The Algebraic Construction of Integrable Hierarchies, Solitons and Backlund Transformation

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International Institute of Physics, 2018

Aim

- Discuss the General structure of *time evolution integrable* equations associated to *graded Affine Lie algebraic* structure, e.g., *sinh-Gordon, mKdV*, etc.
- Representation Theory of Infinite Dimensional Algebras and the Systematic Construction of *Soliton Solutions*.
- Systematic Construction of *Backlund Transformation* for the entire Hierarchy and *Integrable Defects*

- Start with affine Lie Algebra $\hat{\mathcal{G}}$
- Decompose $\hat{\mathcal{G}}$ into graded subspaces, e.g., $\hat{\mathcal{G}} = \oplus_i \mathcal{G}_i$, such that $[\mathcal{G}_i, \mathcal{G}_j] \subset \mathcal{G}_{i+j}$.
- Define constant grade 1 operator $E = E^{(1)} \in \mathcal{G}_1$ and define $\mathcal{K} = Kernel = \{x \in \hat{\mathcal{G}} / [x, E] = 0\}$
- Decompose $\mathcal{G}_0 = \mathcal{K} \oplus \mathcal{M}$

- Define Lax operator

$$L = \partial_x + E^{(1)} + A_0, \qquad A_0 \in \mathcal{M} \subset \mathcal{G}_0 \quad \text{Image}$$

- 2 - Dim. Gauge potentials

$$A_x = E + A_0, \qquad A_{t_{MN}} = D^{(N)} + D^{(N-1)} + \cdots + D^{(0)} + \cdots D^{(-M)}$$

- Zero Curvature Equation for *Hierarchy*

$$[\partial_x + E^{(1)} + A_0, \partial_{t_{MN}} + D^{(N)} + D^{(N-1)} + \cdots + D^{(0)} + \cdots D^{(-M)}] = 0$$

$D^{(j)} \in \mathcal{G}_j$.

Decompose and solve grade by grade, i.e.,

- $$\begin{aligned} [E^{(1)}, D^{(N)}] &= 0, \\ [E^{(1)}, D^{(N-1)}] + [A_0, D^{(N)}] + \partial_x D^{(N)} &= 0, \\ \vdots \quad & \quad \vdots \\ [E^{(1)}, D^{(-1)}] + [A_0, D^{(0)}] + \partial_x D^{(0)} - \partial_{t_{MN}} A_0 &= 0, \\ \vdots \quad & \quad \vdots \\ [A_0, D^{(-M)}] + \partial_x D^{(-M)} &= 0, \end{aligned}$$
- Solving recursively for $D^{(i)}$ we get the eqn. of motion

$$\partial_{t_{MN}} A_0 - \partial_x D^{(0)} - [A_0, D^{(0)}] - [E^{(1)}, D^{(-1)}] = 0,$$

Example: The mKdV Hierarchy

- Choose $\mathcal{G} = sl(2)$ with generators $\{h, E_{\pm\alpha}\}$
- Grading operator Q, e.g. $Q = 2\lambda\frac{d}{d\lambda} + \frac{1}{2}h$
- *semi-simple* element $E = E^{(1)} = E_\alpha + \lambda E_{-\alpha}$
- Decomposition of Affine Lie Algebra into graded subspaces,

$$\begin{aligned}\mathcal{G}_{2m} &= \{h^{(m)} = \lambda^m h\},\\ \mathcal{G}_{2m+1} &= \{\lambda^m(E_\alpha + \lambda E_{-\alpha}),\ \lambda^m(E_\alpha - \lambda E_{-\alpha})\}\end{aligned}$$

$m = 0, \pm 1, \pm 2, \ldots$ where $[\mathcal{G}_i, \mathcal{G}_j] \subset \mathcal{G}_{i+j}$.

For $M = 0$, Zero Curvature can be decomposed and solved grade by grade, i.e.,

- $$\begin{aligned} [E^{(1)}, D^{(N)}] &= 0, \\ [E^{(1)}, D^{(N-1)}] + [A_0, D^{(N)}] + \partial_x D^{(N)} &= 0, \\ \vdots \quad &\quad \vdots \\ [A_0, D^{(0)}] + \partial_x D^{(0)} - \partial_{t_N} A_0 &= 0, \end{aligned}$$

- In particular, highest grade component, i.e.,

  $$[E^{(1)}, D^{(N)}] = 0,$$

  implies $D^{(N)} = E^{(N)} \in \mathcal{K}_{2n+1}$ is const. and therefore $N = 2n + 1$.

