Exact logarithmic connectivities in 2d critical percolation (and the Ising model)

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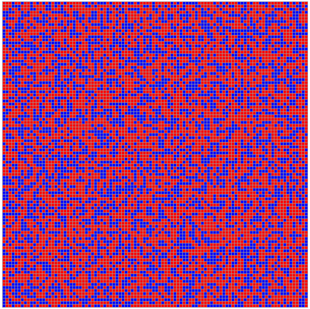
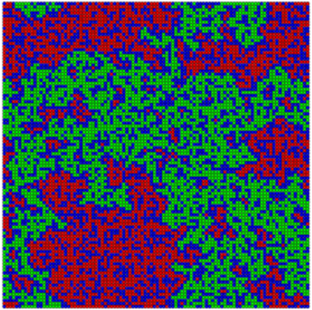
1. Introduction

Invitation: 2d percolation

- A stat. mech models such as Ising, Q-state Potts model or percolation can be formulated in terms of random clusters





- **Percolation:** $L \times L$ random matrix of $0, 1$. Phase transition signaled by the finite probability of one point being connected to the boundary
- No symmetry, however at $p = p_c$ configurations are scale (conformally) invariant! [Langlands-Pouliot-Saint Aubin 92]

Invitation: 2d percolation (see also G. Delfino Ann. Phys. 360 (2015) and Cardy's book)

- Critical exponents obtained from mapping to the Q-color Potts model (Coulomb Gas) +BPZ (i.e. Conformal Field Theory)
- **Textbook example:**

$$\begin{pmatrix}\text{prob. to be connected} \\ \text{to the boundary}\end{pmatrix} \simeq A(p-p_c)^{\frac{5}{36}}$$

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$$\begin{pmatrix} \text{prob. to be connected} \\ \text{to the boundary} \end{pmatrix} \simeq A(p - p_c)^{\frac{5}{36}}$$

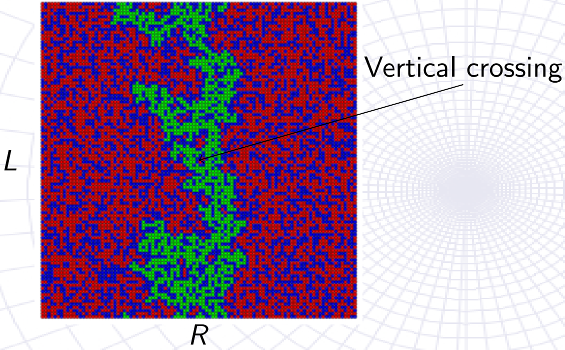
- **However (not in textbooks):** Let Z the percolation partition function on a $L \times L$ matrix

$$Z(L) = 1 \Rightarrow \text{as a CFT, percolation has zero central charge!}$$

- Earlier influential works: Nienhuis, Duplantier-Saluer, Dotsenko-Fateev (especially for the Q-color Potts model)

A deeper insight: Cardy formula

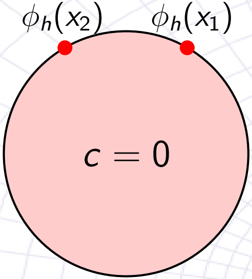
- Back in 1992, J. Cardy showed how to adapt techniques of Boundary Conformal Field Theory to percolation (or a $c = 0$ CFT)



$$\pi_v(\eta) = 3\frac{\Gamma\left(\frac{2}{3}\right)}{\Gamma\left(\frac{1}{3}\right)^2}\eta^{1/3}\ {}_2F_1\left(\tfrac{1}{3}, \tfrac{2}{3}, \tfrac{4}{3}, \eta\right); \quad \eta = f(L/R)$$

- A result proved by S. Smirnov (Field's medal) in 2009

Gurarie and Ludwig b number at $c = 0$



- Consider a CFT with $c = 0$ on a bounded planar domain
- Map to the upper half plane and $\phi_h(z)$ be a boundary field

