^{\mathrm{a}}}}$$

$$C^{\mathrm{a}}_{\mathbb{V}}(y_0, y_1) = \frac{K' \mathrm{e}^{\frac{\pi m}{n}\Delta^{\mathrm{a}}}}{(\mathrm{e}^{\frac{\pi m}{n}} - 1)^{2\Delta^{\mathrm{a}}}}$$

$$\xrightarrow{m \gg n} K' \mathrm{e}^{-\frac{\pi m}{n}\Delta^{\mathrm{a}}}$$

2) using the **transfer matrix**:

$$C^{\mathrm{a}}_{\mathbb{V}}(y_0, y_1) \xrightarrow{m \gg n} \tilde{K}\left(\frac{\Lambda_1}{\Lambda_0}\right)^{m/2} = \tilde{K}\,\mathrm{e}^{-\frac{\pi m}{n}(\Delta_{1,2} - \overbrace{\Delta_{1,1}}^{=0})} = \tilde{K}\,\mathrm{e}^{-\frac{\pi m}{n}\Delta_{1,2}}$$

- **Conformal weight:** $\Delta^{\mathrm{a}} = \Delta_{1,2}$

- For $\beta = 0$: $\Delta_{1,2} = -\frac{1}{8}$

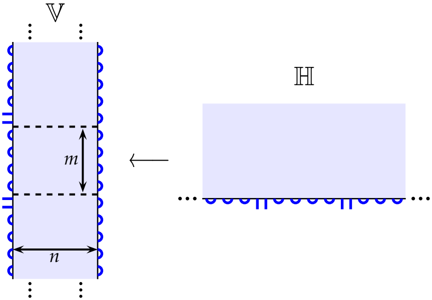
Correlators of type b

- Compute $C^{\mathrm{b}}_{\mathbb{V},}(y_0, y_1)$ in two ways:

1) using **conformal invariance:**

$$C^{\mathrm{b}}_{\mathbb{H}}(z_0, z_1) = \frac{K_0}{|z_0 - z_1|^{2\Delta^{\mathrm{b},0}}} + \frac{K_1}{|z_0 - z_1|^{2\Delta^{\mathrm{b},1}}}$$

$$C^{\mathrm{b}}_{\mathbb{V}}(y_0, y_1) \xrightarrow{m \gg n} K_0' \mathrm{e}^{-\frac{\pi m}{n}\Delta^{\mathrm{b},0}} + K_1' \mathrm{e}^{-\frac{\pi m}{n}\Delta^{\mathrm{b},1}}$$



2) using the **transfer matrix**:

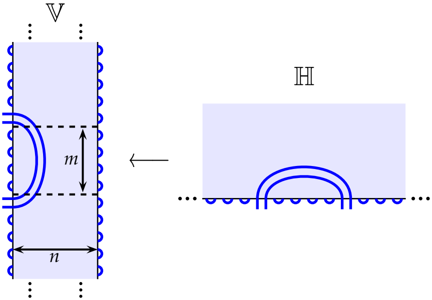
$$C^{\mathrm{b}}_{\mathbb{V}}(y_0, y_1) \xrightarrow{m \gg n} \tilde{K}_0 \left(\frac{\Lambda_0}{\Lambda_0}\right)^{m/2} + \tilde{K}_1 \left(\frac{\Lambda_2}{\Lambda_0}\right)^{m/2} = \tilde{K}_0\, \mathrm{e}^{-\frac{\pi m}{n}\Delta_{1,1}} + \tilde{K}_1\, \mathrm{e}^{-\frac{\pi m}{n}\Delta_{1,3}}$$

- **Conformal weights:** $\quad \Delta^{\mathrm{b},0} = \Delta_{1,1} \qquad \Delta^{\mathrm{b},1} = \Delta_{1,3}$

- This boundary condition change **is not a primary field**

Correlators of type b

- Compute $C^{\mathrm{b}}_{\mathbb{V},}(y_0, y_1)$ in two ways:



1) using **conformal invariance:**

$$C^{\mathrm{b}}_{\mathbb{H}}(z_0, z_1) = \frac{K_0}{|z_0 - z_1|^{2\Delta^{\mathrm{b},0}}} + \frac{K_1}{|z_0 - z_1|^{2\Delta^{\mathrm{b},1}}}$$

$$C^{\mathrm{b}}_{\mathbb{V}}(y_0, y_1) \xrightarrow{m \gg n} K_0' \mathrm{e}^{-\frac{\pi m}{n}\Delta^{\mathrm{b},0}} + K_1' \mathrm{e}^{-\frac{\pi m}{n}\Delta^{\mathrm{b},1}}$$

