Supersymmetric fermion chain with staggered interaction

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Motivation

- Extend family of integrable lattice supersymmetric models
- Construct lattice SUSY and integrable version of Kitaev model
- Topological phases

Fermions

Usual fermionic creation and annihilation operators c_j and $c_j^\dagger$,

$$\{c_i^\dagger, c_j\} = \delta_{ij}, \qquad \{c_i^\dagger, c_j^\dagger\} = \{c_i, c_j\} = 0$$

These operators act in a fermionic Fock space spanned by ket vectors of the form

$$|\tau\rangle = \prod_{i=1}^{L} \left(c_i^\dagger\right)^{\tau_i} |\emptyset\rangle,$$

where the product is ordered and $|\emptyset\rangle$ is the vacuum state defined by $c_i|\emptyset\rangle = 0$.

Example with $L = 8$:

$$|\circ \bullet \circ \circ \bullet \circ \circ \circ\rangle = c_2^\dagger c_5^\dagger |\emptyset\rangle = -c_5^\dagger c_2^\dagger |\emptyset\rangle$$

Hardcore fermions

The fermionic number and hole operators are defined as

$$\mathcal{N}_i = c_i^\dagger c_i, \qquad \mathcal{P}_i = 1 - \mathcal{N}_i.$$

We have

$$\mathcal{N}_i|\tau\rangle = \tau_i|\tau\rangle.$$

Exclude adjacent fermions:

$$\begin{aligned} d_i^\dagger &= \mathcal{P}_{i-1} c_i^\dagger \mathcal{P}_{i+1} \\ d_i &= \mathcal{P}_{i-1} c_i \mathcal{P}_{i+1} \end{aligned}$$

Example with $L = 8$:

$$\begin{aligned} d_1^\dagger|\circ\bullet\bullet\bullet\bullet\circ\circ\circ\rangle &= 0 \\ d_7^\dagger|\circ\bullet\bullet\bullet\bullet\circ\circ\circ\rangle &= |\circ\bullet\bullet\bullet\bullet\circ\bullet\circ\rangle \\ d_7|\circ\bullet\bullet\bullet\bullet\circ\bullet\circ\rangle &= |\circ\bullet\bullet\bullet\bullet\circ\circ\circ\rangle \end{aligned}$$

Supersymmetry

SUSY charges:

$$Q = \sum_{i=1}^{L} d_i, \qquad Q^\dagger = \sum_{i=1}^{L} d_i^\dagger \qquad Q^2 = 0.$$

and Hamiltonian

$$H := \{Q^\dagger, Q\} = \sum_{i=1}^{L} \left(d_i^\dagger d_i + d_i d_i^\dagger + d_i^\dagger d_{i+1} + d_{i+1}^\dagger d_i \right).$$

These equations define supersymmetric quantum mechanics or $\mathcal{N} = 2$ supersymmetric quantum field theory

H is a lattice realisation (Fendley, Nienhuis, Schoutens, de Boer)

Example states:

$$|\circ\bullet\circ\circ\bullet\circ\circ\circ\rangle, \qquad |\circ\bullet\circ\circ\bullet\circ\bullet\circ\rangle$$

Supersymmetry

Alternative form for H:

$$H = L - 2f + \sum_{i=1}^{L} \left(d_i^\dagger d_{i+1} + d_{i+1}^\dagger d_i + d_i^\dagger d_i d_{i+2}^\dagger d_{i+2} \right),$$

The Hamiltonian as a chemical potential of 2 per fermion, a hopping term and a repulsive potential.

- Supersymmetry implies that $E_0 = 0$ and nonzero eigenvalues are positive.

- All singlets have $E = 0$ and all other states are doublets.

Mapping to XXZ chain

Consider the mapping:

$$\circ \bullet \circ \quad \rightarrow \quad +, \qquad \circ\circ \quad \rightarrow \quad -$$

This maps states with length L to length $N = L - f$:

$$|\circ \bullet \circ \circ \bullet \circ \circ \circ \circ\rangle \rightarrow |+ - + - - -\rangle$$
$$|\circ \bullet \circ \circ \bullet \circ \bullet \circ\rangle \rightarrow |+ - + + -\rangle$$

$$H = L - 2f + \sum_{i=1}^{L} \left(d_i^\dagger d_{i+1} + d_{i+1}^\dagger d_i + d_i^\dagger d_i d_{i+2}^\dagger d_{i+2} \right),$$

is equivalent to

$$H = \frac{1}{2} \sum_{j=1}^{N} \left[\sigma_j^x \sigma_{j+1}^y + \sigma_j^y \sigma_{j+1}^y - \Delta \sigma_j^z \sigma_{j+1}^z \right] + 3N/4,$$

with $\Delta = -1/2$ (Note different lenghts N vs L).