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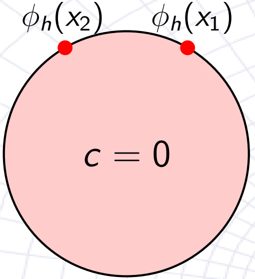


- Consider a CFT with $c = 0$ on a bounded planar domain
- Map to the upper half plane and $\phi_h(z)$ be a boundary field

- There is an obvious problem for the OPE at $c = 0$ if $h \neq 0$

$$\lim_{z \to 0} \phi_h(z)\phi_h(0) = \frac{1}{z^{2h}} \left[1 + \frac{2h}{c} z^2 \overbrace{T(0)}^{\text{stress tensor}} + \dots \right]$$

- Mixing of the null field $T(z)$ with another (logarithmic) field $t(z)$ of the same dimensions [Gurarie-Ludwig 04]

$$\lim_{z \to 0} \phi_h(z)\phi_h(0) = \frac{1}{z^{2h}} \left[1 + \frac{h}{b} z^2 (t(0) + log(z) T(0)) + \dots \right]$$

An useful device: Q-color Potts model [Wu 82]

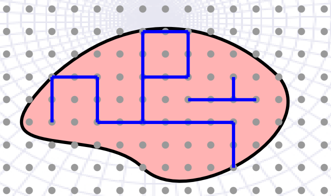
- The Hamiltonian is given by $(J > 0)$

$$H_Q = -J \sum_{\langle x,y \rangle} \delta_{s(x),s(y)}, \qquad s(x) = 1, \ldots, Q$$

- Graph expansion [Fortuin-Kasteleyn 70], here $p = 1 - e^{-J}$

$$p^{\#\text{ bonds}}(1-p)^{\#\text{ empty bonds}}$$

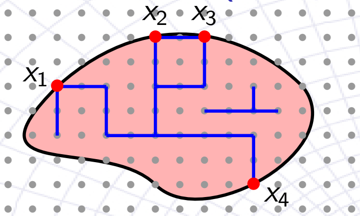
$$Z_Q = \sum_{\mathcal{G}}$$



$$\times\, Q^{\#\text{ clusters}}$$

- $Q = 1$ corresponds to percolation and $Q = 2$ to the Ising model. First order for $Q > 4$ in 2d.

Connectivities (i.e. geometric correlators)



- Mark n points on the boundary of the domain
- How they are partitioned into clusters?

- From solving sum rules for probabilities [Delfino-V. '11]

lin. ind. connectivities = # non-singleton non-crossing partitions

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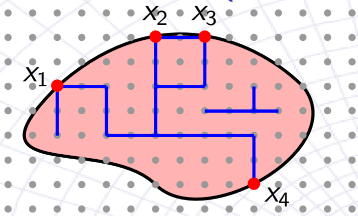

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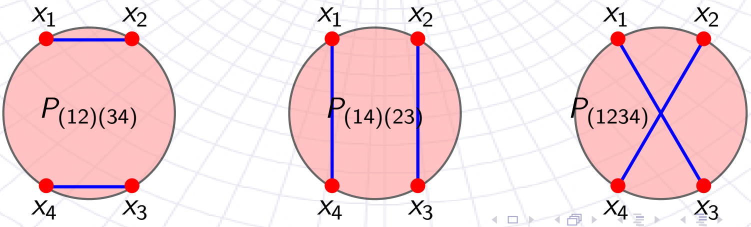
- Example: $n = 4$, probabilities of the following configurations

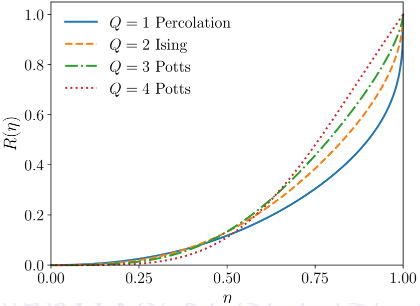


Universal ratio R

- From the three connectivities we can construct an universal ratio R

$$R = \frac{P_{(14)(23)}}{P_{(14)(23)} + P_{(12)(23)} + P_{(1234)}}$$

- R can be measured in **Monte Carlo** simulations (and calculated exactly from CFT)



