Limit behavior of tilings via Algebraic Combinatorics

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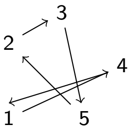
Algebraic Combinatorics: basics

Permutations: $\pi = 43512$



Symmetric group S_n: Permutations $\pi : [1..n] \rightarrowtail [1..n]$ under composition.

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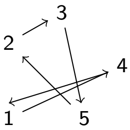

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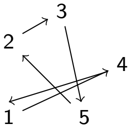

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Irreducible representations of the symmetric group S_n:

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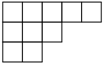
are the **Specht modules** $\mathbb{S}_\lambda$, indexed by

integer partitions $\lambda \vdash n$:

$\lambda = (\lambda_1, \ldots, \lambda_\ell)$, $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_\ell > 0$, $\lambda_1 + \lambda_2 + \cdots = n$

Young diagram of λ:  Here $\lambda = (5, 3, 2)$

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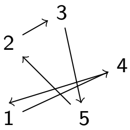

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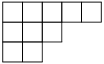
Young diagram of λ:  Here $\lambda = (5, 3, 2)$

Basis for $\mathbb{S}_\lambda$: S_tandard_ **Y**_oung_ **T**_ableaux_ of shape λ: $\lambda = (3, 2)$

$\begin{array}{|c|c|c|}\hline 1&2&3\\ \hline 4&5\\ \cline{1-2}\end{array}$ $\begin{array}{|c|c|c|}\hline 1&2&4\\ \hline 3&5\\ \cline{1-2}\end{array}$ $\begin{array}{|c|c|c|}\hline 1&2&5\\ \hline 3&4\\ \cline{1-2}\end{array}$ $\begin{array}{|c|c|c|}\hline 1&3&4\\ \hline 2&5\\ \cline{1-2}\end{array}$ $\begin{array}{|c|c|c|}\hline 1&3&5\\ \hline 2&4\\ \cline{1-2}\end{array}$

Young Tableaux and Schur functions

Irreducible representations of the **symmetric group** S_n: **Specht modules** $\mathbb{S}_\lambda$

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Irreducible (polynomial) representations of the **General Linear group** $GL_N(\mathbb{C})$:

Weyl modules V_λ, indexed by highest weights λ, $\ell(\lambda) \leq N$.

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Irreducible representations of the **symmetric group** S_n: **Specht modules** $\mathbb{S}_\lambda$

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Irreducible (polynomial) representations of the **General Linear group** $GL_N(\mathbb{C})$:

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Schur functions: characters of V_λ
$Tr_{V_\lambda}(diag(x_1, .., x_N)) = s_\lambda(x_1, .., x_N)$
Weyl's determinantal formula:

$$s_\lambda(x_1, \ldots, x_N) = \frac{\det \left[x_i^{\lambda_j + N - j} \right]_{ij=1}^{N}}{\prod_{i<j}(x_i - x_j)}$$

Semi-Standard Young tableaux of shape λ :

$$s_{(2,2)}(x_1, x_2, x_3) = \underset{\begin{array}{|c|c|}\hline 1&1\\\hline 2&2\\\hline\end{array}}{x_1^2x_2^2} + \underset{\begin{array}{|c|c|}\hline 1&1\\\hline 3&3\\\hline\end{array}}{x_1^2x_3^2} + \underset{\begin{array}{|c|c|}\hline 2&2\\\hline 3&3\\\hline\end{array}}{x_2^2x_3^2} + \underset{\begin{array}{|c|c|}\hline 1&1\\\hline 2&3\\\hline\end{array}}{x_1^2x_2x_3} + \underset{\begin{array}{|c|c|}\hline 1&2\\\hline 2&3\\\hline\end{array}}{x_1x_2^2x_3} + \underset{\begin{array}{|c|c|}\hline 1&2\\\hline 3&3\\\hline\end{array}}{x_1x_2x_3^2}.$$

Schur functions in statistical mechanics

Characters of $U(\infty)$, boundary of the Gelfand-Tsetlin graph

1	1	1	2	2	...
2	2	3	...		
...					

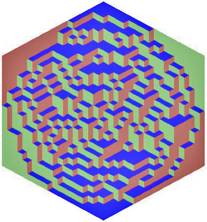
Alternating Sign Matrices (ASM)/ 6-Vertex model:

$$\begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 1 & -1 & 1 \\ 1 & -1 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}$$

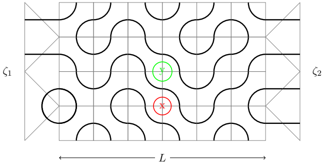
Normalized Schur functions:

$$S_\lambda(x_1, \ldots, x_k; N) = \frac{s_\lambda(x_1, \ldots, x_k, 1^{N-k})}{s_\lambda(1^N)}$$

Lozenge tilings:



Dense loop model:



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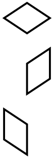
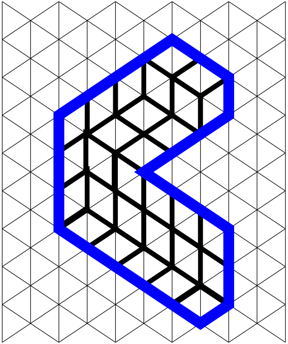


Tilings of a domain Ω (on a triangular lattice) with elementary rhombi of 3 types ("lozenges").



