Fredkin model

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Introduction

Spin chains are important for high energy theory: N. Nekrasov, S. Sahatashvill. arXiv:0908.4052

K. Costello, E. Witten, M. Yamazaki related Chern-Simons Theory to Yang-Baxter equation arXiv:1709.09993

C. Herzog described entanglement entropy by holography arXiv:1605.01404

I will lecture about entanglement in spin chains.

Properties

For massive theories [gap-full Hamiltonians] the area law is valid Srednicki 1993. In 1D the entropy of a large block of spins approaches a constant as the size of the block x increases. VBS is a good example.

For massless theories the entropy can increase without a bound. For example in CFT

$$EE \sim (c/3)\, logx$$

1994 Holzhey, Larsen and Wilczek. This is applicable to XXX chain of spins 1/2:

$$H_{XXX} = -\sum_{j}(1 - \vec{\sigma}_{j+1} \cdot \vec{\sigma}_{j+2})$$

Can we change local interaction in order to increase EE ?

Fredkin model

- The model represents a chain of spin k - 1/2 with a fully local hamiltonian (next nearest neighbour interaction). The spin 1/2 case is particularly simple.
- The model exibits an unusually high level of entanglement for a local spin chain model.
- The model is Frustration Free, which allows us to obtain an exact expression for the ground state.
- The ground state of the model is described by Combinatorics closely related to the Catalan numbers.

The spin 1/2 Hamiltonian

$$H_{bulk} = \sum_j (1+\sigma_j^z)(1-\vec{\sigma}_{j+1}\cdot\vec{\sigma}_{j+2}) + (1-\vec{\sigma}_j\cdot\vec{\sigma}_{j+1})(1-\sigma_{j+2}^z)$$

- The hamiltonian above can be rewritten in terms of the *Fredkin gate* F_{ijk} which permutes sites j and k if site i is in the state $|\uparrow\rangle$. It is an involutive permutation matrix, and as such $1-F_{ijk}$ is a hermitian projector. In terms of this we have:

$$H_{bulk} = \sum_j (1-F_{j,j+1,j+2}) + (1-\sigma_{j+2}^x F_{j+2,j+1,j}\sigma_{j+2}^x) = \sum_j 1-f_j$$

- The $f_i = F_{j,j+1,j+2}\sigma_{j+2}^x F_{j+2,j+1,j}\sigma_{j+2}^x$ generate a representation of an infinite coxeter group that generalizes the permutation group.
- We will also study the model with added boundary terms $H_\partial = |\downarrow_1\rangle\langle\downarrow_1| + |\uparrow_N\rangle\langle\uparrow_N|$ which make the ground state nondegenerate.

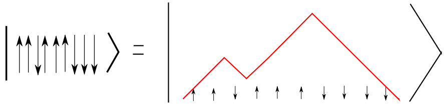
The spin 1/2 Hamiltonian: initial remarks

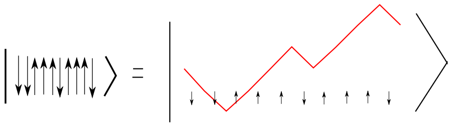
$$H_{bulk} = \sum_j (1 + \sigma_j^z)(1 - \vec{\sigma}_{j+1} \cdot \vec{\sigma}_{j+2}) + (1 - \vec{\sigma}_j \cdot \vec{\sigma}_{j+1})(1 - \sigma_{j+2}^z)$$

- The Hamiltonian is symmetric if we map the lattice site j to $N - j$ and $S_z \to -S_z$, if we interpret S_z as spins along the lattice this is just reflection symmetry.
- The Hamiltonian commutes with $Z = \sum_j \sigma_z$
- H is positive semidefinite.
- Without boundary terms, the state $|\uparrow\uparrow\uparrow \cdots\rangle$ is a ground state of every term. The Hamiltonian is unfrustrated.
- In the 1-magnon subspace(all spins except one pointing up except one) our Hamiltonian reduces to the Heisenberg XXX Hamiltonian $H = \sum_j 1 - \vec{\sigma}_j \cdot \vec{\sigma}_{j+1}$. Therefore, the two have the same spin wave solutions.

Classifying the ground states: view basis states as paths

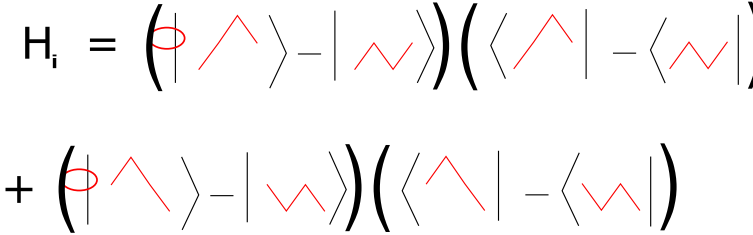
- Identify basis states with paths on an integer lattice assigning spin up to a step up and spin down to a step down, ex: $|\uparrow\downarrow\uparrow\rangle = |/\backslash/\rangle$,$|\downarrow\uparrow\downarrow\rangle = |\backslash/\backslash\rangle$.
- This map defines a path uniquely up to a constant shifting of the height axis, by convention we will set the height of the lowest point of the path to zero.





