On the scaling limit of the Fateev-Zamolodchikov spin chain

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Fateev-Zamolodchikov (1982) spin chain

- $$\boxed{\mathbb{H}_{\rm FZ} = -\sum_{s=1}^{N}\sum_{l=1}^{n-1}\frac{(\mathtt{X}_s)^l + (\mathtt{U}_s\mathtt{U}_{s+1}^{\dagger})^l}{n\sin(\frac{\pi l}{n})}}$$

  $\mathtt{X}$, $\mathtt{U}$ – $n\times n$ matrices cyclic matrices: $\mathtt{X}^n = \mathtt{U}^n = 1,\ \mathtt{XU} = \omega\ \mathtt{UX}$, i.e.

  $$\mathtt{X}^{\alpha}_{\beta} = \delta_{\alpha+1,\beta(\mathrm{mod}\, n)}\ , \qquad \mathtt{U}^{\alpha}_{\beta} = \omega^{\alpha}\ \delta_{\alpha,\beta}, \qquad \omega = \mathrm{e}^{-\frac{2\pi \mathrm{i}}{n}}$$

- $\mathbb{Z}_n$ invariance: $Z = \prod_{s=1}^{N}\mathtt{X}_s,\quad [\mathbb{H}_{\rm FZ}, \mathtt{Z}] = 0 \quad\Longrightarrow\quad Z = \omega^{M_+}$

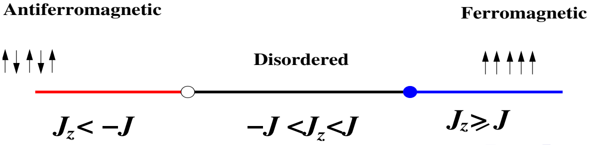
- Twisted boundary conditions: $\mathtt{U}_{N+1} = \omega^{M_-}\ \mathtt{U}_1\ , \quad \mathtt{X}_{N+1} = \mathtt{X}_1$

- **Scaling limit?**

XXZ spin $\frac{1}{2}$ chain

$$\mathbb{H}_{\mathrm{XYZ}} = -\sum_{k=1}^{N} \left(J_x S_k^x S_{k+1}^x + J_y S_k^y S_{k+1}^y + J_z S_k^z S_{k+1}^z \right)$$

- Spin $\frac{1}{2}$: $S^a = \frac{1}{2}\,\sigma^a$
- $J_x = J_y = J > 0$
- XXZ spin-$\frac{1}{2}$ chain is an exactly solvable model [Bethe'31, Lieb'67, Sutherland'67]

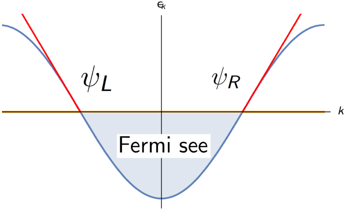


Thirring'58 (Tomanaga'50-Luttinger'63) model

Jordan-Wigner transformation: $\Psi_n^\dagger = \prod_{j<n} \sigma_j^z \sigma_n^-\ , \quad \Psi_n = \prod_{j<n} \sigma_j^z \sigma_n^+$

$$\mathbb{H}_{\rm XXZ} = -\sum_n 2J\,(\Psi_n^\dagger \Psi_{n+1} + \Psi_{n+1}^\dagger \Psi_n) + J_z\,(1 - 2\Psi_n^\dagger\Psi_n)(1 - 2\Psi_{n+1}^\dagger \Psi_{n+1})$$

Scaling limit $(N,\ J \to \infty,\ \mathtt{R} = N/J - \text{fixed})$



$$\Psi_n \approx \mathrm{e}^{\frac{\mathrm{i}\pi n}{2}}\ \psi_R + \mathrm{e}^{-\frac{\mathrm{i}\pi n}{2}}\ \psi_L$$

$$\mathbb{H}_{\rm XXZ} \to \mathbf{H}_{\rm Thirring} = \int_0^{\mathrm{R}} \mathrm{d}x\,\Big(\mathrm{i}\,\psi_L^\dagger \partial_x \psi_L - \mathrm{i}\,\psi_R^\dagger \partial_x \psi_R + \underbrace{\mathrm{g}_{4\mathrm{F}}}_{\mathrm{g}_{4\mathrm{F}}(J_z/J)}\,\psi_L^\dagger \psi_R^\dagger \psi_L \psi_R\Big)$$

Bosonization

Massless Thirring model = Massless Gaussian model

$$\mathbf{H}_{\rm Gauss} = \int_0^R \frac{\mathrm{d}x}{4\pi}\ \left((\partial_t\Phi)^2 + (\partial_x\Phi)^2\right)$$

$$\Phi(t,x) = \varphi(t+x) + \bar\varphi(t-x)$$

$$\varphi(z) = \varphi_0 + \frac{2\pi z}{R}\,\hat p + \mathrm{i}\sum_{n\neq 0}\frac{a_n}{n}\,\mathrm{e}^{-\frac{2\pi \mathrm{i} n}{R}z}\ , \qquad \bar\varphi(\bar z) = \bar\varphi_0 + \frac{2\pi\bar z}{R}\,\hat{\bar p} + \ldots$$

$$[a_n, a_m] = \ \tfrac{n}{2}\,\delta_{n+m,0}\ , \qquad [\varphi_0, \hat p] = \tfrac{\mathrm{i}}{2}$$

$$[\bar a_n, \bar a_m] = \ \tfrac{n}{2}\,\delta_{n+m,0}\ , \qquad [\bar\varphi_0, \hat{\bar p}] = \tfrac{\mathrm{i}}{2}$$

Fock space $\mathcal{F}_p$:

$$a_{-n_k}\ldots a_{-n_1}\,|\,p\,\rangle \qquad (n_1,\ldots,n_k>0)$$

$$a_n|\,p\,\rangle = 0 \quad (n>0)\,, \qquad \hat p|\,p\,\rangle = p\,|\,p\,\rangle\ .$$

$$H\ :\quad \mathcal{F}_p\otimes\bar{\mathcal{F}}_{\bar p}\mapsto\mathcal{F}_p\otimes\bar{\mathcal{F}}_{\bar p}$$