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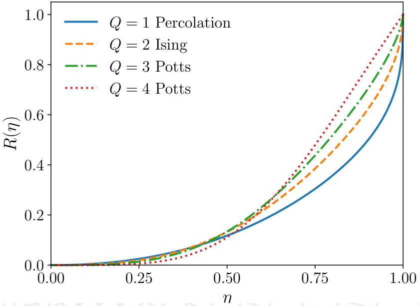

- **Why only η?** From conformal symmetry, R can be calculated in the UHP ($z_1 < z_2 < z_3 < z_4$)

$$R = R(\eta), \quad \eta = \frac{z_{21} z_{43}}{z_{31} z_{42}}, \quad 0 < \eta < 1$$

- Under a conformal map to a new geometry $w = f(z)$ it does not change (conformally invariant)

2. Some CFT details

(Boundary) CFT for the Potts model

- Q-color Potts conformal field theory [Dotsenko-Fateev 84]

$$c = 1 - \frac{6}{p(p+1)} \quad \text{and} \quad Q = 4\cos^2\left[\frac{\pi}{p+1}\right]$$

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- Operators at the boundary have scaling dimensions [Cardy 89]

$$h_{r,s} = \frac{[r(p+1) - sp]^2 - 1}{4p(p+1)}$$

$Q = 2$ $\mathbb{Z}_2$ Ising

s \ r	1	2	3	4	5
5	$\frac{5}{2}$	1	$\frac{1}{6}$	0	$\frac{1}{2}$
4	$\frac{21}{16}$	$\frac{5}{16}$	$\frac{1}{6}$	$-\frac{1}{48}$	$\frac{5}{16}$
3	$\frac{1}{2}$	0	$\frac{1}{6}$	1	$\frac{5}{2}$
2	$\frac{1}{16}$	$\frac{1}{16}$	$\frac{35}{48}$	$\frac{33}{16}$	$\frac{65}{116}$
1	0	$\frac{1}{2}$	$\frac{5}{3}$	$\frac{7}{2}$	6

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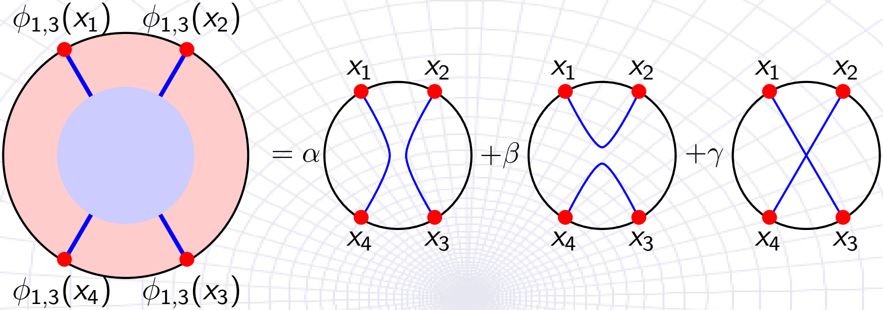
$\phi_{1,2s+1}$ anchors s clusters at the boundary [Saleur-Duplantier 87]

Connectivities from CFT



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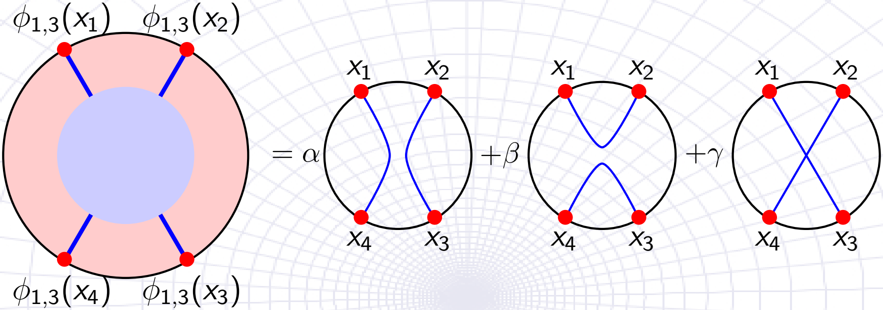