2) using the **transfer matrix**:

$$C^{\mathrm{b}}_{\mathbb{V}}(y_0, y_1) \xrightarrow{m \gg n} \tilde{K}_0 \left(\frac{\Lambda_0}{\Lambda_0}\right)^{m/2} + \tilde{K}_1 \left(\frac{\Lambda_2}{\Lambda_0}\right)^{m/2} = \tilde{K}_0\, \mathrm{e}^{-\frac{\pi m}{n}\Delta_{1,1}} + \tilde{K}_1\, \mathrm{e}^{-\frac{\pi m}{n}\Delta_{1,3}}$$

- **Conformal weights:** $\Delta^{\mathrm{b},0} = \Delta_{1,1}$ $\Delta^{\mathrm{b},1} = \Delta_{1,3}$

- This boundary condition change **is not a primary field**

Correlators of type b for $\beta = 0$

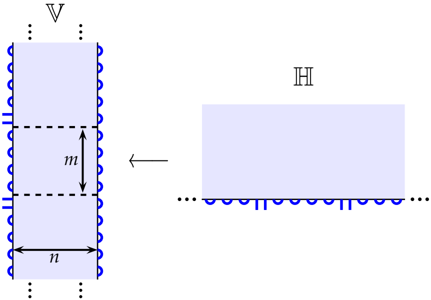
- For $\beta = 0$: $\Delta_{1,1} = \Delta_{1,3} = 0$
- Rank 2 Jordan cell in the transfer matrix: $D(\frac{\lambda}{2}) \simeq \begin{pmatrix} \Lambda_0 & 1 & 0 \\ 0 & \Lambda_0 & 0 \\ 0 & 0 & \ddots \end{pmatrix}$
- Compute $C^{\mathrm{b}}_{\mathbb{V}}(y_0, y_1)$ using the **transfer matrix:**

$$C^{\mathrm{b}}_{\mathbb{V}}(y_0, y_1) \xrightarrow{m \gg n} \tilde{K}_0 m + \tilde{K}_1 + \dots$$

- Using **conformal invariance:**

$$C^{\mathrm{b}}_{\mathbb{H}}(z_0, z_1) = \frac{K_0 \ln|z_0 - z_1| + K_1}{|z_0 - z_1|^{2\Delta^{\mathrm{b}}}}$$

$$C^{\mathrm{b}}_{\mathbb{V}}(z_0, z_1) = \mathrm{e}^{-\frac{\pi m}{n}\Delta^{\mathrm{b}}}(K_0' m + \tilde{K}_1')$$



- **Logarithmic field with $\Delta^{\mathrm{b}} = 0$**

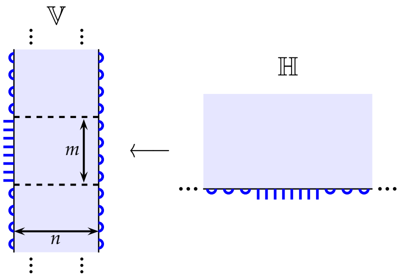
Correlators of type c

- Compute $C^{c}_{\mathbb{V}}(y_0, y_1)$ in two ways:

  1) using **conformal invariance:**

$$C^{c}_{\mathbb{H}}(z_0, z_1) = \frac{K}{|z_0 - z_1|^{2\Delta^c}}$$

$$C^{c}_{\mathbb{V}}(y_0, y_1) \xrightarrow{m \gg n} K' \mathrm{e}^{-\frac{\pi m}{n}\Delta^c}$$



  2) using the transfer matrix of the **Blob algebra:** (Jacobsen, Saleur '08)

$$C^{c}_{\mathbb{V}}(y_0, y_1) \xrightarrow{m \gg n} \tilde{K}\,\mathrm{e}^{-\frac{\pi m}{n}\Delta_{0,\frac{1}{2}}}$$