Scaling limit of the XXZ spin $\frac{1}{2}$ chain

$$\mathbb{H}_{\rm XXZ} = -\frac{1-g}{2\sin(\pi g)} \sum_{k=1}^{N} \left(\sigma_k^x \sigma_{k+1}^x + \sigma_k^y \sigma_{k+1}^y + \Delta\, \sigma_k^z \sigma_{k+1}^z \right)$$

$$\Delta = \cos(\pi g) \qquad (0 < g < 1)$$

- Twisted boundary conditions: $\sigma_1^{\pm} = \mathrm{e}^{\pm 2\pi \mathrm{i}\theta}\ \sigma_{N+1}^{\pm}$
- $\mathbf{S^z} = \frac{1}{2} \sum_{j=1}^{N} \sigma_j^z$: $\quad [\mathbb{H}_{\rm XXZ}, \mathbf{S^z}] = 0.$
- In the limit $N \to \infty$

$$\mathbb{H}_{\rm XXZ}\big|_{\theta, S^z} = N\mathcal{E}_0 + \tfrac{2\pi}{N}\ (L_0 + \bar{L}_0)\big|_{\mathcal{F}_p \otimes \bar{\mathcal{F}}_{\bar{p}}} + o\big(N^{-1}\big)$$

$$L_0 = p^2 - \tfrac{1}{24} + 2\sum_{n>0} a_{-n} a_n \quad , \qquad \bar{L}_0 = \bar{p}^2 - \tfrac{1}{24} + \ldots$$

$$p = \tfrac{\theta - gS^z}{2\sqrt{g}} \ , \qquad \bar{p} = \tfrac{\theta + gS^z}{2\sqrt{g}}$$

Fateev-Zamolodchikov spin chain

- $$\mathbb{H}_{\rm FZ} = -\sum_{s=1}^{N}\sum_{l=1}^{n-1}\frac{(\mathtt{X}_s)^l + (\mathtt{U}_s\mathtt{U}^\dagger_{s+1})^l}{n\sin(\frac{\pi l}{n})}$$

$$\mathtt{X}^n = \mathtt{U}^n = 1,\ \ \mathtt{XU} = \omega\ \mathtt{UX} \qquad \big(\omega = \mathrm{e}^{-\frac{2\pi \mathrm{i}}{n}}\big)$$

- $\mathbb{Z}_n$ charge: $Z = \prod_{s=1}^{N}\mathtt{X}_s, \quad [\mathbb{H}_{\rm FZ}, \mathtt{Z}] = 0 \quad \Longrightarrow \quad Z = \omega^{M_+}$
- Twisted boundary conditions: $\mathtt{U}_{N+1} = \omega^{M_-}\ \mathtt{U}_1\ , \quad \mathtt{X}_{N+1} = \mathtt{X}_1$

$$\mathbb{P} = \delta^{\alpha_1}_{\beta_2}\delta^{\alpha_2}_{\beta_3}\ldots\delta^{\alpha_N}_{\beta_1+M_-}\ \ : \qquad [\mathbb{H}_{\rm FZ}, \mathbb{P}] = 0$$

- Scaling limit:

$$\mathbb{H}_{\rm FZ}\big|_{M_-,M_+} = N\mathcal{E}_0 + \tfrac{2\pi}{N}\ (L_0 + \bar{L}_0)\big|_{\boxed{V\otimes\bar{V}}} + o\big(N^{-1}\big)$$

$$\mathbb{P}\big|_{M_-,M_+} = \exp\big(\tfrac{2\pi \mathrm{i}}{N}\,(L_0 - \bar{L}_0)\big)\big|_{\boxed{V\otimes\bar{V}}}$$

Yang-Baxter Algebra

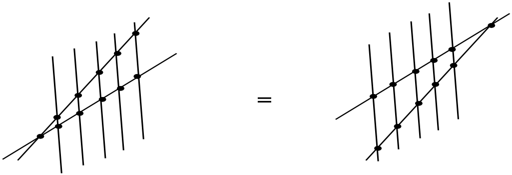
$$\mathcal{L}(\lambda) = \begin{pmatrix} \lambda\, q^{+\frac{\mathrm{h}}{2}} - \lambda^{-1}\, q^{-\frac{\mathrm{h}}{2}} & (q - q^{-1})\ \mathrm{e}_- \\ (q - q^{-1})\ \mathrm{e}_+ & \lambda\, q^{-\frac{\mathrm{h}}{2}} - \lambda^{-1}\, q^{+\frac{\mathrm{h}}{2}} \end{pmatrix}$$

$$U_q(\mathfrak{sl}_2): \quad [\mathrm{h},\, \mathrm{e}_\pm] = \pm 2\, \mathrm{e}_\pm\ , \qquad [\mathrm{e}_+,\, \mathrm{e}_-] = \frac{q^{\mathrm{h}} - q^{-\mathrm{h}}}{q - q^{-1}}$$

[Kulish, Reshetikhin, Sklyanin'81]

- $\boldsymbol{R}_{6V}(\lambda_2/\lambda_1)\, (\mathcal{L}(\lambda_1) \otimes \mathbf{1})\, (\mathbf{1} \otimes \mathcal{L}(\lambda_2)) = (\mathbf{1} \otimes \mathcal{L}(\lambda_2))\, (\mathcal{L}(\lambda_1) \otimes \mathbf{1})\, \boldsymbol{R}_{6V}(\lambda_2/\lambda_1)$

- $\boldsymbol{M}(\lambda) = \overset{\leftarrow\ N}{\prod_{s=1}} \mathcal{L}^{(\mathrm{s})}(\lambda)$



$$\boldsymbol{R}_{6V}(\lambda_2/\lambda_1)\, (\boldsymbol{M}(\lambda_1) \otimes \mathbf{1})\, (\mathbf{1} \otimes \boldsymbol{M}(\lambda_2)) = (\mathbf{1} \otimes \boldsymbol{M}(\lambda_2))\, (\boldsymbol{M}(\lambda_1) \otimes \mathbf{1})\, \boldsymbol{R}_{6V}(\lambda_2/\lambda_1)$$