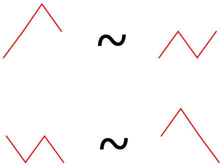
Classifying the ground states: rewrite the Hamiltonian

- We rewrite the Hamiltonian in terms of projectors



Classifying the ground states: Defining an equivalence relation on paths

- Our strategy is then to define a local equivalence relation on paths. We say that two paths are equivalent if they are related by a sequence of the Fredkin moves below.
- The equivalence relation allows us to move a /\ peak to any point in the path.
- The moves manifestly conserve the lowest point and the endpoints of the path.



The Schmidt decomposition in spin 1/2

- The ground state $|C(N)\rangle$ has a Schmidt decomposition $\sum_m \sqrt{p_m}\,|C_{0,m}(L)\rangle \otimes |C_{m,0}(N-L)\rangle$.
- The Schmidt coefficients in our case are given by $p_m = \frac{|C_{0,m}(L)||C_{0,m}(N-L)|}{C(N)}$ which is nonzero for $L+m$ even.
- Using some combinatorics we get

$$|C_{a,b}(L)| = \binom{L}{\frac{L+a+b}{2}} - \binom{L}{\frac{L+a+b}{2}+1}$$

for $L+a+b$ even, $0 \le a+b < L$, which we can plug into our expression.

- Plugging this into the Schmidt coefficients and approximating binomials with Gaussians, one gets

$$p_h \ \frac{h^2}{Z} \exp\left(-h^2 \frac{n(l-n)}{n}\right)$$

The entanglement entropy in spin 1/2

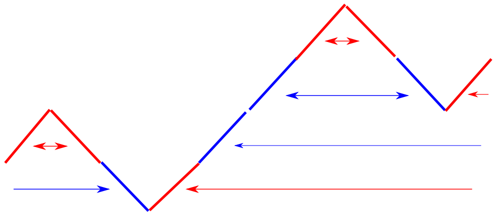
- The Schmidt rank, is the number of nonzero Schmidt coefficients.
- The entanglement entropy is defined as $S = \sum_m -p_m Log(p_m)$.
- The Schmidt coefficients themselves are also known as the entanglement spectrum.
- The Schmidt rank in our case is $\lfloor \frac{L}{2} \rfloor$.
- Our approximate entanglement spectrum is isomorphic to the Maxwell-Boltzmann speed distribution. This has an entropy $S \approx \frac{1}{2} Log(\frac{L(N-L)}{N}) + O(C)$. Comparing numerically to exact values, one gets a constant term of roughly 0.437 bits.

Spin 3/2 and up: colored Dyck walks

- In spin 3/2 and higher, we can consider colored paths.
- In the case of 3/2, we can for example identify $m = 3/2$ with a red up step, $m = 1/2$ with a blue up step, $m = -\ 1/2$ with a blue down step, and $m = -3/2$ with a red down step.
- To analyze what happens when we consider colored paths and why this enables polynomial rather than logaritmic entropy growth, we need to mention the concept of matched steps and colored paths.

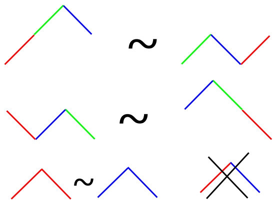
Spin 3/2 and up: colorings and matchings

- An up step and a down step are *matched* if the the up step is of the form $(i, j) \to (i, j+1)$ and the down step is the first down step occurring after our up step which is of the form $(i', j+1) \to (i'+1, j)$. Equivalently, two steps are matched if the subpath between them is a Dyck path.
- A properly colored path is a colored path such that matched steps have the same color.



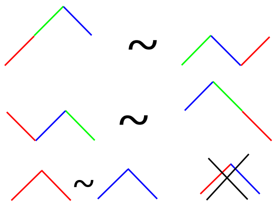
Spin 3/2 and up: colored Fredkin Moves, coloring rules

- Fredkin moves will move peaks along with their colors.
- In addition to the Fredkin moves, we introduce the coloring rules, which allow us to recolor matched peaks and which forbid matched pairs from having different colors.



Spin 3/2 and up: colored Fredkin Moves, coloring rules

- The colored Fredkin moves allow us to reduce any path with colored steps to one where all matched steps are adjacent to their match. The coloring rules can then be applied to recolor matched pairs or to exclude invalid path colorings.
- This allows us to define equivalence classes of colored paths which are defined only by their endpoints, and the colors of the unmatched steps to their left and right.