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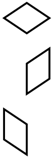

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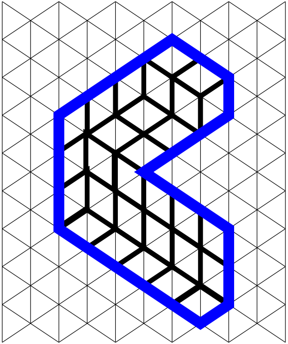
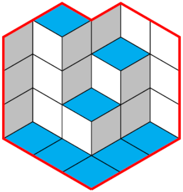
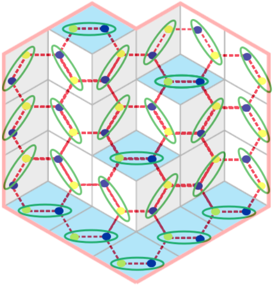
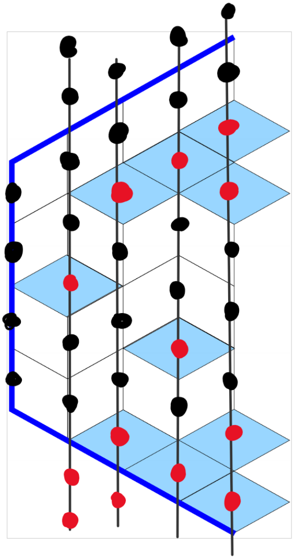




Dimer covers on the hexagonal grid

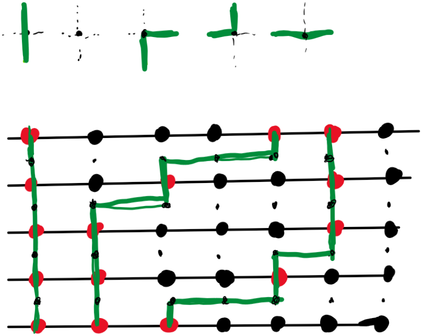


Particles and 5 vertex model



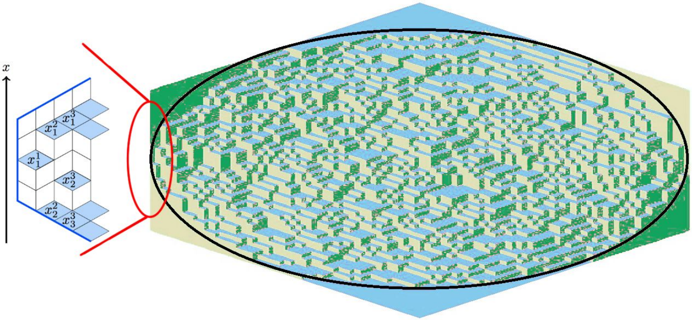
5 vertex model

<-> non-intersecting lattice paths



Classical questions: limit behavior

Question: Fix Ω in the plane and let *grid size* $\to 0$, what are the properties of *uniformly random* tilings of Ω?

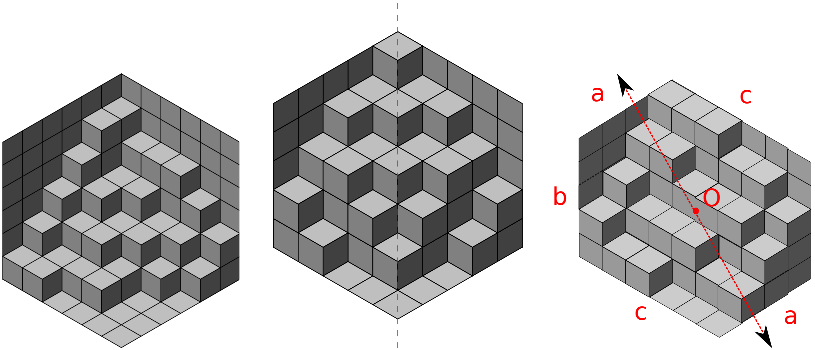


Frozen regions (polygonal domains), "limit shapes" of the surface of the height function (plane partition).
([Cohn–Larsen–Propp, 1998], [Kenyon–Okounkov, 2005], [Cohn–Kenyon–Propp, 2001; Kenyon-Okounkov-Sheffield, 2006])
Behavior near boundary: Gaussian Unitary Ensemble eigenvalues,
conjectured by [Okounkov-Reshetikhin, 2006], proofs – hexagon [Johansson-Nordenstam, 2006], [Gorin-Panova, 2013]

Unrestricted (uniform) vs symmetric

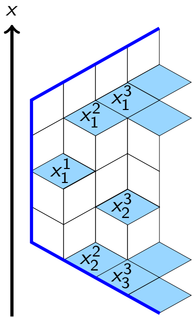
Tilings of the hexagon $a \times b \times c \times a \times b \times c$, s.t.

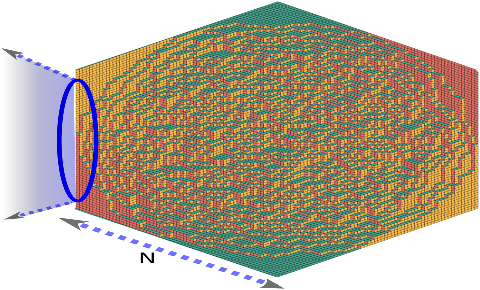
Unrestricted | Vertically symmetric | Centrally symmetric



Limit behavior: fluctuations near the boundary, limit surface, CLT?

Behavior near the flat boundary:





Horizontal lozenges near a flat boundary:

$$\begin{array}{ccccccccc} & & & & x_1^1 & & & & \\ & & & \le & & \le & & & \\ & & x_2^2 & & & & x_1^2 & & \\ & \le & & \le & & \le & & \le & \\ x_3^3 & & & & x_2^3 & & & & x_1^3 \end{array}$$

Question: Joint distribution of $\{x_j^i\}_{i=1}^k$ as $N \to \infty$ (rescaled)?