Transfer-matrix for the XXZ spin chain

$$T(\lambda) = \mathrm{Tr}\Big[\, \boldsymbol{M}(\lambda)\, q^{(\mathfrak{a}+\mathfrak{b}h)\sigma_3} \,\Big]\,, \qquad h = \sum_s \mathtt{h}^{(s)} \;\;:\quad [T(\lambda), T(\lambda')] = 0$$

- 2D irrep of $U_q(\mathfrak{sl}_2)$: $\quad [\mathtt{h},\, \mathtt{e}_\pm] = \pm\, 2\, \mathtt{e}_\pm \;, \qquad [\mathtt{e}_+,\, \mathtt{e}_-] = \frac{q^{\mathtt{h}} - q^{-\mathtt{h}}}{q - q^{-1}}$

Casimir : $\frac{1}{2}\big[(q+q^{-1})\,(q^{\mathtt{h}}+q^{-\mathtt{h}}) + (q-q^{-1})^2\;(\mathtt{e}_-\mathtt{e}_+ + \mathtt{e}_+\mathtt{e}_-)\,\big] = q^{2\ell+1} + q^{-2\ell-1}$

$\boxed{\ell = \frac{1}{2}}$: $\quad \mathtt{e}_\pm = \sigma_\pm \;, \qquad \mathtt{h} = \sigma_3, \qquad \mathfrak{a} = \frac{\theta}{\mathfrak{g}} \;:\quad [T(\lambda), \mathbb{H}_{\mathrm{XXZ}}] = 0$

- $T - Q$-equation

$$Q(\lambda)\, T(\lambda) \;=\; \big(1-\lambda^2 q^{-1}\big)^N\, Q(\lambda q) \;+\; \big(1-\lambda^2 q\big)^N\, Q(\lambda q^{-1})$$

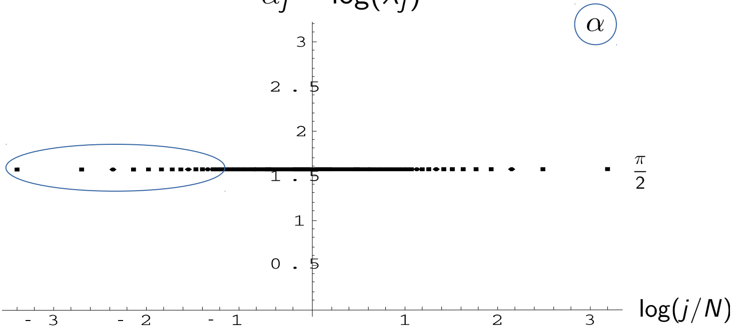
BA equations

$$\left[\frac{1-\lambda_k^2\, q}{1-\lambda_k^2\, q^{-1}}\right]^N \;=\; -q^{-2S^z}\, \mathrm{e}^{2\mathrm{i}\theta} \prod_{j=1}^{\frac{N}{2}+S^z} \frac{\lambda_j^2 - \lambda_k^2\, q^2}{\lambda_j^2 - \lambda_k^2\, q^{-2}}$$

Bethe roots for the vacuum state for $|J_z| < J$

$$\left(\frac{\sinh\left(\alpha_n + \frac{\mathrm{i}\pi g}{2}\right)}{\sinh\left(\alpha_n - \frac{\mathrm{i}\pi g}{2}\right)}\right)^N = -\mathrm{e}^{2\pi\mathrm{i}\theta} \prod_{j=1}^{\frac{N}{2}+S^z} \frac{\sinh(\alpha_n - \alpha_j + \mathrm{i}\pi g)}{\sinh(\alpha_n - \alpha_j - \mathrm{i}\pi g)} \qquad (\lambda_j = \mathrm{e}^{\alpha_j})$$

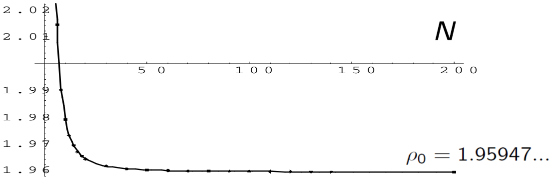
$$\alpha_j = \log(\lambda_j)$$



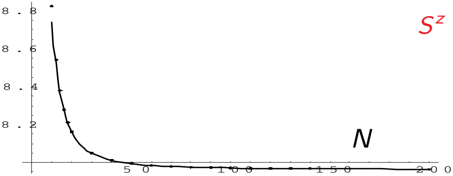
Ground state for $S^z = \theta = 0\,,\ g = \frac{1}{3}\ \ (J_z/J = \frac{1}{2}),\ N = 200$

Bethe roots in the limit $N \to \infty$

$$\rho_n = -\lim_{N\to\infty} \left(N^{2-2g}\ \lambda_{n+1}^2 \right), \quad n = 0, 1, 2 \ldots - \textbf{is fixed}$$



$S^z = \theta = 0\,,\ g = \frac{1}{3}$



$\rho_1 = 7.96362\ldots$

ODE/IM correspondence

Dorey, Tateo'98, Bazhanov, Lukyanov, Zamolodchikov'98]

Let $\{\, E_n \,\}_{n=0}^{\infty}$ be an ordered spectral set of eigenvalues of

$$\left[\, -\frac{\mathrm{d}^2}{\mathrm{d}z^2} + \frac{\ell(\ell+1)}{z^2} + z^{2\alpha} - E \,\right] \psi = 0 \ ,$$

then

$$\lim_{N\to\infty} \left(N^{2-2g}\ \lambda_{n+1}^2 \right) = -\left[\frac{\sqrt{\pi}\,\Gamma\left(1+\frac{1}{2\alpha}\right)}{2\,\Gamma\left(\frac{3}{2}+\frac{1}{2\alpha}\right)} \right]^{\frac{2\alpha}{1+\alpha}} E_n$$

provided

$$\alpha = \frac{1}{g} - 1 \,, \qquad \ell = \frac{\theta}{g} - S^z - \frac{1}{2} \ .$$