Solving recursively for $D^{(i)}$ we get the eqn. of motion

$$\partial_{t_N} A_0 - \partial_x D^{(0)} - [A_0, D^{(0)}] = 0,$$

Examples: $A_0 = v(x, t_N)h$,

$N = 3$,

$$4\partial_{t_3} v = v_{3x} - 6v^2 v_x, \qquad mKdV$$

$N = 5$,

$$16\partial_{t_5} v = v_{5x} - 10v^2 v_{3x} - 40 v v_x v_{2x} - 10 v_x^3 + 30 v^4 v_x,$$

$N = 7$,

$$\begin{aligned} 64\partial_{t_7} v &= v_{7x} - 182 v_x v_{2x}^2 - 126 v_x^2 v_{3x} - 140 v v_{2x} v_{3x} \\ &- 84 v v_x v_{4x} - 14 v^2 v_{5x} + 420 v^2 v_{3x} + 560 v^3 v_x v_{2x} \\ &+ 70 v^4 v_{3x} - 140 v^6 v_x \end{aligned}$$

$\cdots etc$

Remark

- Vacuum Solution is $v = const = 0$
- zero curvature for vacuum solution becomes

$$[\partial_x + E^{(1)}, \partial_{t_N} + E^{(N)}] = 0, \qquad [E^{(1)}, E^{(N)}] = 0$$

- and imply pure gauge potentials, i.e.,

$$A_{x,vac} = E^{(1)} = T_0^{-1}\partial_x T_0, \qquad A_{t_N,vac} = E^{(N)} = T_0^{-1}\partial_{t_N} T_0$$

- where

$$T_0 = e^{xE^{(1)}} e^{t_N E^{(N)}}$$

- Zero Curvature Equation for *Negative Hierarchy*

$$[\partial_x + E^{(1)} + A_0, \partial_{t_{-n}} + D^{(-n)} + D^{(-n+1)} + \cdots + D^{(-1)}] = 0.$$

- Lowest grade projection,

$$\partial_x D^{(-n)} + [A_0, D^{(-n)}] = 0$$

yields a nonlocal equation for $D^{(-n)}$. **No condition upon n.**

- The second lowest projection of grade $-n+1$ leads to

$$\partial_x D^{(-n+1)} + [A_0, D^{(-n+1)}] + [E^{(1)}, D^{(-n)}] = 0$$

and determines $D^{(-n+1)}$.

- The same mechanism works recursively until we reach the zero grade equation

$$\partial_{t_{-n}} A_0 + [E^{(1)}, D^{(-1)}] = 0$$

which gives the *time evolution* for the field in A_0 according to time t_{-n}.

- Simplest Example $t_{-n} = t_{-1}$.

$$\begin{aligned} \partial_x D^{(-1)} + [A_0, D^{(-1)}] &= 0, \\ \partial_{t_{-1}} A_0 - [E^{(1)}, D^{(-1)}] &= 0. \end{aligned} \tag{1}$$

- Compact Solution is

$$D^{(-1)} = B^{-1} E^{(-1)} B, \qquad A_0 = B^{-1} \partial_x B, \qquad B = \exp(\mathcal{G}_0)$$

- The time evolution is then given by the Leznov-Saveliev equation,

$$\partial_{t_{-1}} \left(B^{-1} \partial_x B \right) = [E^{(1)}, B^{-1} E^{(-1)} B]$$

- For $\hat{sl}(2)$ with principal gradation $Q = 2\lambda\frac{d}{d\lambda} + \frac{1}{2}h$, yields the sinh-Gordon equation (*relativistic*)

$$\partial_{t_{-1}}\partial_x\phi = e^{2\phi} - e^{-2\phi}, \qquad B = e^{\phi h}.$$

where $\quad t_{-1} = z, \quad x = \bar{z}, \quad A_0 = vh = \partial_x\phi h.$

- **No restriction for Negative even Hierarchy**

- Next simplest example $t = t_{-2}$ [1]

$$
\begin{aligned}
\partial_x D^{(-2)} + [A_0, D^{(-2)}] &= 0, \\
\partial_x D^{(-1)} + [A_0, D^{(-1)}] + [E^{(1)}, D^{(-2)}] &= 0, \\
\partial_{t_{-2}} A_0 - [E^{(1)}, D^{(-1)}] &= 0.
\end{aligned}
$$