Conjecture [Okounkov–Reshetikhin, 2006]:
Fixed boundary: The joint distribution converges to a *GUE*-corners (aka *GUE*-minors) process: eigenvalues of GUE matrices.

Proofs: hexagonal domain [Johansson-Nordenstam, 2006], more general domains [Gorin-P,2012], [Novak, 2014], unbounded [Mkrtchyan, 2013], symmetric tilings [P, 2014, 2015]

Behavior near the flat boundary:GUE

GUE: matrices $A = [A_{ij}]_{i,j}$: $A = \overline{A^T}$

$\mathrm{Re}A_{ij}, \mathrm{Im}A_{ij}$ – i.i.d. $\sim \mathcal{N}(0, 1/2)$, $i \neq j$

A_{ii} – i.i.d. $\sim \mathcal{N}(0, 1)$

$$\begin{pmatrix} A_{11} & A_{12} & A_{13} & A_{14} \\ A_{21} & A_{22} & A_{23} & A_{24} \\ A_{31} & A_{32} & A_{33} & A_{34} \\ A_{41} & A_{42} & A_{43} & A_{44} \end{pmatrix}$$

$(x_1^k \leq x_2^k \leq \cdots \leq x_k^k)$ – eigenvalues of $[A_{i,j}]_{i,j=1}^k$

Interlacing condition: $x_{i-1}^j \leq x_{i-1}^{j-1} \leq x_i^j$

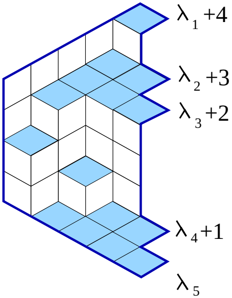
$$\begin{array}{ccccccc} x_1^4 & & x_2^4 & & x_3^4 & & x_4^4 \\ & x_1^3 & & x_2^3 & & x_3^3 & \\ \leq & & x_1^2 & & x_2^2 & & \geq \\ & & & x_1^1 & & & \end{array}$$

The *joint distribution* of $\{x_i^j\}_{1 \leq i \leq j \leq k}$ is the
GUE–corners (also, GUE-minors) process, $=: \mathbb{GUE}_k$.

Tilings setup

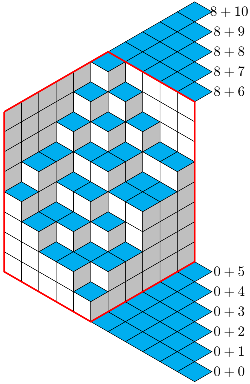
Domain $\Omega_{\lambda(N)}$:
positions of the N horizontal lozenges on right boundary are:

$$\lambda(N)_1 + N - 1 > \lambda(N)_2 + N - 2 > \cdots > \lambda(N)_N$$



$\lambda(5) = (4, 3, 3, 0, 0)$

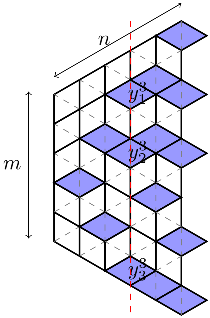
($\frac{1}{N}\Omega_{\lambda(N)}$ is *not necessarily a finite polygon as* $N \to \infty$, e.g. $\lambda(N) = (N, N-1, \ldots, 2, 1)$)



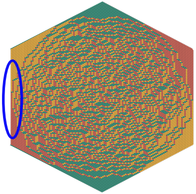
$\lambda = (\underbrace{a, \ldots, a}_{c}, \underbrace{0, \ldots, 0}_{b})$

$\leftrightarrow a \times b \times c$... hexagon.

Behavior near the flat left boundary



Line $k = 3$



Theorem

Let $Y_n^k = (y_1^k, \ldots, y_k^k)$ – horizontal lozenges on kth line of a uniformly random tiling $T \in \mathcal{T}_n$. As $n \to \infty$ the collection

$$\left\{ \frac{Y_n^j - \mu_n}{\sqrt{n\sigma_n}} \right\}_{j=1}^{k} \to \mathbb{GUE}_k$$

weakly as RVs, where

- *$\mathcal{T}_n$ – all tilings of a hexagon [Johansson-Nordenstam].*
- *$\mathcal{T}_n = \Omega_{\lambda(n)}$ – $\mu_n = E(f)$, $\sigma_n = S(f)$, "$f(t) = \lim_{n\to\infty} \frac{\lambda(n)_{nt}}{n}$" [Gorin-P, 2013].*
- *$\mathcal{T}_n$ – vertically symmetric lozenge tilings of a $n \times m \times n$.. hexagon, $a = \lim_{n\to\infty} m/n$, $\mu_n = m/2$, $\sigma_n = \frac{a^2+2a}{8}$ [P, 2014].*
- *$\mathcal{T}_n$ – centrally-symmetric tilings of a $a \times b \times c$... hexagon with $a = 2qn$, $b = 2pn$, $c = 2(1-q)n$: $\mu_n = 2pqn$ and $\sigma_n = 2pq(1-q)(1+p)$ [P, 2015+].*

Limit shape (surface)

Theorem (P)

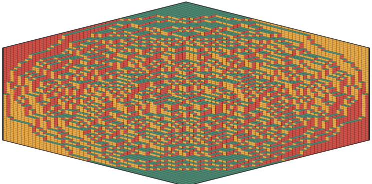
Let $H_n(u,v)$ – height function of a uniformly random tiling from a set $\mathcal{T}_n$, i.e.