Include particle hole symmetry

Include particle-hole symmetry $d_i^\dagger \leftrightarrow e_i$ with

$$e_i = \mathcal{N}_{i-1} c_i \mathcal{N}_{i+1}$$

Example

$$e_3 |\circ \bullet \bullet \bullet \bullet \circ \circ \circ\rangle = -|\circ \bullet \circ \bullet \bullet \circ \circ \circ\rangle$$

SUSY charges:

$$Q^\dagger = \sum_{i=1}^{L} (d_i^\dagger + e_i), \qquad Q = \sum_{i=1}^{L} (d_i + e_i^\dagger), \qquad Q^2 = 0.$$

and Hamiltonian

$$\begin{aligned} H := \{Q^\dagger, Q\} = \sum_{i=1}^{L} & \left(d_i^\dagger d_i + d_i d_i^\dagger + d_i^\dagger d_{i+1} + d_{i+1}^\dagger d_i \right) \\ & + \left(e_i^\dagger e_i + e_i e_i^\dagger + e_i^\dagger e_{i+1} + e_{i+1}^\dagger e_i \right) \\ & + \left(e_i^\dagger d_{i+1}^\dagger + d_{i+1} e_i + e_{i+1}^\dagger d_i^\dagger + d_i e_{i+1} \right). \end{aligned}$$

Creation and hopping of domain walls

Cross terms do not conserve fermions:

$$e_i^\dagger d_{i+1}^\dagger \, |\ldots \bullet \,|\, \circ \circ \circ \ldots\rangle \quad \mapsto \quad |\ldots \bullet \bullet \bullet \,|\, \circ \ldots\rangle$$
$$d_i e_{i+1} \, |\ldots \circ \,|\, \bullet \bullet \bullet \ldots\rangle \quad \mapsto \quad -|\ldots \circ \circ \circ \,|\, \bullet \ldots\rangle$$

But domain walls are conserved

Hopping terms:

$$d_{i+1}^\dagger d_i \, |\ldots \circ \,|\, \bullet \,|\, \circ \circ \ldots\rangle \quad \mapsto \quad |\ldots \circ \circ| \bullet \,|\, \circ \ldots\rangle$$
$$e_{i+1} e_i^\dagger \, |\ldots \bullet \,|\, \circ \,|\, \bullet \bullet \ldots\rangle \quad \mapsto \quad -|\ldots \bullet \bullet| \circ \,|\, \bullet \ldots\rangle$$

- 10 domain walls hop with two steps
- 01 domain walls hop with a minus sign
- nearby domain walls interact

Fendley-Schoutens model

Consider the following supersymmetry generator

$$Q_{FS} = c_2^\dagger c_1 + \sum_{k=1}^{L/2-1} \left(e^{\mathrm{i}\alpha_{2k-2}} c_{2k-1}^\dagger + e^{\mathrm{i}\alpha_{2k}} c_{2k+1}^\dagger \right) c_{2k},$$

where

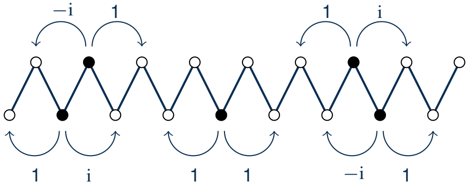
$$\alpha_k = \frac{\pi}{2} \sum_{j=1}^{k} (-1)^j n_j, \qquad Q_{FS}^2 = \left(Q_{FS}^\dagger \right)^2 = 0.$$

The Hamiltonian is built up in the usual way

$$\begin{aligned} H_{FS} &= \{Q_{FS},\, Q_{FS}^\dagger\} \\ &= \sum_{j=1}^{L-1} (c_{j+1}^\dagger p_j c_{j-1} + c_{j-1}^\dagger p_j c_{j+1} + \mathrm{i} c_{j+1}^\dagger n_j c_{j-1} - \mathrm{i} c_{j-1}^\dagger n_j c_{j-1}) \\ &\quad - 2 \sum_{j=1}^{L-1} n_j n_{j+1} + 2F_1 + 2F_2 - \sum_{j=1}^{L/2} n_{2j-1}, - \sum_{j=1}^{L/2} n_{2j}, \end{aligned}$$

Fermions are hopping on two chains coupled in a zig-zag fashion.