- Propose solution of the form

$$
\begin{aligned}
D^{(-2)} &= c_{-2}\lambda^{-1} h, \\
D^{(-1)} &= a_{-1}\left(\lambda^{-1}E_{\alpha} + E_{-\alpha}\right) + b_{-1}\left(\lambda^{-1}E_{\alpha} - E_{-\alpha}\right).
\end{aligned}
$$

[1] JFG, G Starvaggi França, G R de Melo and A H Zimerman, *J. of Phys.* **A42**,(2009), 445204

- Get $c_{-2} = const$ and

$$\begin{aligned} a_{-1} + b_{-1} &= 2c_{-2} \exp(-2d^{-1}v) d^{-1} \left(\exp(2d^{-1}v) \right), \\ a_{-1} - b_{-1} &= -2c_{-2} \exp(2d^{-1}v) d^{-1} \left(\exp(-2d^{-1}v) \right), \end{aligned}$$

where $A_0 = vh = \partial_x \phi h$ and $d^{-1}v = \int^x v(x')dx' = \phi$.

$t = t_{-2}$ Equation

- Equation of motion is (integral eqn.)

$$\partial_{t_{-2}} v = -2c_{-2} e^{-2d^{-1}v} d^{-1} \left(e^{2d^{-1}v} \right) - 2c_{-2} e^{2d^{-1}v} d^{-1} \left(e^{-2d^{-1}v} \right)$$

where $d^{-1}v = \int^{x} v(x')dx' = \phi$.

- Constant Vacuum for $t = t_{-2}$ equation
  1) Let $v = 0, \longrightarrow d^{-1}0 = \alpha = const$

$$0 + 2c_{-2}e^{-2\alpha}\int e^{2\alpha} + 2c_{-2}e^{2\alpha}\int e^{-2\alpha} \neq 0,$$

for $c_{-2} \neq 0$.

2) $v = v_0, \qquad d^{-1}v_0 = v_0 x$

$$0 + 2c_{-2}e^{-2v_0x}\int e^{2v_0x} + 2c_{-2}e^{2v_0x}\int e^{-2v_0x} = 0$$

Notice that, for $c_{-2} \neq 0$, $v = 0$ is not solution of the evolution equation and therefore $A_0 = 0$ does not satisfy the zero curvature representation for $t = t_{-2}$.

The **Soliton solutions** are constructed from the vacuum solution by **gauge transformation** (which preserves the zero curvature condition), i.e.,

$$A_\mu = \Theta^{-1} A_{\mu,vac} \Theta + \Theta^{-1} \partial_\mu \Theta,$$

where

$$A_\mu = T^{-1}\partial_\mu T, \qquad T = T_0 \Theta, \qquad A_{\mu,vac} = T_0^{-1}\partial_\mu T_0$$

we may choose $\Theta = \Theta_+ = e^{\theta_0} e^{\theta_1} \cdots$ or $\Theta = \Theta_- = e^{\theta_{-1}} e^{\theta_{-2}} \cdots$, $\theta_i \in \mathcal{G}_i$.

It then follows that $T = \Theta_+ T_0 = \Theta_- T_0 g$,

$$\Theta_-^{-1}\Theta_+ = T_0^{-1} g T_0, \qquad e^{\theta_0} = Be^{\nu \hat{c}}$$

In order to introduce highest weight states $|\lambda_i>, i = 0, 1$, need to extended the loop to the fully **central extended** Kac-Moody algebra

$$[h^{(m)}, h^{(n)}] = \hat{c} m \delta_{m+n,0}$$

$$[h^{(m)}, E_{\pm\alpha}^{(n)}] = \pm 2 E_{\pm\alpha}^{(n+m)}, \qquad [E_{\alpha}^{(m)}, E_{-\alpha}^{(n)}] = h^{(m+n)} + \hat{c} m \delta_{m+n,0},$$

and introduce ν field associated to $\hat{c}$, i.e.,

$$B \to Be^{\nu \hat{c}}, \qquad A_0 \to A_0 + \partial_x \nu \hat{c}$$

such that

$$< \lambda | Be^{\nu \hat{c}} | \lambda > \quad = \quad < \lambda | T_0^{-1} g T_0 | \lambda > .$$