$$H_n(u,v) = \frac{1}{n} y^{\lfloor nu \rfloor}_{\lfloor nv \rfloor} - v,$$

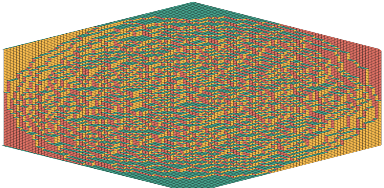
where y_i^k is the vertical height of the ith horizontal lozenge on the kth vertical line (left to right). For all $1 \geq u \geq v \geq 0$, as $n \to \infty$ we have that $H_n(u,v)$ converges uniformly in probability to a deterministic function $L(u,v)$ ("the limit shape"), which can be computed explicitly... when $\mathcal{T}_n$ is

- *$\mathcal{T}_n$ – polygonal domain [Cohn, Kenyon, Larsen, Propp, Okounkov]*
- *$\mathcal{T}_n = \Omega_{\lambda(n)}$ for "nice" family $\lambda(n)$ [Bufetov-Gorin].*
- *$\mathcal{T}_n$ – symmetric tilings [P, 2014].*
- *$\mathcal{T}_n$ – centrally symmetric tilings [P, 2015+].*

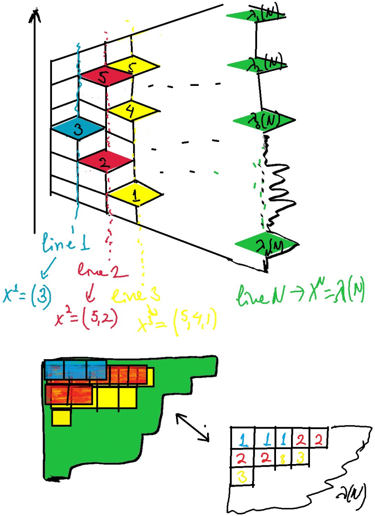
Symmetric:



General:



Tilings probability: combinatorics and SSYTs



Lozenge tilings with right boundary $\lambda(N)$

$\Longleftrightarrow$

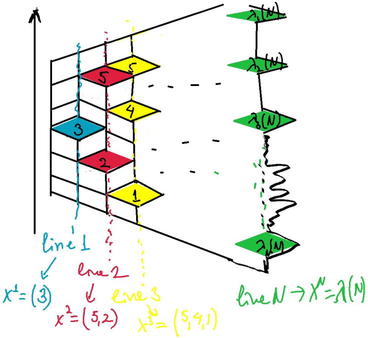
Semi-Standard Young Tableaux T of shape $\lambda(N)$ and entries $1, \ldots, N$.

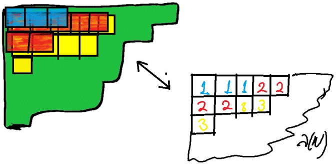
Tilings with horizontal lozenges on vertical line k at positions $x^k = \eta_1, \ldots, \eta_k$

$\Longleftrightarrow$

SSYTs T whose entries $1..k$ have shape η

Tilings probability: combinatorics and SSYTs





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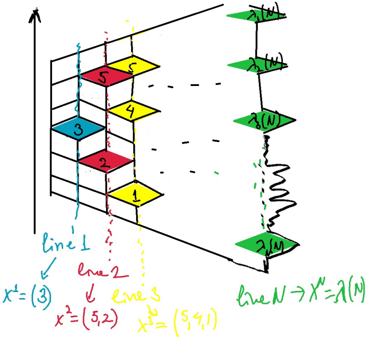
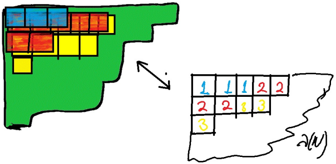
Number of SSYTs of shape ν, entries $1...\ell$ = $s_\nu\underbrace{(1, \dots, 1)}_{\ell}$.

$$\mathrm{Prob}\{x^k(\lambda) = \eta\} = \frac{s_\eta(1^k)s_{\lambda/\eta}(1^{N-k})}{s_\lambda(1^N)},$$

Tilings probability: combinatorics and SSYTs





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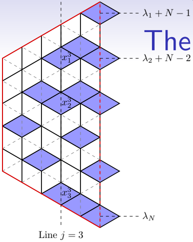
$$\mathrm{Prob}\{x^k(\lambda) = \eta\} = \frac{s_\eta(1^k) s_{\lambda/\eta}(1^{N-k})}{s_\lambda(1^N)},$$

Proposition[Gorin-P] For any variables $y_1, \ldots, y_k$, the **Schur Generating Function** of x^k is

$$\mathbb{E}\left(\frac{s_{x^k}(y_1, \ldots, y_k)}{s_{x^k}(\underbrace{1, \ldots, 1}_{k})} \right) = \frac{s_\lambda(y_1, \ldots, y_k, \overbrace{1, \ldots, 1}^{N-k})}{s_\lambda(\underbrace{1, \ldots, 1}_{N})} =: S_\lambda(y_1, \ldots, y_k).$$



The explicit Schur Generating Functions[1]

$\mathcal{T}_n$ – set of tilings, $x^j(T)$ – horizontal lozenge positions on line j of $T \in \mathcal{T}_n$