- **Conformal weight:** $\Delta^c = \Delta_{0,\frac{1}{2}}$

- For $\beta = 0$: $\Delta_{0,\frac{1}{2}} = -\frac{3}{32}$

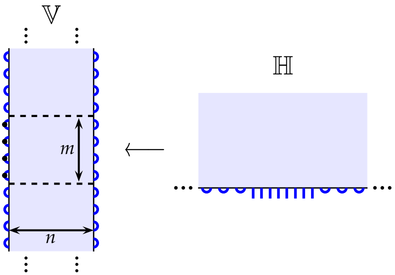
Correlators of type c

- Compute $C^{\mathrm{c}}_{\mathbb{V}}(y_0, y_1)$ in two ways:

  1) using **conformal invariance:**

$$C^{\mathrm{c}}_{\mathbb{H}}(z_0, z_1) = \frac{K}{|z_0 - z_1|^{2\Delta^{\mathrm{c}}}}$$

$$C^{\mathrm{c}}_{\mathbb{V}}(y_0, y_1) \xrightarrow{m \gg n} K' \mathrm{e}^{-\frac{\pi m}{n}\Delta^{\mathrm{c}}}$$



  2) using the transfer matrix of the **Blob algebra:** (Jacobsen, Saleur '08)

$$C^{\mathrm{c}}_{\mathbb{V}}(y_0, y_1) \xrightarrow{m \gg n} \tilde{K}\, \mathrm{e}^{-\frac{\pi m}{n}\Delta_{0,\frac{1}{2}}}$$

- **Conformal weight:** $\Delta^{\mathrm{c}} = \Delta_{0,\frac{1}{2}}$

- For $\beta = 0$: $\Delta_{0,\frac{1}{2}} = -\frac{3}{32}$

Correlators of type d,e,f

- Compute $C_{\mathbb{V}}^{\mathrm{d,e,f}}(y_0, y_1)$ in two ways:

1) using **conformal invariance:**

$$C_{\mathbb{V}}^{\mathrm{d,e,f}}(y_0, y_1) \xrightarrow{m \gg n} K' \mathrm{e}^{-\frac{\pi m}{n} \Delta^{\mathrm{d,e,f}}}$$

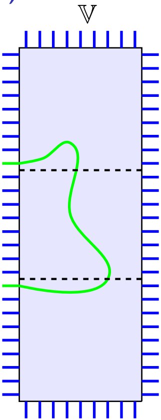
2) using the transfer matrix of the **2-blob Temperley-Lieb algebra:**

(Dubail, Jacobsen, Saleur '09)

$$C_{\mathbb{V}}^{\mathrm{d}}(y_0, y_1) \xrightarrow{m \gg n} \tilde{K} \mathrm{e}^{-\frac{\pi m}{n} \Delta_{1,0}}$$

$$C_{\mathbb{V}}^{\mathrm{e}}(y_0, y_1) \xrightarrow{m \gg n} \tilde{K} \mathrm{e}^{-\frac{\pi m}{n} \Delta_{1,-1}}$$

$$C_{\mathbb{V}}^{\mathrm{f}}(y_0, y_1) \xrightarrow{m \gg n} \tilde{K} \mathrm{e}^{-\frac{\pi m}{n} \Delta_{\frac{2\theta}{\lambda}+1, \frac{2\theta}{\lambda}+1}}$$



- **Conformal weights:** $\Delta^{\mathrm{d}} = \Delta_{1,0}$ $\quad \Delta^{\mathrm{e}} = \Delta_{1,-1}$ $\quad \Delta^{\mathrm{f}} = \Delta_{\frac{2\theta}{\lambda}+1, \frac{2\theta}{\lambda}+1}$

- For $\beta = 0$: $\quad \Delta_{1,0} = \frac{3}{8} \quad \Delta_{1,-1} = 1 \quad \Delta_{\frac{4\theta}{\pi}+1, \frac{4\theta}{\pi}+1} = \frac{\theta}{\pi}\left(1 + \frac{2\theta}{\pi}\right)$

Correlators of type d,e,f

- Compute $C_{\mathbb{V}}^{\mathrm{d,e,f}}(y_0, y_1)$ in two ways:

  1) using **conformal invariance:**

  $$C_{\mathbb{V}}^{\mathrm{d,e,f}}(y_0, y_1) \xrightarrow{m \gg n} K' \mathrm{e}^{-\frac{\pi m}{n}\Delta^{\mathrm{d,e,f}}}$$