- The solution for mKdV hierarchy is then given by

$$\begin{aligned} e^{-\nu} &= \ < \lambda_0 | T_0^{-1} g T_0 | \lambda_0 > \ \equiv \ \tau_0, \\ e^{-\phi-\nu} &= \ < \lambda_1 | T_0^{-1} g T_0 | \lambda_1 > \ \equiv \ \tau_1 \end{aligned}$$

and hence, $v = -\partial_x \ln \left(\frac{\tau_0}{\tau_1}\right)$, $\qquad v = \partial_x \phi$
where $T_0 = e^{xA_{x,vac}} e^{t_M A_{t_M,vac}}$, $\qquad g = e^{F(\gamma)}$.
Taking $F(\gamma)$ is an eigenvector (vertex operator) of $E^{(M)} = A_{t_M,vac}$ and $E^{(1)} = A_{x,vac}$, i.e.,

$$[E^{(M)}, F(\gamma)] = w_M(\gamma) F(\gamma) .$$

it follows that

$$T_0^{-1} e^{F(\gamma)} T_0 = e^{\rho(x,t_N;\gamma) F(\gamma)}, \quad \rho(x, t_N; \gamma) = e^{x w_1 + t_N w_N},$$

- We find that the *one-soliton solution* of the form,

$$\tau_0 = 1 + C_0\rho\left(\gamma, v_0\right), \qquad \tau_1 = 1 + C_1\rho\left(\gamma, v_0\right)$$

solves all eqns. within the positive mKdV hierarchy for $w_1 = 2\gamma, \quad w_N = 2\gamma^N$, i.e.,

$$v = -\partial_x \ln\left(\frac{1 + C_1\rho}{1 + C_0\rho}\right).$$

where

$$\rho\left(\gamma, v_0\right) = \exp\left\{2\gamma x + 2\gamma^N t_N\right\}.$$

- The same works for *multi-soliton* solutions, ie., $g = \Pi e^{F_i(\gamma_i)}$.

$$T_0^{-1} \Pi e^{F_i(\gamma_i)} T_0 = \Pi e^{\rho_i(x,t_N;\gamma_i) F_i(\gamma_i)}, \quad \rho_i(x,t_N) = e^{2\gamma_i x + 2\gamma_i^N t_N}.$$

- For negative hierarchy [2] and constant vacuum solution $v = v_0 - \partial_x \ln\left(\frac{1+C_1\rho}{1+C_0\rho}\right)$.

$$\rho\left(\gamma, v_0\right) = \exp\left\{2\gamma x + \frac{2\gamma t_{-m}}{v_0\left(\gamma^2 - v_0^2\right)^{m/2}}\right\}.$$

[2] JFG, G Starvaggi França, G R de Melo and A H Zimerman, *J. of Phys.* **A42**,(2009), 445204

Gauge-Backlund Transformation for Sinh-Gordon

Assume now that two field configurations ϕ_1 and ϕ_2 embedded in $A_{x,mKdV}(\phi_1)$ and $A_{x,mKdV}(\phi_2)$ are related by a Backlund gauge transformation, i.e.,

$$K(\phi_1, \phi_2) A_{x,mKdV}(\phi_1) = A_{x,mKdV}(\phi_2) K(\phi_1, \phi_2) + \partial_x K(\phi_1, \phi_2),$$

holds for

$$K(\phi_1, \phi_2) = \begin{bmatrix} 1 & -\frac{\beta}{2\lambda} e^{-(\phi_1+\phi_2)} \\ -\frac{\beta}{2} e^{(\phi_1+\phi_2)} & 1 \end{bmatrix}$$

provided Backlund transformation is satisfied, i.e.,

$$\partial_x (\phi_1 - \phi_2) = -\beta \sinh(\phi_1 + \phi_2), \qquad v_i \equiv \partial_x \phi_i.$$

For the sinh-Gordon, the equations of motion

$$\partial_{t_{-1}} \partial_x \phi_a = 2 \sinh 2\phi_a, \quad a = 1, 2$$

we to introduce the time component of the Bäcklund transformation,

$$\partial_{t_{-1}} (\phi_1 + \phi_2) = \frac{4}{\beta} \sinh (\phi_2 - \phi_1) . \tag{3}$$