Spin 3/2 and up: Hamiltonian

- These rules can be implemented by the following SU(k)-invariant Hamiltonian of the form $H = H_F + H_X + H_\partial$. We will look at the three terms in turn.
- H_F, which implements the Fredkin moves, can be expressed using the operators P_j^+, P_j^- which project onto up/down steps at sites j without regard for colors, and the cyclic permutation operators $C_{i,j,k}$ which cyclically permute the sites i,j,k. It is then defined as:

$$
\begin{aligned}
H_F = \sum_{j=1}^{N-2} P_j^+ P_{j+1}^+ P_{j+2}^- + P_j^+ P_{j+1}^- P_{j+2}^+ - \\
- P_j^+ P_{j+1}^+ P_{j+2}^- C_{j,j+1,j+2} - C_{j,j+1,j+2}^\dagger P_j^+ P_{j+1}^+ P_{j+2}^- + \\
+ P_j^+ P_{j+1}^- P_{j+2}^- + P_j^- P_{j+1}^+ P_{j+2}^- - \\
- P_j^+ P_{j+1}^- P_{j+2}^- C_{j,j+1,j+2}^\dagger - C_{j,j+1,j+2} P_j^+ P_{j+1}^- P_{j+2}^-
\end{aligned}
$$

Spin 3/2 and up: Hamiltonian

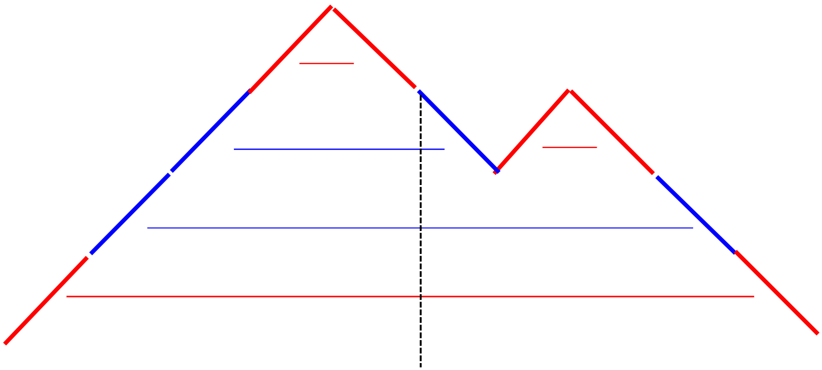
- The matching term H_X, is defined using local SU(k) generators T_j^a acting on the color space. Up steps lie in the fundamental representation of SU(k), while down step colors lie in its conjugate repesentation.
- The color matching then simply corresponds to projecting the colors of matched pairs onto SU(k) singlets.
- The Hamiltonian then is symmetric under SU(k), since it manifestly commutes with generators $T^a = \sum_j T_j^a$.

$$H_X = \sum_{j=1}^{N-1} P_j^+ P_{j+1}^- \left[\sum_a (T_j^a + T_{j+1}^a)^2 \right]$$

$$H_\partial = P_1^- + P_N^+$$

Spin 3/2 and up: Ground state

- The ground state with boundary terms is then the sum of properly colored Dyck paths.
- The ground states with open boundaries are sums of properly colored paths, with a fixed coloring of the unmatched pairs.

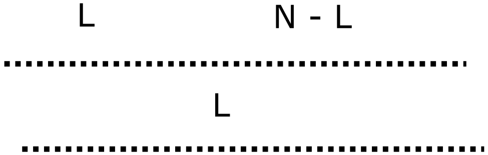


Spin 3/2 and up: Schmidt decomposition

- The Schmidt coefficients in our case are closely linked to the spin 1/2 case.
- Schmidt decomposition looks like $\sum_{m,c} \sqrt{q_{m,c}} \, |C_{m,c}(L)\rangle \otimes |C_{m,c}(N-L)\rangle$ where c sums over all colorings of the steps matched across the boundary.
- we have $q_{m,c} = k^{-m} p_m$ with degeneracy k^m due the the free color index.
- The Schmidt rank is thus $\sum_h k^{2h}$ for even half-chain lengths and $\sum_h k^{2h+1}$ for odd half-chains.

Spin 3/2 and up: Entanglement entropy

- With the k^m degeneracy, the entropy can simply be written as $\sum_m -p_m \log(k^m p_m) = \sum_m -p_m[\log(p_m) + m \log(k)]$
- The first term is just the spin 1/2 entropy, while the second is $\log(k)$ times the expectation value of the path height, which scales as $O(\sqrt{N})$
- We can work out the coefficients, which eventually gives us $\frac{2}{\sqrt{\pi}} \log(k) \sqrt{2 \frac{L(N-L)}{N}}$.
- However, this last term has an unexpected symmetry.
- It turns out to be shift-invariant...



Reference

- O. Salberger and V. Korepin, Entangled spin chain,Rev. Math. Phys. 29(10) (2017) 1750031, 20 pp
- D. E. Kharzeev, E. M. Levin. Deep inelastic scattering as a probe of entanglement, Phys. Rev. D 95, 114008 (2017)