MGF: $$\mathbb{E}\left[\frac{s_{x^k(T)}(y_1,\ldots,y_k)}{s_{x^k(T)}(\underbrace{1,\ldots,1}_{k})} \;\middle|\; T \sim Unif(\mathcal{T}_n) \right] = \sum_{\nu} \frac{s_\nu(y_1,\ldots,y_k)}{s_\nu(1^k)} \Pr(x^k(T) = \nu) = \ldots$$

- $= S_{\lambda(n)}(y_1,\ldots,y_k) = \frac{s_{\lambda(n)}(y_1,\ldots,y_k,1^{n-k})}{s_{\lambda(n)}(1^n)}$ for $\mathcal{T}_n = \Omega_{\lambda(n)}$.
- $= \prod_i y_i^{m/2} \cdot \frac{\mathfrak{so}_{(\frac{m}{2})^n}(y_1,\ldots,y_k,1^{n-k})}{\mathfrak{so}_{(\frac{m}{2})^n}(1^n)}$ for $\mathcal{T}_n$ – symmetric tilings of $n \times m \times n$....
- $= S_{(\frac{b}{2})^{a/2}}(y_1,\ldots,y_k)^2$ for $\mathcal{T}_n$ – centrally symmetric tilings of $a \times b \times c$... hexagon.

[1] from [Gorin-Panova, *Ann. Prob.*], [Panova, *Comm. Math. Phys*], [Panova, in prep]

Tilings probability III: MGF asymptotics

Proposition (Gorin-P)

$$\mathbb{E}\left[\frac{s_{\nu-\delta_k}(y_1,\ldots,y_k)}{s_{\nu-\delta_k}(\underbrace{1,\ldots,1}_{k})} \quad \nu \sim \mathbb{GUE}_k\right] = \exp\left(\frac{1}{2}(y_1^2+\cdots+y_k^2)\right),$$

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Compare:

$$S_\lambda(y_1,\ldots,y_k) = \mathbb{E}_{tiling}\left(\frac{s_{x^k}(y_1,\ldots,y_k)}{s_{x^k}(\underbrace{1,\ldots,1}_{k})}\right)$$

Proposition (Gorin-P)

For any k real numbers $h_1,\ldots,h_k$ and $\lambda(N)/N \to f$ we have:

$$\lim_{N\to\infty} S_{\lambda(N)}\left(e^{\frac{h_1}{\sqrt{NS(f)}}},\ldots,e^{\frac{h_k}{\sqrt{NS(f)}}}\right)e^{\left(-\frac{E(f)}{\sqrt{NS(f)}}\sum_{i=1}^k h_i\right)} = \exp\left(\frac{1}{2}\sum_{i=1}^k h_i^2\right).$$

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Theorem. Let $\Upsilon^k_{\lambda(N)} = \{x^k, x^{k-1},\ldots\}$ –collection of positions of the horizontal lozenges on lines $k, k-1,\ldots,1$ of tiling from $\Omega_{\lambda(N)}$, then

$$\frac{\Upsilon^k_{\lambda(N)} - NE(f)}{\sqrt{NS(f)}} \to \mathbb{GUE}_k \text{ (GUE-corners process of rank } k).$$

The limit surface

Counting measure:

$$m[\mu] := \frac{1}{L} \sum_{i=1}^{L} \delta\left(\frac{\mu_i + L - i}{L} \right),$$

Random measure on μs: $\rho^n(\mu)$ (e.g. $= \mathrm{Prob}\{x^k(T) = \mu\}$ for $T \in \mathcal{T}_n$), $m[\rho]$ – pushforward.

$$S_\rho(u_1, \ldots, u_k) := \sum_{\mu} \rho(\mu) \frac{s_\mu(u_1, \ldots, u_k)}{s_\mu(1^k)} = \mathbb{E}\left[\frac{s_{x^k(T)}(y_1, \ldots, y_k)}{s_{x^k(T)}(\underbrace{1, \ldots, 1}_{k})} \,\middle|\, T \sim Unif(T_n) \right]$$

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Theorem[Bufetov-Gorin,2014] Suppose that ρ^N is s.t. for every r

$$\lim_{N\to\infty} \frac{1}{N} \ln\left(S_{\rho^N}(u_1,\ldots,u_r,1^{N-r})\right) = Q(u_1) + \cdots + Q(u_r),$$

uniformly in a $\mathbb{C}$ nbhd of (1^r), Q – analytic. Then the random measures $m[\rho^N]$ converge, as $N \to \infty$, in probability to a deterministic measure M on $\mathbb{R}$ with moments

$$\int_{\mathbb{R}} t^p M(dt) = \sum_{\ell=0}^{p} \binom{p}{\ell} \frac{1}{(\ell+1)!} \frac{\partial^\ell}{\partial u^\ell} u^p Q'(u)^{p-\ell}\Big|_{u=1}$$

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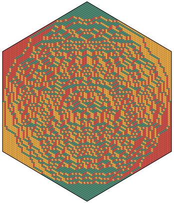
Our cases: MGF = normalized Schur $S_{\lambda(n)}$, SO characters, etc.
Asymptotics using [Gorin-P, 2013] for fixed r:

$$\lim_{n\to\infty} \frac{1}{n} \ln S_{\lambda(n)}(u_1,\ldots,u_r) = \sum_{i=1}^{r} \lim_{n\to\infty} \frac{1}{n} \ln S_{\lambda(n)}(u_i) = \sum_{i=1}^{r} \Phi(u_i)$$

Limit surface for symmetric tilings



Theorem (P, 2014)

Let $n, m \in \mathbb{Z}$, such that $m/n \to a$ as $n \to \infty$, where $a \in (0, +\infty)$. Let $H_n(u, v)$ – height function of a symmetric tiling of $n \times m \times n\ldots$ hexagon,i.e.

$$H_n(u, v) = \frac{1}{n} y^{\lfloor nu \rfloor}_{\lfloor nv \rfloor} - v.$$

For all $1 \geq u \geq v \geq 0$, as $n \to \infty$:
$H_n(u, v)$ converges unif. in prob. to a deterministic function $L(u, v)$ ("the limit surface").