Scaling limit of Q and T

- $$Q_{\rm CFT} = \lim_{N\to\infty} N^{(1-g)(S^z-\frac{\theta}{g})}\, Q\big(N^{g-1}\,\lambda\big) = \lambda^{\frac{\theta}{g}-S^z}\prod_{j=0}^{\infty}\Big(1-\frac{\lambda^2}{\rho_j}\Big)$$

  $$T_{\rm CFT} = \lim_{N\to\infty}\big(-\lambda\big)^N\, T\big(N^{g-1}\,\lambda\big)$$

  $$Q_{\rm CFT},\ T_{\rm CFT}\ :\ \mathcal{F}_P \mapsto \mathcal{F}_p \qquad \Big(p=\tfrac{\theta-gS^z}{2\sqrt{g}}\Big)$$

- $$\bar{Q}_{\rm CFT} = \lim_{N\to\infty} N^{(g-1)(S^z+\frac{\theta}{g})}\,\big[Q\big(N^{1-g}\,\lambda\big)/(N^{1-g}\lambda)^N\big]$$

  $$\bar{T}_{\rm CFT} = \lim_{N\to\infty}(-\lambda)^{-N}\, T\big(N^{1-g}\lambda\big)$$

  $$\bar{Q}_{\rm CFT},\ \bar{T}_{\rm CFT}\ :\ \bar{\mathcal{F}}_{\bar{p}} \mapsto \bar{\mathcal{F}}_{\bar{p}} \qquad \Big(\bar{p}=\tfrac{\theta+gS^z}{2\sqrt{g}}\Big)$$

Heisenberg representation of $U_q(\mathfrak{sl}_2)$ [Izergin, Korepin'81]

- $U_q(\mathfrak{sl}_2)$: $\quad [\mathtt{h},\, \mathsf{e}_\pm] = \pm\, 2\, \mathsf{e}_\pm \,, \qquad [\mathsf{e}_+,\, \mathsf{e}_-] = \frac{q^{\mathtt{h}} - q^{-\mathtt{h}}}{q - q^{-1}}$

$$\mathsf{e}_\pm = \mathrm{e}^{\mp \frac{1}{2}\mathtt{Q}}\ \frac{\sinh\big(\,\frac{\mathtt{P}}{2} \pm \frac{\mathrm{i}\hbar}{4}(2\ell+1)\,\big)}{\sin\big(\frac{1}{2}\hbar\big)}\ \mathrm{e}^{\mp \frac{1}{2}\mathtt{Q}}\,, \quad \mathtt{h} \ = -\frac{2\mathrm{i}}{\hbar}\ \mathtt{P}$$

$$[\,\mathtt{Q}\,,\mathtt{P}\,] = \mathrm{i}\,\hbar\,, \qquad q = \mathrm{e}^{\frac{\mathrm{i}\hbar}{2}}$$

- ℓ is arbitrary and is related to the value of the quantum Casimir

$$\tfrac{1}{2}\,\Big[(q+q^{-1})\,\big(q^{\mathtt{h}}+q^{-\mathtt{h}}\big) + (q-q^{-1})^2\ \big(\mathsf{e}_-\mathsf{e}_+ + \mathsf{e}_+\mathsf{e}_-\big)\,\Big] = q^{2\ell+1} + q^{-2\ell-1}$$

- $\boldsymbol{M}(\lambda) = \overleftarrow{\prod}_{s=1}^{N} \boldsymbol{\mathcal{L}}^{(\mathrm{s})}(\lambda)\,, \qquad \boldsymbol{\mathcal{L}}(\lambda) = \begin{pmatrix} \lambda\, q^{+\frac{\mathtt{h}}{2}} - \lambda^{-1}\, q^{-\frac{\mathtt{h}}{2}} & (q-q^{-1})\ \mathsf{e}_- \\ (q-q^{-1})\ \mathsf{e}_+ & \lambda\, q^{-\frac{\mathtt{h}}{2}} - \lambda^{-1}\, q^{+\frac{\mathtt{h}}{2}} \end{pmatrix}$

$$T(\lambda) = \mathrm{Tr}\Big[\,\boldsymbol{M}(\lambda)\, q^{(\mathfrak{a}+\mathfrak{b}h)\sigma_3}\,\Big]\,, \qquad h = \sum_s \mathtt{h}^{(s)}\ : \qquad [T(\lambda),\, T(\lambda')] = 0$$

T-operator for the FZ spin chain

[Bazhanov, Stroganov'89, Baxter, Bazhanov, Perk'90]

$$T(\lambda) = (-\lambda)^N \,\mathrm{Tr}\left[\overleftarrow{\mathcal{P}}\Big(\prod_{s=1}^{N}\big(\boldsymbol{\mathcal{L}}_-(\mathtt{U}_s,\mathtt{X}_s) - \lambda^2\boldsymbol{\mathcal{L}}_+(\mathtt{U}_s,\mathtt{X}_s)\big)\Big)\begin{pmatrix} q^{-a}\,\mathtt{Z}^{-1} & 0 \\ 0 & q^{+a}\end{pmatrix}\right]$$

$$\boldsymbol{\mathcal{L}}_- = \begin{pmatrix} 1 & 0 \\ -\mathrm{i}\,\big(q^{-\ell-1} - q^{+\ell-1}\,\mathtt{X}\big)\,\mathtt{U}^{-1} & \mathtt{X}\end{pmatrix}, \quad \boldsymbol{\mathcal{L}}_+ = \begin{pmatrix} \mathtt{X} & \mathrm{i}\,\big(q^{1+\ell} - q^{1-\ell}\,\mathtt{X}\big)\,\mathtt{U} \\ 0 & 1\end{pmatrix}$$

$$\mathtt{X} = \mathrm{e}^{\mathtt{P}}, \quad \mathtt{U} = \mathrm{e}^{\mathtt{Q}} \;:\quad \mathtt{UX} = q^2\,\mathtt{XU}, \qquad \mathtt{Z} = \prod_{s=1}^{N}\mathtt{X}_s$$