For higher graded time evolutions the time component of the Backlund transformation can be derived from the appropriated time component of the two dimensional gauge potential. [3] e.g.,

$$K(\phi_1,\phi_2)A_{t_N,mKdV}(\phi_1) = A_{t_N,mKdV}(\phi_2)K(\phi_1,\phi_2) + \partial_{t_N}K(\phi_1,\phi_2),$$

which for $t = t_3$ leads to

$$\begin{aligned}\partial_{t_3}\phi_2 \quad - \quad & \partial_{t_3}\phi_1 = \frac{\beta}{4}(\partial_x^2\phi_1 + \partial_x^2\phi_2)\cosh(\phi_1+\phi_2) \\ - \quad & \frac{\beta}{8}(\partial_x\phi_1 + \partial_x\phi_2)^2\sinh(\phi_1+\phi_2) - \frac{\beta^3}{8}\sinh^3(\phi_1+\phi_2).\end{aligned}$$

[3] See JFG, AL Retore and AH Zimerman arXiv:1501.00865, arXiv:1505.01024

Consider now

$$g_1 = \begin{pmatrix} \zeta & 1 \\ \zeta & -1 \end{pmatrix}, \qquad g_2(v,\epsilon) = \begin{pmatrix} 1 & \epsilon \\ -\epsilon v & -v + 2\epsilon\zeta \end{pmatrix}, \quad \zeta^2 = \lambda,$$

which transforms

$$A_{x,mKdV} = E^{(1)} + v(x,t_N)h = \begin{pmatrix} v & 1 \\ \lambda & -v \end{pmatrix},$$

into

$$A_{x,KdV} \;=\; g_2 g_1 \left(A_{x,mKdV}\right) g_1^{-1} g_2^{-1} - \partial_x g_2 g_2^{-1} = \begin{pmatrix} \zeta & -1 \\ J & -\zeta \end{pmatrix}$$

where $J = \epsilon\partial_x v - v^2, \quad \epsilon^2 = 1.$

Following the same line of reasoning propose now

$$\tilde{K}(J_1, J_2)A_{\mu,KdV}(J_1) = A_{\mu,KdV}(J_2)\tilde{K}(J_1, J_2) + \partial_\mu \tilde{K}(J_1, J_2),$$

which can be constructed from K, i.e.,

$$\tilde{K} = g_2(v_2, \epsilon_2)\left(g_1 K(\phi_1, \phi_2) g_1^{-1}\right) g_2(v_1, \epsilon_1)^{-1}$$

and depend upon ϵ_1, ϵ_2.

For $\epsilon_1 = -\epsilon_2 = \epsilon$ we found

$$\tilde{K}(J_1, J_2, \beta) = -\frac{1}{\zeta}\begin{pmatrix} -\zeta + \frac{1}{2}Q & 1 \\ \frac{-\beta^2}{4} + \frac{1}{4}Q^2 & \zeta + \frac{1}{2}Q \end{pmatrix},$$

where

$$Q = \epsilon(v_1 + v_2) + \frac{\beta}{2}(e^{(\phi_1+\phi_2)} + e^{-(\phi_1+\phi_2)}) = w_1 - w_2$$

and $J_i = \partial_x w_i, i = 1, 2$[4] which generates to the Backlund transformation for the KdV hierarchy

$$J_1 + J_2 = \partial_x P = \frac{\beta^2}{2} - \frac{(w_1 - w_2)^2}{2}, \qquad P = w_1 + w_2.$$

[4] see JFG, AL Retore and AH Zimerman, 2016

- Affine Algebraic structure, i.e., $\hat{\mathcal{G}}, Q, E^{(n)}$ provide a systematic method in deriving integrable nonlinear equations, *Integrable Hierarchies*.

- Provide the construction and classification of Soliton Solutions via *Dressing Method*.

- How to adapt Dressing method to construct *periodic solutions* (Jacobi Theta functions). where

$$\tau_a = \sum_{k=-\infty}^{+\infty} e^{2\pi i\eta k^2}\rho^k, \qquad \eta = \text{deform.} \quad \text{parameter}$$

  c.f. soliton where

$$\tau_0 = 1 + \rho, \qquad \tau_1 = 1 - \rho$$

- Provide the Systematic construction of *Backlund Transformation* for higher members of same hierarchy.