For any fixed $u \in (0, 1)$, $L(u, v)$ is the distribution function of the measure $\mathbf{m}$, given by its moments:

$$\int_{\mathbb{R}} t^r \mathbf{m}(dt) = \sum_{\ell=0}^{r} \binom{r}{\ell} \frac{1}{(\ell+1)!} u^{-r+\ell} \frac{\partial^\ell}{\partial z^\ell} z^p \Phi'_a(z)^{p-\ell} \Big|_{z=1},$$

where $\Phi_a(e^y) = y\frac{a}{2} + 2\phi(y; a) - 2$ and...

$$h(y) = \frac{1}{4}\left((e^y + 1) + \sqrt{(e^y + 1)^2 + 4(a^2 + a)(e^y - 1)^2}\right)$$

$$\phi(y; a) = (\frac{a}{2} + 1)\ln\left(h(y) - (\frac{a}{2} + 1)(e^y - 1)\right) - (\frac{a}{2} + \frac{1}{2})\ln\left(h(y) - (\frac{a}{2} + \frac{1}{2})(e^y - 1)\right) + \frac{a}{2}\ln\left(h(y) + \frac{a}{2}(e^y - 1)\right) - (\frac{a}{2} - \frac{1}{2})\ln\left(h(y) + (\frac{a}{2} - \frac{1}{2})(e^y - 1)\right)$$

Theorem (P, 2015+)

The scaled height function $H_n(u, v)$ of a centrally symmetric tiling of an $a \times b \times c\ldots$ hexagon converges uniformly in probability to a deterministic function $L(u, v)$ – the limit surface, as $n \to \infty$, where $n = \frac{a+c}{2}$ and $a/n, b/n$ – approx constant.
The limit surface coincides with the limit surface for the uniformly random tilings of the hexagon (without symmetry constraints).

Behind the scene: asymptotics of symmetric functions

$$S_{\lambda(N)}(x_1,\ldots,x_k) := \frac{s_{\lambda(N)}(x_1,\ldots,x_k,\overbrace{1,\ldots,1}^{N-k})}{s_{\lambda(N)}(\underbrace{1,\ldots,1}_{N})} \qquad (similarly, othercharacters$$

Theorem [Gorin-P] For any partition λ and any $x \in \mathbb{C} \setminus \{0,1\}$ we have

$$S_\lambda(x;N,1) = \frac{(N-1)!}{(x-1)^{N-1}} \frac{1}{2\pi \mathbf{i}} \oint_C \frac{x^z}{\prod_{i=1}^N (z-(\lambda_i+N-i))} dz,$$

Theorem[Gorin-P] If $\frac{\lambda(N)}{N} \to f\left(\frac{i}{N}\right)$ [under certain convergence conditions], for all fixed $y \neq 0$:

$$\lim_{N\to\infty} \frac{1}{N} \ln S_{\lambda(N)}(e^y;N,1) = yw_0 - \mathcal{F}(w_0) - 1 - \ln(e^y-1),$$

where $\mathcal{F}(w;f) = \int_0^1 \ln(w-f(t)-1+t)dt$, w_0 – root of $\frac{\partial}{\partial w}\mathcal{F}(w;f) = y$. If $\frac{\lambda(N)}{N} \to f\left(\frac{i}{N}\right)$ ["other" conv. cond.], for any fixed $h \in \mathbb{R}$:

$$S_{\lambda(N)}(e^{h/\sqrt{N}};N,1) = \exp\left(\sqrt{N}E(f)h + \frac{1}{2}S(f)h^2 + o(1)\right),$$

$$\text{where } E(f) = \int_0^1 f(t)dt, \quad S(f) = \int_0^1 (f(t)-t+1/2)^2 dt - 1/6 - E(f)^2.$$

Multivariate: [Gorin-P] Let $D_{i,1} = x_i \frac{\partial}{\partial x_i}$, $\Delta-$ Vandermonde det, then

$$S_\lambda(x_1,\ldots,x_k;N) = \prod_{i=1}^k \frac{(N-i)!}{(N-1)!(x_i-1)^{N-k}} \times \frac{\det\left[D_{i,1}^{j-1}\right]_{i,j=1}^k}{\Delta(x_1,\ldots,x_k)} \prod_{j=1}^k S_\lambda(x_j;N,1)(x_j-1)^{N-1}.$$

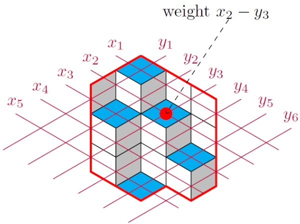
Corollary[Gorin-P]

$$\text{If } \frac{\ln\left(S_{\lambda(N)}(x;N,1)\right)}{N} \to \Psi(x) \quad \text{unif. on a compact } M \subset \mathbb{C}. \text{ Then for any } k$$

$$\lim_{N\to\infty} \frac{\ln\left(S_{\lambda(N)}(x_1,\ldots,x_k;N,1)\right)}{N} = \Psi(x_1)+\cdots+\Psi(x_k)$$

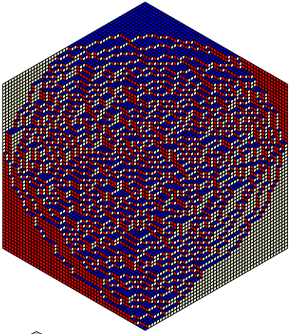
uniformly on M^k. More informally, under various regimes of convergence for $\lambda(N)$ we have $S_{\lambda(N)}(x_1,\ldots,x_k) \sim S_{\lambda(N)}(x_1)\cdots S_{\lambda(N)}(x_k)$.