- n-dimensional of irrep for the Heisenberg group $\mathtt{UX} = q^2\ \mathtt{XU}$:

$$\mathtt{X}^{\alpha}_{\beta} = \delta_{\alpha+1,\beta\,(\mathrm{mod}\,n)}\,, \qquad \mathtt{U}^{\alpha}_{\beta} = \omega^{\alpha}\ \delta_{\alpha,\beta}, \qquad \omega = q^{-2} = \mathrm{e}^{-\frac{2\pi\mathrm{i}}{n}}$$

- $\boxed{\ell = -\frac{1}{2}}$: $\quad [\mathbb{H}_{\mathrm{FZ}}, T(\lambda)] = 0$

$$\mathbb{H}_{\mathrm{FZ}} = -\sum_{s=1}^{N}\sum_{l=1}^{n-1}\frac{(\mathtt{X}_s)^l + (\mathtt{U}_s\mathtt{U}^{\dagger}_{s+1})^l}{n\sin(\frac{\pi l}{n})}\,, \quad \mathtt{U}_N = q^{-2M_-}\mathtt{U}_1, \;\; \mathtt{Z} = q^{-2M_+}$$

BA equations for the FZ spin chain

- **T-Q relations**

  Odd n

  $$T(\mu)\,Q_\pm(\mu) = (1 \mp q^{-\frac{1}{2}}\,\mu)^{2N}\ \ Q_\mp(q^{-1}\mu) + (1 \mp q^{+\frac{1}{2}}\,\mu)^{2N}\ \ Q_\mp(q^{+1}\mu)$$

  Even n:

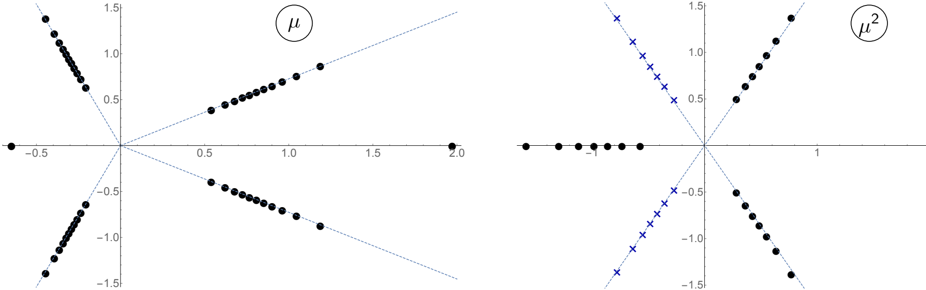
  $$T(\mu)\,Q_-(\mu) = Q_+(q^{-1}\mu) + Q_+(q^{+1}\mu)$$
  $$T(\mu)\,Q_+(\mu) = (1 - q^{-1}\,\mu^2)^{2N}\,Q_-(q^{-1}\mu) + (1 - q^{+1}\,\mu^2)^{2N}\,Q_-(q^{+1}\mu)$$

- **BA equations** $(\mathtt{U}_N = q^{-2M_-}\mathtt{U}_1,\ \ \mathtt{Z} = q^{-2M_+})$

$$\prod_{i=1}^{(n-1)N-2M_+} \frac{\mu_i + q^{-1}\,\mu_l}{\mu_i + q^{+1}\,\mu_l} = -q^{2\mathfrak{m}} \left(\frac{1 - q^{+\frac{1}{2}}\,\mu_l}{1 - q^{-\frac{1}{2}}\,\mu_l}\right)^{2N} \qquad (n-\text{odd})$$

$$\prod_{i=1}^{\frac{nN}{2}-M_+} \frac{v_i - q^{-2}\,w_l}{v_i - q^{+2}\,w_l} = -q^{2\mathfrak{m}}, \quad \prod_{i=1}^{\frac{(n-2)N}{2}-M_+} \frac{w_i - q^{-2}\,v_l}{w_i - q^{+2}\,v_l} = -q^{2\mathfrak{m}} \left(\frac{1 - q^{+1}\,v_l}{1 - q^{-1}\,v_l}\right)^{2N} \quad (n-\text{even})$$

$\mathfrak{m} = M_+ - M_-$ [similar to Albertini'92; Ray'97]

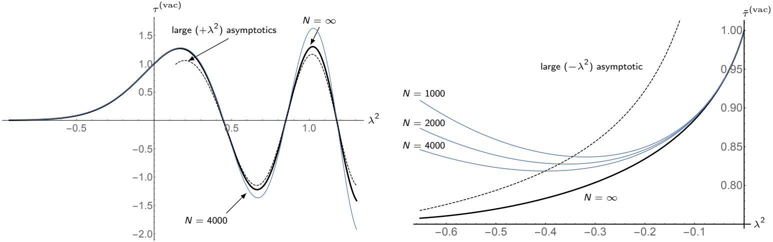


On the left panel, the roots of $Q_+^{(\mathrm{vac})}(\mu)$ are depicted in the complex plane for $n = 5$, $M_+ = M_- = 1$ and $N = 12$. On the right panel, the roots of $Q_+^{(\mathrm{vac})}$ (circles) and $Q_-^{(\mathrm{vac})}$ (crosses) as functions of μ^2 are shown for $n = 6$, $M_+ = 2$, $M_- = 1$ and $N = 8$.