  2) using the transfer matrix of the **2-blob Temperley-Lieb algebra:**

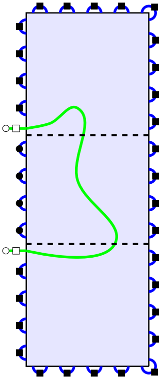
  (Dubail, Jacobsen, Saleur '09)

  $$C_{\mathbb{V}}^{\mathrm{d}}(y_0, y_1) \xrightarrow{m \gg n} \tilde{K}\,\mathrm{e}^{-\frac{\pi m}{n}\Delta_{1,0}}$$

  $$C_{\mathbb{V}}^{\mathrm{e}}(y_0, y_1) \xrightarrow{m \gg n} \tilde{K}\,\mathrm{e}^{-\frac{\pi m}{n}\Delta_{1,-1}}$$

  $$C_{\mathbb{V}}^{\mathrm{f}}(y_0, y_1) \xrightarrow{m \gg n} \tilde{K}\,\mathrm{e}^{-\frac{\pi m}{n}\Delta_{\frac{2\theta}{\lambda}+1,\frac{2\theta}{\lambda}+1}}$$

$\mathbb{V}$



- **Conformal weights:** $\Delta^{\mathrm{d}} = \Delta_{1,0}$ $\quad \Delta^{\mathrm{e}} = \Delta_{1,-1}$ $\quad \Delta^{\mathrm{f}} = \Delta_{\frac{2\theta}{\lambda}+1,\frac{2\theta}{\lambda}+1}$

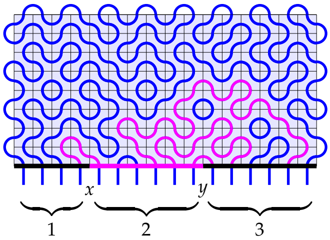
- For $\beta = 0$: $\quad \Delta_{1,0} = \frac{3}{8} \quad \Delta_{1,-1} = 1 \quad \Delta_{\frac{4\theta}{\pi}+1,\frac{4\theta}{\pi}+1} = \frac{\theta}{\pi}(1 + \frac{2\theta}{\pi})$

Valence bond entanglement entropy

- Correlator of type f on the upper-half plane:

$$C^{\mathrm{f}}_{\mathbb{H}}(z_0, z_1) = \frac{K}{|z_0 - z_1|^{2\Delta^{\mathrm{f}}}}$$

$$\Delta^{\mathrm{f}} = \Delta_{\frac{2\theta}{\lambda}+1, \frac{2\theta}{\lambda}+1}$$



$n_{12} = 1$
$n_{23} = 3$
$n_{13} = 1$

- Reminder: $\beta = 2\cos\lambda$ $\tau = \frac{\cos(\frac{\lambda}{2}+\theta)}{\cos(\frac{\lambda}{2})}$

- Valence bond entanglement entropy: (Jacobsen, Saleur '08)

$$\langle n_{12} + n_{23} \rangle = \frac{\sum_\sigma (n_{12}+n_{23})\beta^{n_\beta}}{\sum_\sigma \beta^{n_\beta}} = \frac{\mathrm{d}(\ln C^{\mathrm{f}}_{\mathbb{H}}(x,y))}{\mathrm{d}\tau}\bigg|_{\tau=1}$$

$$= \frac{2(\lambda/\pi)}{\pi(1-\lambda/\pi)} \frac{\cos(\lambda/2)}{\sin(\lambda/2)} \ln(y-x)$$

Outlook

Overview:

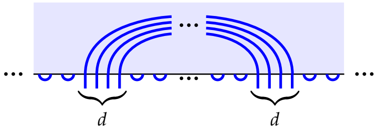
- perfect agreement between exact results and CFT predictions
- corner free energy for the logarithmic field: $\sim \ln(\ln n)$

Possible future avenues: extend the method to

- more types of two-points functions
- exact derivations for other values of β
- periodic boundary conditions
- correlators in the bulk

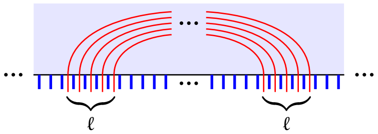
More two-point correlators

- Correlator of d defects in a Dirichlet boundary:



$$\longrightarrow \qquad \Delta = \Delta_{1,1+d}$$

- Correlator of ℓ clusters in a Neumann boundary:



$$\longrightarrow \qquad \Delta = \Delta_{1,1-\ell}$$