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$$\langle \phi_{1,3}(z_1)\phi_{1,3}(z_2)\phi_{1,3}(z_3)\phi_{1,3}(z_4)\rangle_{\mathbb{H}} = \frac{1}{(z_{12}z_{34})^{2h}} \overbrace{\frac{G(\eta)}{(1-\eta)^{2h}}}^{\text{conf. block}}$$

- The function $G(\eta)$ solves a **third** order ODE [BPZ 83]

Frobenius series vs Conformal blocks

- The differential equation for $G(\eta)$ is ($h = h_{1,3}$)

$$6(1-h)h^2(-1+2\eta)G(\eta) + \left[(2(-1+\eta)\eta - 3h\left(1-5\eta+5\eta^2\right) + h^2\left(3-19\eta+19\eta^2\right)\right]G'(\eta) + (-1+\eta)\eta\left[(-2+4h+4\eta-8h\eta)G''(\eta) + (-1+\eta)\eta G'''(\eta)\right] = 0.$$

- The 3d space of solutions is spanned by

1.Frobenius series $G_\rho(\eta) = \sum_{k=1}^{N} a_k \eta^{\rho+k}$	⟷	2.Conformal blocks [Al. Zam 87]	⟷	3. Geometric connectivities $P_{(12)(34)}$, $P_{(14)(23)}$, $P_{(1234)}$

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- The exponent ρ corresponds to the leading singularity produced in the OPE when $z_1 \to z_2$ (care when roots are separated by integers!)

$$\overbrace{\rho}^{\text{no cluster}}\ \overbrace{(\rho - h_{1,3})}^{\text{one clust.}}\overbrace{(\rho - h_{1,5})}^{\text{two clust.}} = 0$$

Conformal Blocks vs Connectivities

- Remember that we want to calculate

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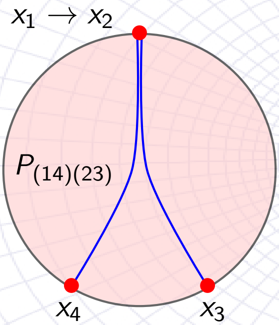

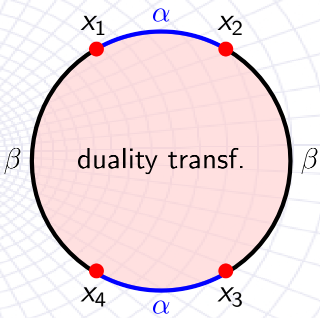

Num. $\to$ conf. block of $\phi_{1,5}$

Den. $\to$ (linear comb. of $\phi_{1,5}$ and identity conf. blocks symm. under $x_1 \leftrightarrow x_3$)

3. Results

Percolation ($Q = 1$ or $c = 0$)

- We have both Frobenius series up to 10^5 coefficients and **closed forms**

$$R_{Q=1}(\eta) = A_1 \frac{G_2(\eta)}{G_0(\eta)}, \quad A_1 = \frac{3^{7/6}\pi\Gamma\left(\frac{5}{9}\right)\Gamma\left(\frac{8}{9}\right)\Gamma\left(\frac{7}{3}\right)}{4\cos(13\pi/18)\Gamma\left(-\frac{2}{9}\right)\Gamma\left(\frac{1}{6}\right)\Gamma\left(\frac{11}{6}\right)}.$$

- Where the **numerator** (i.e. the $\phi_{1,5}$ conformal block) is expressed through a (rather complicated) ${}_3F_2$

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- Where the **numerator** (i.e. the $\phi_{1,5}$ conformal block) is expressed through a (rather complicated) ${}_3F_2$
- The **denominator** is the regularized identity conformal block at $c = 0$

$$G_0(\eta) = -\frac{8}{45}\log(\eta)G_2(\eta) + \left(1 - \frac{2}{3}\eta + \frac{119}{225}\eta^2 + \frac{152}{2025}\eta^3 + o(\eta^3)\right).$$