Scaling limit of the transfer-matrix

$$\tau(\lambda) = \underset{N\to\infty}{\mathrm{s}\lim}\ F^{(N)}(\lambda)\, T^{(N)}\Big(\big(\tfrac{\pi}{N}\big)^{\frac{1}{n}}\,\lambda\Big)$$

$$F^{(N)}(\lambda) = \begin{cases} \exp\Big(\sum_{l=1}^{[\frac{n}{2}]} \frac{\pi^{\frac{2l}{n}}}{l\cos(\frac{\pi l}{n})}\, N^{1-\frac{2l}{n}}\,\lambda^{2l}\Big) & (n=3,\,5,\,\ldots) \\ \Big(\frac{N\mathrm{e}}{\pi}\Big)^{\frac{4}{n}\lambda^n}\ \exp\Big(\sum_{l=1}^{[\frac{n}{2}]-1} \frac{\pi^{\frac{2l}{n}}}{l\cos(\frac{\pi l}{n})}\, N^{1-\frac{2l}{n}}\,\lambda^{2l}\Big) & (n=2,\,4,\ldots) \end{cases}$$



On the left panel, a plot of $\tau^{(\mathrm{vac})}$ for $n=3$, $M_+=1$, $M_-=0$ compared to its large $(+\lambda^2)$. On the right panel, $\tilde\tau^{(\mathrm{vac})} = \tau^{(\mathrm{vac})}\exp\big(2\pi\,(-\lambda^2)^{\frac{3}{2}}\big)$ is plotted and compared with the large $(-\lambda^2)$ asymptotic. The scaling function was numerically estimated by interpolating to $N=\infty$ the data for $N=500, 1000, 2000, 4000$.

Universal R-matrix

The algebraic structure underlying YB relation was clarified within the theory of Hopf algebras [Drinfeld'86]. A basic example is $U_q(\widehat{\mathfrak{g}})$ [Drinfeld'86; Jimbo'86].

- The universal R-matrix: $\mathcal{R} \in U_q(\widehat{\mathfrak{b}}_+) \otimes U_q(\widehat{\mathfrak{b}}_-)$

$$\mathcal{R}^{12}\,\mathcal{R}^{13}\,\mathcal{R}^{23} = \mathcal{R}^{23}\,\mathcal{R}^{13}\,\mathcal{R}^{12} \qquad (*)$$

- If we consider the evaluation homomorphism of $U_q(\widehat{\mathfrak{g}})$ to the loop algebra $U_q(\mathfrak{g})[\lambda, \lambda^{-1}]$ and specify an N-dimensional matrix representation of $U_q(\mathfrak{g})$, then

$$\boldsymbol{L}(\lambda) = \big(\pi(\lambda) \otimes 1\big)[\mathcal{R}]$$

is a $U_q(\widehat{\mathfrak{b}}_-)$-valued $N \times N$ matrix whose entries depend on an auxiliary parameter λ.

- (*) becomes the Yang-Baxter relation

$$\boldsymbol{R}(\lambda_2/\lambda_1)\,\big(\boldsymbol{L}(\lambda_1) \otimes \mathbf{1}\big)\,\big(\mathbf{1} \otimes \boldsymbol{L}(\lambda_2)\big) = \big(\mathbf{1} \otimes \boldsymbol{L}(\lambda_2)\big)\,\big(\boldsymbol{L}(\lambda_1) \otimes \mathbf{1}\big)\boldsymbol{R}(\lambda_2/\lambda_1)$$

$$\boldsymbol{R}(\lambda_2/\lambda_1) = \big(\pi(\lambda_1) \otimes \pi(\lambda_2)\big)[\mathcal{R}]$$

Universal R-matrix for $U_q(\widehat{sl}(2))$ [Khoroshkin, Tolstoy'92]

$$(h_i, x_i, y_i) \in U_q(\widehat{sl}(2)) \qquad (i = 0, 1)$$

$$[h_i, x_j] = -a_{ij}\ x_j\ , \quad [h_i, y_j] = a_{ij}\, y_j, \quad [y_i, x_j] = \delta_{ij}\ \tfrac{q^{h_i} - q^{-h_i}}{q - q^{-1}}$$

$$x_i^3 x_j - [3]_q x_i^2 x_j x_i + [3]_q x_i x_j x_i^2 - x_j x_i^3 = 0\ , \quad (x \mapsto y) \quad (q - \text{Serre relations})$$

- The evaluation homomorphism $U_q(\widehat{\mathfrak{sl}_2}) \mapsto U_q(\mathfrak{sl}_2)[\lambda, \lambda^{-1}]$

$$y_0 \mapsto \lambda\, q^{-\frac{\mathtt{h}}{2}}\, \mathtt{e}_+\ , \qquad y_1 \mapsto \lambda\, q^{\frac{\mathtt{h}}{2}}\, \mathtt{e}_-\ , \qquad h_0 \mapsto \mathtt{h}\ , \qquad h_1 \mapsto -\mathtt{h}$$

$$U_q(\mathfrak{sl}_2): \quad [\mathtt{h},\, \mathtt{e}_\pm] = \pm\, 2\, \mathtt{e}_\pm\ , \qquad [\mathtt{e}_+,\, \mathtt{e}_-] = \tfrac{q^{\mathtt{h}} - q^{-\mathtt{h}}}{q - q^{-1}}$$

- $\boldsymbol{L}(\lambda) = \big(\pi(\lambda) \otimes 1\big)[\mathcal{R}] = \Big[1 + \lambda\, (q - q^{-1})\, (x_0 E_+ + x_1 E_-)$

$$+\lambda^2\, \tfrac{(q - q^{-1})^2}{1 + q^2} \big(x_0^2 E_+^2 + x_1^2 E_-^2 + \frac{q^2\, x_0 x_1 - x_1 x_0}{1 - q^{-2}}\ E_+ E_- +$$

$$+ \frac{q^2\, x_1 x_0 - x_0 x_1}{1 - q^{-2}}\ E_- E_+\big) + \ldots\Big]\, q^{-\frac{1}{2}\mathtt{h}\, h_0} \qquad (E_\pm = q^{\pm\frac{\mathtt{h}}{2}}\, \mathtt{e}_\pm)$$

1 field rep for $U_q(\widehat{\mathfrak{b}}_-)$ [Bazhanov, Lukyanov, Zamolodchikov'94]

$$x_0 = \frac{1}{q - q^{-1}} \int_0^R \mathrm{d}z\, V^+(z)\ , \qquad x_1 = \frac{1}{q - q^{-1}} \int_0^R \mathrm{d}z\, V^-(z)$$