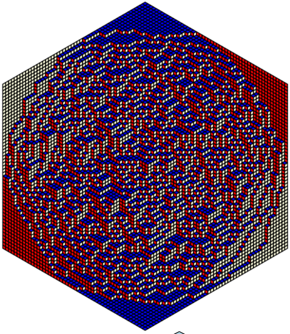
Multivariate local weights



$$\text{Total weight} = \prod_{\lozenge \text{ at } (i,j)} (x_i - y_j)$$

$$(x_1 - y_1)(x_2 - y_3)(x_3 - y_5)(x_3 - y_2)(x_5 - y_5).$$





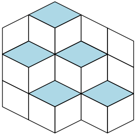
$\lozenge$ at $(i,j) = 2N - (i+j)$ $\qquad$ $\lozenge = 1$

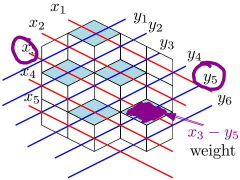
Lozenge tilings with multivariate weights

Plane partitions with base μ, height d

weights of horizontal lozenges $= x_i - y_j$

3	2	1
2	1	





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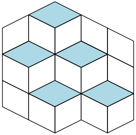
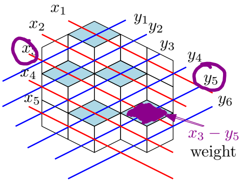




Theorem (Morales-Pak-P)

Consider tilings with base μ and height d, we have that

$$\sum_{T \in \Omega_{\mu,d}} \prod_{(i,j)\in T} (x_i - y_j) = \det[A_{i,j}(\mu,d)]_{i,j=1}^{d+\ell(\mu)},$$

where

$$A_{i,j}(\mu,d) := \begin{cases} \dfrac{(x_i-y_1)\cdots(x_i-y_{d+\ell(\mu)-j})}{(x_i-x_{i+1})\cdots(x_i-x_{d+\ell(\mu)})}, & \textit{when } j = \ell(\mu)+1,\ldots,\ell(\mu)+d, \\ \dfrac{(x_i-y_1)\cdots(x_i-y_{\mu_j+d})}{(x_i-x_{i+1})\cdots(x_i-x_{d+j})}, & \textit{when } j = i-d,\ldots,\ell(\mu), \\ 0, & \textit{when } j < i-d. \end{cases}$$

Corollary (Krattenthaler, Stanley etc)

Consider the set $PP(\mu, d)$ of plane partitions of base μ and entries less than or equal to d. Then their volume generating function is given by the following determinantal formula

$$\sum_{P \in PP(\mu,d)} q^{|P|} = q^{\sum_r r\mu_r} \det[C_{i,j}]_{i,j=1}^{\ell+d},$$

where

$$C_{i,j} = \begin{cases} \dfrac{(-1)^{d+\ell-i} q^{(d-i)(d+\ell-j)-\frac{(d-i+\ell)(d-i-\ell-1)}{2}}}{(q;q)_{d+\ell-i}}, & \text{when } j = \ell+1, \ldots, \ell+d, \\ \dfrac{(-1)^{d+j-i} q^{(d-i)(\mu_j+d)-\frac{(d+j-i)(d-i-j-1)}{2}}}{(q;q)_{d+j-i}}, & \text{when } j = i-d, \ldots, \ell, \\ 0, & \text{when } j < i-d, \end{cases}$$

where $(q;q)_m = (1-q)\cdots(1-q^m)$ is the $q-$Pochhammer symbol.

Theorem (Morales-Pak-P)

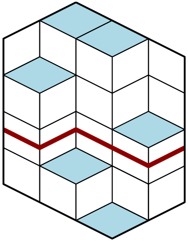
Consider tilings of the $a \times b \times c \times a \times b \times c$ (base $a \times b$, height c) hexagon with horizontal lozenges having weights $x_i - y_j$, i.e. tilings $\Omega_{a,b,c}$ with rectangular base $\mu = a \times b$ and height c. The partition function is given by

$$Z(a,b,c) := \sum_{T \in \Omega_{a,b,c}} \prod_{(i,j) \in T} (x_i - y_j) = \det \left[\begin{cases} \frac{(x_i - y_1)\cdots(x_i - y_{c+a-j})}{(x_i - x_{i+1})\cdots(x_i - x_{c+a})} & \text{if } j > a \\ \frac{(x_i - y_1)\cdots(x_i - y_{b+c})}{(x_i - x_{i+1})\ldots(x_i - x_{c+j})} & \text{if } j = i-c, \ldots, a \\ 0, & j < i - c \end{cases} \right]_{i,j=1}^{a+c}$$

Consider a path $P(d_1, \ldots)$ consisting of vertical lozenges (i.e. not the horizontal lozenges) passing through the points (i, d_i) (ith vertical line, distance of the midpoint $d_i + 1/2$ from the top axes) (necessarily $|d_i - d_{i+1}| \leq 1$, $d_i \leq d_{i+1}$ if $i \leq b$ and $d_i \geq d_{i+1}$ if $i > b$, and $d_1 = d_{a+b}$).
The probability that such path exists is given by

$$\text{Prob(path)} = \frac{\det[A_{i,j}(\mu, d)] \det[\bar{A}_{i,j}(\bar{\mu}, c-d-1)]}{Z}$$

where $d := d_1$, $\ell(\mu) = b$, $\mu_1 = a$ and μ is given by its diagonals – $(d_1 - d, d_2 - d, \ldots)$, and $\bar{\mu}$ is the complement of μ in $a \times b$. The matrix $\bar{A}$ is defined as in previous Theorem with the substitution of x_i by $x_{a+c+1-i}$ and y_j by $y_{b+c+1-j}$.