The interaction between the chains is statistical: the hopping amplitude picks up an extra i or $-\mathrm{i}$ factor, if the fermion "hops over" another fermion on the other chain.



The eigenenergies of H_{FS} and H_{free} are

$$E = 2 \sum_{a=1}^{f_1+f_2} \left(1 + \cos(2p_a)\right),$$

$$p_a = m_a \frac{\pi}{L+1}, \qquad m_a \in \{1, 2, \ldots, L/2\}$$

where p_a are called momenta, f_1 and f_2 are the number of fermions on the respective chains.

Two models:

$$Q_{FS}^{\dagger} = c_1^{\dagger} c_2 + \sum_{k=1}^{L/2-1} c_{2k+1}^{\dagger} \left(e^{\mathrm{i}\alpha_{2k-1}} c_{2k} + e^{\mathrm{i}\alpha_{2k}} c_{2k+2} \right),$$

$$Q^{\dagger} = \sum_{i=1}^{L-1} (d_i + e_i^{\dagger})$$

turn out to be equivalent

We would like to find the map between Q and Q_{FS}. Assume that it is given by a transformation T,

$$Q = T Q_{FS} T^{\dagger}, \qquad T = P \Gamma M,$$

The operator M turns creation and annihilation of domain walls into hopping of domain walls. The second operator Γ turns domain walls into particles and the third operator P fixes the phase factors.

P is highly nontrivial (generalised Jordan Wigner transform)

Explicit operators

M turns creation and annihilation of domain walls into hopping of domain walls.

$$M = \prod_{i=0}^{n-1} \delta_{4i+1}\delta_{4i+2}, \qquad \delta_j = i\,(c_j - c_j^\dagger)$$

with consequence

$$M(L-H) = HM,$$

The second operator Γ turns domain walls into particles

$$\Gamma = \prod_{i=1}^{L-1} \left(p_i + n_i(c_{i+1}^\dagger + c_{i+1})\right).$$

Domain wall model

Cross terms do not conserve fermions:

$$e_i^\dagger d_{i+1}^\dagger \, |\ldots \bullet \,|\, \circ\circ\circ \ldots\rangle \quad \mapsto \quad |\ldots \bullet\bullet\bullet \,|\, \circ \ldots\rangle$$
$$d_i e_{i+1} \, |\ldots \circ \,|\, \bullet\bullet\bullet \ldots\rangle \quad \mapsto \quad -|\ldots \circ\circ\circ \,|\, \bullet \ldots\rangle$$

But domain walls are conserved

Hopping terms:

$$d_{i+1}^\dagger d_i \, |\ldots \circ \,|\, \bullet \,|\, \circ\circ \ldots\rangle \quad \mapsto \quad |\ldots \circ\circ | \, \bullet \,|\, \circ \ldots\rangle$$
$$e_{i+1} e_i^\dagger \, |\ldots \bullet \,|\, \circ \,|\, \bullet\bullet \ldots\rangle \quad \mapsto \quad -|\ldots \bullet\bullet | \, \circ \,|\, \bullet \ldots\rangle$$

- 10 domain walls hop with two steps
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- nearby domain walls interact

Spectrum

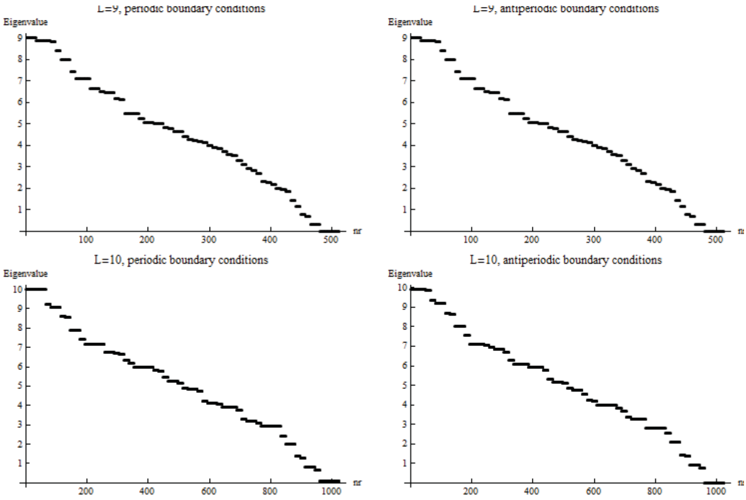

Figure 2: Comparison of spectra for system sizes $L = 7 - 10$ for periodic and antiperiodic boundary conditions.