The vertex operators $V^{\pm}(z) = \mathrm{e}^{\mp 2\mathrm{i}\beta\varphi}(z)$ are built from the bosonic field

$$\varphi(z) = \varphi_0 + \frac{2\pi z}{R}\,\hat{p} + \mathrm{i} \sum_{n\neq 0} \frac{a_n}{n}\,\mathrm{e}^{-\frac{2\pi \mathrm{i} n}{R} z}$$

$$[a_n, a_m] = \tfrac{n}{2}\,\delta_{n+m,0}\ , \qquad [\varphi_0, \hat{p}] = \tfrac{\mathrm{i}}{2}$$

$$h_0 = \tfrac{2}{\beta}\,\hat{p} \qquad (q = \mathrm{e}^{-\mathrm{i}\pi\beta^2})$$

Fock space $\mathcal{F}_p$ (the highest weight module of the Heisenberg algebra)

$$x_0 : \ \ \mathcal{F}_p \mapsto \mathcal{F}_{p-\beta}\ , \qquad x_1 : \ \ \mathcal{F}_p \mapsto \mathcal{F}_{p+\beta}$$

The matrix elements of $\boldsymbol{L}(\lambda)$ are operators in $\oplus_{n=-\infty}^{\infty} \mathcal{F}_{p+n\beta}$.

Using the commutation relations

$$V^{\sigma_1}(z_1)\, V^{\sigma_2}(z_2) = q^{2\sigma_1\sigma_2}\, V^{\sigma_2}(z_2)\, V^{\sigma_1}(z_1)\,, \qquad z_2 > z_1 \qquad (\sigma_{1,2} = \pm)$$

the monomials built from the generators x_0, x_1 can be expressed in terms of the ordered integrals

$$J(\sigma_1, \ldots, \sigma_m) = \int\limits_{R>z_1>z_2>\ldots>z_m>0} \mathrm{d}z_1 \ldots \mathrm{d}z_m\, V^{\sigma_1}(z_1) \ldots V^{\sigma_m}(z_m)$$

$$\boldsymbol{L}(\lambda) = \sum_{m=0}^{\infty} \lambda^m \sum_{\sigma_1\ldots\sigma_m=\pm} \left(q^{\frac{\mathtt{h}}{2}\sigma_1} \mathsf{e}_{\sigma_1}\right) \ldots \left(q^{\frac{\mathtt{h}}{2}\sigma_m} \mathsf{e}_{\sigma_m}\right)\, J(\sigma_1, \ldots, \sigma_m)\, \mathrm{e}^{\mathrm{i}\pi\beta\,\hat{\boldsymbol{p}}\,\mathtt{h}}$$

$$= \overleftarrow{\mathcal{P}} \exp\left(\lambda \int_0^R \mathrm{d}z\, \left(V^+\, q^{\frac{\mathtt{h}}{2}}\, \mathsf{e}_+ + V^-\, q^{-\frac{\mathtt{h}}{2}}\, \mathsf{e}_-\right)\right) \mathrm{e}^{\mathrm{i}\pi\beta\,\hat{\boldsymbol{p}}\,\mathtt{h}}$$

$$V^{\pm}(z_2)\, V^{\mp}(z_1)\big|_{z_2\to z_1+0} \sim (z_2 - z_1)^{-2\beta^2}$$

$J(\sigma_1, \ldots, \sigma_m)$ are well define for $0 < \beta^2 < \frac{1}{2}$

Scaling limit of the XXZ transfer-matrix

- $$\tau(\lambda) = \mathrm{Tr}\Big[\overleftarrow{\mathcal{P}}\exp\Big(\lambda\int_0^R \mathrm{d}x\,\big(V^+\,\sigma_+ + V^-\,\sigma_-\big)\Big)\,\mathrm{e}^{-2\mathrm{i}\pi\beta p\sigma_3}\Big]$$

$$\tau(\lambda)\ :\quad \mathcal{F}_p \mapsto \mathcal{F}_p\,,\qquad [\tau(\lambda),\tau(\lambda')] = [\tau(\lambda), L_0] = 0$$

- In the limit $N \to \infty$

$$\mathbb{H}_{\mathrm{XXZ}}\big|_{\theta,S^z} = N\mathcal{E}_0 + \tfrac{2\pi}{N}\,(L_0 + \bar{L}_0)\Big|_{\boxed{\mathcal{F}_p\otimes\bar{\mathcal{F}}_{\bar{p}}}} + o\big(N^{-1}\big)$$

$$L_0 = p^2 - \tfrac{1}{24} + 2\sum_{n>0} a_{-n}a_n\ ,\qquad \bar{L}_0 = \bar{p}^2 - \tfrac{1}{24} + \ldots$$

$$p = \tfrac{\theta - gS^z}{2\sqrt{g}}\ ,\qquad \bar{p} = \tfrac{\theta + gS^z}{2\sqrt{g}}$$

- $$\tau(\lambda) = T_{CFT}(\lambda) = \lim_{N\to\infty}\big(-\lambda\big)^N\,T_{XXZ}\big(N^{g-1}\,\lambda\big)\qquad(\beta^2 = g)$$

3 fields rep for $U_q(\widehat{\mathfrak{b}}_-)$ [Feigin, Semikhatov'01]

- The Borel subalgebra $U_q(\widehat{\mathfrak{b}}_-) \subset U_q(\widehat{\mathfrak{sl}}_2)$ admits a realization with

$$x_0 = \frac{1}{q-q^{-1}} \int_0^R \mathrm{d}z\, V^+(z)\,, \qquad x_1 = \frac{1}{q-q^{-1}} \int_0^R \mathrm{d}z\, V^-(z)$$

$$h_0 = -4\mathrm{i}b\,\hat{p}_3$$

The vertices $V^\pm$ are built from three bosonic fields φ_1, φ_2, φ_3:

$$V^\pm = \frac{1}{2b^2}\left(\mathrm{i}b\,\partial\varphi_3 + \alpha_2\,\partial\varphi_2 \pm \alpha_1\,\partial\varphi_1\right)\mathrm{e}^{\pm\frac{\varphi_3}{b}}$$

$$\alpha_1^2 + \alpha_2^2 - b^2 = \frac{1}{2}$$

$$q = \mathrm{e}^{\frac{\mathrm{i}\hbar}{2}} \qquad \text{with} \qquad \hbar = \frac{\pi}{2b^2}$$

- $V^{\sigma_2}(z_2)\,V^{\sigma_1}(z_1) \sim (z_2 - z_1)^{-2-\sigma_1\sigma_2/(2b^2)} \qquad (\sigma_{1,2} = \pm)$

The path ordered exponent expression for $\boldsymbol{L}(\lambda)$ is ill defined.