- The coefficient of the logarithm is according to **Gurarie and Ludwig**

$$\frac{h^2}{b} \xrightarrow{h=\frac{1}{3}} b = -\frac{5}{8}$$

Monte Carlo vs CFT for percolation

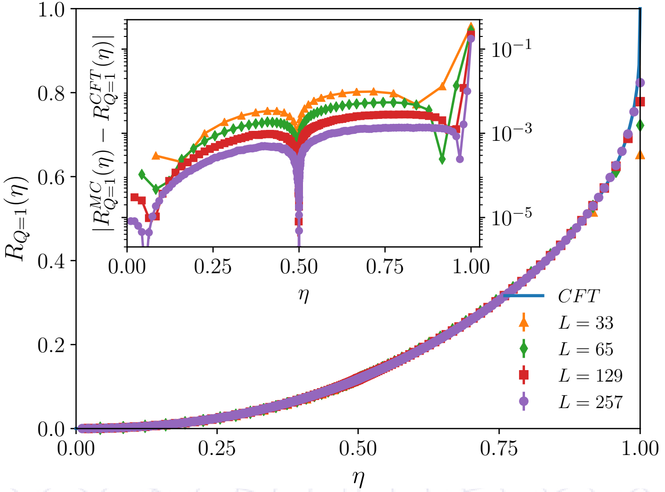


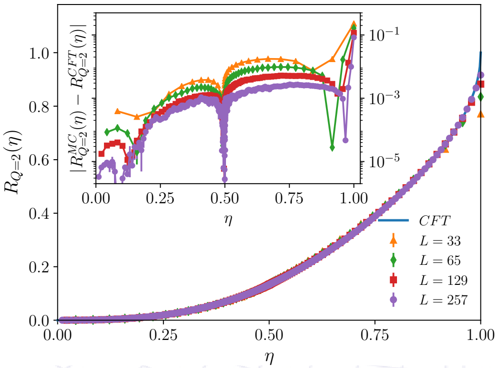
Figure: Simulations on a triangular lattice

Ising ($Q = 2$ or $c = 1/2$)

- We have a more explicit result [Gori-V. 17]

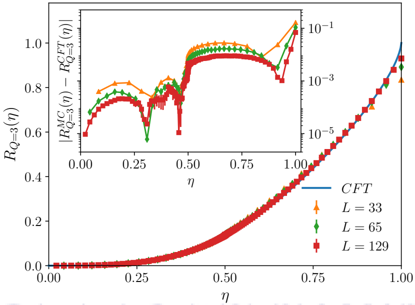
$$R_{Q=2}(\eta) = A_2 \int_0^\eta g(\eta')$$

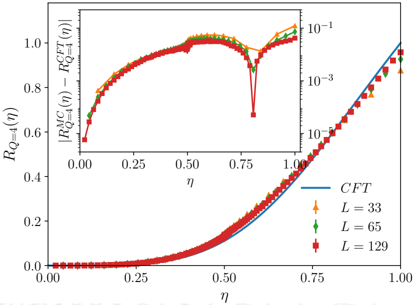
- where g contains elliptic integrals of first and second kind



- **Logarithmic** singularity for $\eta \to 1$
- Collision between $\phi_{1,5}$ and null vector at level two of $\phi_{1,3}$

Q=3 ($c = 4/5$) and Q=4 ($c = 1$)





- Exact value for $R_{Q=3}$

$$-\frac{(1-\eta)^{2/3}\left(1-\frac{2\eta}{3}+\eta^2\right)}{1-\frac{4\eta}{3}+\frac{4\eta^2}{3}}$$

- Exact value for $R_{Q=4}$

$$\left(\frac{\eta^2}{1-\eta+\eta^2}\right)^2$$

Summary

- Exact CFT results for four-point boundary connectivities in the Q-color Potts model
- Explicit logarithmic singularities at $c = 0$ (percolation) and $c = 1/2$ (Ising)
- First correlation function at $c = 0$ where Gurarie and Ludwig b number appears explicitly

Thank you!