Spectrum

The spectrum has a large degeneracy in powers of 2:

	4.618	8	9	0.000	32		4.380	8	10	0.087	16		5.829
	5.000	4		0.317	8		4.619	8		0.098	32		6.000
6	0.268	16		0.326	8		4.642	8		0.113	16		6.186
	2.000	16		0.671	8		4.783	8		0.637	16		6.337
	3.732	16		0.776	8		4.829	8		0.824	32		6.644
	6.000	16		1.129	8		5.000	16		1.289	16		6.721
7	0.000	16		1.446	8		5.064	8		1.362	16		6.727
	0.523	8		1.853	8		5.065	8		2.000	32		7.176
	1.177	8		1.937	8		5.257	8		2.429	16		7.177
	1.818	8		2.001	8		5.468	16		2.916	16		7.183
	3.000	8		2.195	8		5.494	8		2.922	32		7.443
	3.098	8		2.255	8		6.125	8		2.938	16		7.902
	3.198	8		2.300	8		6.168	8		3.069	16		8.572
	3.721	8		2.705	8		6.449	8		3.176	32		8.607
	4.095	8		2.831	8		6.461	8		3.292	16		9.078
	4.282	8		2.933	8		6.500	8		3.736	16		9.250
	4.555	8		3.097	8		6.653	16		3.898	16		10.000
	6.247	8		3.301	8		7.075	8		3.902	32		
	6.312	8		3.523	8		7.087	8		4.069	16		
	6.975	8		3.566	8		7.097	8		4.098	32		
	7.000	8		3.721	8		7.435	8		4.209	16		

Table 2: Multiplicities of eigenvalues for systems with periodic boundary conditions for $L = 3 - 10$.

Symmetries

Supersymmetry:

$$[H,\ Q] = 0, \qquad [H,\ Q^\dagger] = 0.$$

Particle-hole symmetry:

$$\Gamma = \prod_{i=1}^{L} \gamma_i, \qquad \gamma_i = c_i + c_i^\dagger, \qquad [\Gamma, H] = 0.$$

Domain wall – non-domain wall

$$E = \prod_{i=1}^{L/2} (c_{2i} - c_{2i}^\dagger), \qquad [E, H] = 0$$

Symmetries

Shift symmetry

$$S = \sum_{i=1}^{L} n_{i-1}\gamma_i p_{i+1} + p_{i-1}\gamma_i n_{i+1}, \qquad \gamma_i = c_i + c_i^\dagger, \qquad [S, H] = 0.$$

Reflection symmetry

$$M = \prod_{i=0}^{n-1} \delta_{4i+1}\delta_{4i+2}, \qquad \delta_j = \mathrm{i}\,(c_j - c_j^\dagger)$$

with consequence

$$M(L - H) = HM,$$

These are not enough to explain large degeneracy

Integrability

This model turns out to be integrable

Spectrum can be analysed by Bethe ansatz on domain wall dynamics

Conservation of all domain walls, but also of *odd* and *even* domain walls separately

Nested Bethe ansatz:

$$|\Psi(m;k)\rangle = \sum_{\{x_i\}} \sum_{\{p_j\}} \sum_{\epsilon=0,1} \psi_\epsilon(x_1,\ldots,x_m;p_1,\ldots,p_k)|x_1,\ldots,x_m;p_1,\ldots,p_k\rangle_\epsilon,$$

Derive conditions on ψ_ϵ such that

$$H|\Psi(m;k)\rangle = \Lambda|\Psi(m;k)\rangle.$$

Bethe ansatz

Bethe ansatz works and the spectrum is described by:

$$\Lambda = L + \sum_{i=1}^{2n} (z_i^2 + z_i^{-2} - 2).$$

where the numbers z_i satisfy the following equations

$$z_j^L = i^{-L/2} \prod_{k=1}^{m} \frac{u_k - (z_j - 1/z_j)^2}{u_k + (z_j - 1/z_j)^2}, \qquad j = 1, \ldots, 2n$$

$$1 = \prod_{j=1}^{2n} \frac{u_k - (z_j - 1/z_j)^2}{u_k + (z_j - 1/z_j)^2}, \qquad k = 1, \ldots, m.$$

Free fermion on nested level.

Zero energy Cooper pairs?