Parafermion transfer-matrix

$$V^{\pm} = \frac{1}{2b^2}\left(\mathrm{i}b\,\partial\varphi_3 + \alpha_2\,\partial\varphi_2 \pm \alpha_1\,\partial\varphi_1\right)\mathrm{e}^{\pm\frac{\varphi_3}{b}}$$

- $\alpha_1 = \frac{\sqrt{n+2}}{2}\,, \quad \alpha_2 = 0, \quad b = \frac{\sqrt{n}}{2}$
  $\psi^{\pm} = V^{\pm}, \quad \Omega = \mathrm{e}^{\frac{4\pi p_1}{\sqrt{n}}}$

$$\tau(\lambda) = \mathrm{Tr}\left[\overleftarrow{\mathcal{P}}\exp\left(\lambda\int_0^R \mathrm{d}x\,\left(\psi^+\,\sigma_+ + \psi^-\,\sigma_-\right)\right)\Omega^{-\frac{1}{2}\sigma_3}\right]$$

- $\psi^{\pm}$ – fundamental Z_n-parafermion currents [Fateev,Zamolodchikov'85]:

$$\psi^{\pm}(x_2)\,\psi^{\mp}(x_1)\big|_{x_2\to x_1+0} \sim 1\times\left(x_2 - x_1\right)^{-2\Delta_\psi}\,, \qquad \Delta_\psi = 1 - \frac{1}{n}$$

ψ^+ and ψ^- carry the $\mathbb{Z}_n$-charges $+2$ and -2 respectively:

$$\Omega\,\psi^{\pm}\,\Omega^{-1} = \omega^{\pm 2}\,\psi^{\pm}\,, \qquad \omega = \mathrm{e}^{-\frac{2\pi\mathrm{i}}{n}}$$

- Parafermion irreps $\mathcal{V}_{\mathrm{j}}$ with the highest weight

$$|\,\sigma_{\mathrm{j}}\,\rangle : \qquad \Delta_{\mathrm{j}} = \frac{\mathrm{j}\,(n-2\mathrm{j})}{n\,(n+2)} \qquad \Big(\mathrm{j} = 0,\ \tfrac{1}{2},\ 1,\ \tfrac{3}{2},\ldots,\ \tfrac{1}{2}\,\big[\tfrac{n}{2}\big]\Big)$$

$$\mathcal{V}_{\mathrm{j}}^{(\mathfrak{m})} \subset \mathcal{V}_{\mathrm{j}} :\ \ \Delta_{\mathrm{j},\mathfrak{m}} = \frac{\mathrm{j}(\mathrm{j}+1)}{n+2} - \frac{\mathfrak{m}^2}{4n} \qquad (\mathfrak{m} = 2\mathrm{j},\ 2\mathrm{j}-2,\ \ldots \geq 0)$$

- $$\tau(\lambda)\big|_{\mathcal{V}_{\mathrm{j}}^{(\mathfrak{m})}} = \operatorname*{s\,lim}_{N\to\infty}\ F^{(N)}(\lambda)\ T^{(N)}\Big(\big(\tfrac{\pi}{N}\big)^{\frac{1}{n}}\,\lambda\Big)\Big|_{M_+,M_-}, \qquad M_\pm = \tfrac{\mathrm{j}}{2} \pm \mathfrak{m}$$

$$F^{(N)}(\lambda) = \begin{cases} \exp\Big(\sum_{l=1}^{[\frac{n}{2}]} \frac{\pi^{\frac{2l}{n}}}{l\cos(\frac{\pi l}{n})}\ N^{1-\frac{2l}{n}}\ \lambda^{2l}\Big) & (n = 3,\,5,\,\ldots) \\ \big(\frac{N\mathrm{e}}{\pi}\big)^{\frac{4}{n}\lambda^n}\ \ \exp\Big(\sum_{l=1}^{[\frac{n}{2}]-1} \frac{\pi^{\frac{2l}{n}}}{l\cos(\frac{\pi l}{n})}\ N^{1-\frac{2l}{n}}\ \lambda^{2l}\Big) & (n = 2,\,4,\ldots) \end{cases}$$

- Scaling limit:

$$\mathbb{H}_{\mathrm{FZ}}\big|_{M_+,M_-} = N\mathcal{E}_0 + \frac{2\pi}{N}\ (L_0 + \bar{L}_0)\Big|_{\boxed{\mathcal{V}_{\mathrm{j}}^{(\mathfrak{m})} \otimes \bar{\mathcal{V}}_{\mathrm{j}}^{(\mathrm{j})}}} + o\big(N^{-1}\big)$$

$$\mathbb{P}\big|_{M_+,M_-} = \exp\Big(\frac{2\pi\mathrm{i}}{N}\,(L_0 - \bar{L}_0)\Big)\Big|_{\boxed{\mathcal{V}_{\mathrm{j}}^{(\mathfrak{m})} \otimes \bar{\mathcal{V}}_{\mathrm{j}}^{(\mathrm{j})}}}$$

Further developments (hep-th/1706.09941)

- ODE-IM correspondence for the Fateev-Zamolodchikov spin chains
- Non Linear Integral Equations
- Relation to the sausage NLSM