If $z_j^L = 1$, then

$$z_j^L = 1 = \frac{u_1 - (z_j - 1/z_j)^2}{u_1 + (z_j - 1/z_j)^2} \frac{u_2 - (z_j - 1/z_j)^2}{u_2 + (z_j - 1/z_j)^2}$$

$$1 = \prod_{j=1}^{2n} \frac{u_1 - (z_j - 1/z_j)^2}{u_1 + (z_j - 1/z_j)^2} = \prod_{j=1}^{2n} \frac{u_2 - (z_j - 1/z_j)^2}{u_2 + (z_j - 1/z_j)^2}$$

Pair $u_2 = -u_1$ are solutions that do not contribute to

$$\Lambda = L + \sum_{i=1}^{2n} (z_i^2 + z_i^{-2} - 2).$$

Zero energy Cooper pairs can explain large degeneracy in spectrum

Low excitations

For $L = 4n$, first excited states in section with $m = 2n - 2$ *even* domain walls.

This sector is free fermion: $z_j^{4n} = \pm 1$.

Energy gap is equal to

$$\Delta\Lambda_n = 4(1 - \cos(\pi/2n)) \approx \frac{\pi^2}{2n^2}. \tag{1}$$

Energy gap scales as $1/L^2$.

- Can we introduce a mass gap while maintaining supersymmetry?

Add staggering

$$H = \{Q^\dagger, Q\}, \qquad Q = \sum_{i=1}^{L-1} \lambda_i (d_i^\dagger + e_i), \qquad Q^2 = 0,$$

where

$$\lambda_i = 1 + (-1)^i \lambda, \qquad \lambda \in \mathbb{R}.$$

The Hamiltonian is

$$H = H_I + H_{II} + H_{III},$$

with

$$H_I = \sum_{i=1}^{L-1} (1 + (-1)^i \lambda)^2 \left(d_i^\dagger d_i + d_i d_i^\dagger \right) + (1 - \lambda^2) \sum_{i=1}^{L-2} \left(d_i^\dagger d_{i+1} + d_{i+1}^\dagger d_i \right),$$

$$H_{II} = \sum_{i=1}^{L-1} (1 + (-1)^i \lambda)^2 \left(e_i e_i^\dagger + e_i^\dagger e_i \right) + (1 - \lambda^2) \sum_{i=1}^{L-2} \left(e_i e_{i+1}^\dagger + e_{i+1} e_i^\dagger \right),$$

$$H_{III} = (1 - \lambda^2) \sum_{i=1}^{L-2} \left(e_i^\dagger d_{i+1}^\dagger + d_{i+1} e_i + e_{i+1}^\dagger d_i^\dagger + d_i e_{i+1} \right).$$

Nested Bethe Ansatz

$$\Lambda = const + (1-\lambda^2)\sum_{i=1}^{n}(z_i^2 + z_i^{-2})$$

$$\psi(x_1,\ldots,x_{2n};p_1,\ldots p_m) = \sum_{\pi\in S_{2n}} A^{\pi_1\cdots\pi_{2n}} \sum_{\sigma\in S_m} B^{\sigma_1\cdots\sigma_m} \prod_{j=1}^{m}\phi_{p_j}(u_{\sigma_j};\pi)\prod_{j=1}^{n}\left[(\mathrm{i}z_{\pi_{2j-1}})^{x_{2j-1}}z_{\pi_{2j}}^{x_{2j}}\right],$$

$$\phi_p(u;\pi) = g(z_{\pi_p})(-1)^{\lfloor (p-1)/2\rfloor}\prod_{j=1}^{p-1} f(u,z_{\pi_j}).$$

Nested Bethe Ansatz

- Periodic boundaries

$$z_j^L = i^{-L/2} \prod_{k=1}^{m} \frac{v_k - (z_j - 1/z_j)^2 + c}{v_k + (z_j - 1/z_j)^2 - c}, \qquad j = 1, \ldots, n,$$

$$1 = \prod_{k=1}^{m} \frac{v_k - (z_j - 1/z_j)^2 + c}{v_k + (z_j - 1/z_j)^2 - c}, \qquad k = 1, \ldots, m.$$

$$v = \frac{u}{1-\lambda^2}, \qquad c = \frac{4\lambda^2}{1-\lambda^2}.$$

- Open boundaries

$$z_i^{2(L+1)} = -\frac{1-\lambda - z_i^2(1+\lambda)}{1+\lambda - z_i^2(1-\lambda)}, \quad i = 1, \ldots, n, \quad L \text{ even}.$$

$$z_i^{2(L+1)} = 1, \quad i = 1, \ldots, n, \quad L \text{ odd}.$$

No dependence on nested rapidities! ⇒ Very large degeneracy.

Conclusions

- SUSY supersymmetric lattice model, with two very different physical pictures.
- Staggering: maintain integrability and supersymmetry
- Gapped and exponential degeneracy of all energy levels
- Very large symmetry algebra (?)
- Zero energy BCS pairing mechanism
- Topological phases?