$\mu = 31$

$\mu^* = 20$

Counting skew SYTs

Outer shape λ, inner – μ, e.g. for $\lambda = (5,4,4,2), \mu = (2,2,1)$:

		2	3	6
		7	8	
	1	5	10	
4	9			
11				

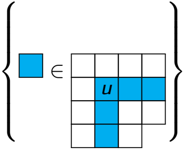
When $\mu = \emptyset$ – straight shape SYTs:

1	3	4	8
2	5	7	
6			

Hook-length formula [Frame-Robinson-Thrall]:

$$\dim \mathbb{S}_\lambda = \#\{\text{SYTs of shape } \lambda\} = f^\lambda = \frac{|\lambda|!}{\prod_{u \in \lambda} h_u} = \frac{8!}{6*4*3*1*4*2*1*1}$$

Hook length of box $u = (i,j) \in \lambda$: $h_u = \lambda_i - j + \lambda'_j - i + 1 = \#$



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		7	8	
	1	5	10	
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11				

Jacobi-Trudi[Feit 1953]:

$$f^{\lambda/\mu} = |\lambda/\mu|! \cdot \det \left[\frac{1}{(\lambda_i - \mu_j - i + j)!} \right]_{i,j=1}^{\ell(\lambda)}.$$

Littlewood-Richardson:

$$f^{\lambda/\mu} = \sum_{\nu} c^{\lambda}_{\mu,\nu} f^{\nu}$$

No product formula, e.g. $\lambda/\mu = \delta_{n+2}/\delta_n$:

			5	6
		1	9	
	2	7		
3	4			
8				

$\leftrightarrow 8 > 3 < 4 > 2 < 7 > 1 < 9 > 5 < 6$

$f^{\delta_{n+2}/\delta_n} = E_{2n+1}$:

$$1 + E_1 x + E_2 \frac{x^2}{2!} + E_3 \frac{x^3}{3!} + E_4 \frac{x^4}{4!} + \ldots = \sec(x) + \tan(x).$$

Euler numbers: $2, 5, 16, 61 \ldots$.

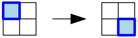
Hook-Length formula for skew shapes

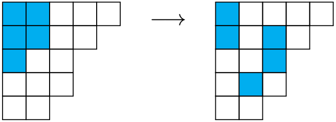
Theorem (Naruse, SLC, September 2014)

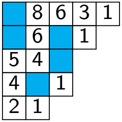
$$f^{\lambda/\mu} = |\lambda/\mu|! \sum_{D \in \mathcal{E}(\lambda/\mu)} \prod_{u \in [\lambda] \setminus D} \frac{1}{h(u)},$$

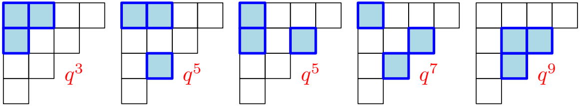
where $\mathcal{E}(\lambda/\mu)$ *is the set of* excited diagrams *of* λ/μ*.*

Excited diagrams:

$\mathcal{E}(\lambda/\mu) = \{D \subset \lambda :$ obtained from μ via  $\}$

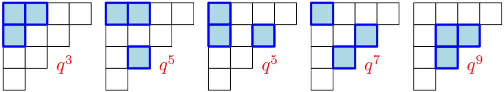


Hook lengths inside λ: 



$$f^{(4321/21)} = 7! \left(\frac{1}{1^4 \cdot 3^3} + \frac{1}{1^3 \cdot 3^3 \cdot 5} + \frac{1}{1^3 \cdot 3^3 \cdot 5} + \frac{1}{1^2 \cdot 3^3 \cdot 5^2} + \frac{1}{1^2 \cdot 3^2 \cdot 5^2 \cdot 7} \right) = 61$$

Hook-Length formula for skew shapes



$$s_{\lambda/\mu}(1, q, q^2, \ldots) = \sum_{T \in SSYT(4321/21)} q^{|T|} = \frac{q^3}{(1-q)^4(1-q^3)^3} + 2 \times \frac{q^5}{(1-q)^3(1-q^3)^3(1-q^5)} + \cdots$$

Theorem (Morales-Pak-P)

For skew SSYTs, we have that

$$s_{\lambda/\mu}(1, q, q^2, \ldots) = \sum_{T \in SSYT(\lambda/\mu)} q^{|T|} = \sum_{D \in \mathcal{E}(\lambda/\mu)} \prod_{(i,j) \in [\lambda] \setminus D} \left[\frac{q^{\lambda'_j - i}}{1 - q^{h(i,j)}} \right].$$

Theorem (Morales-Pak-P)

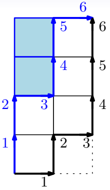
For (reverse) plane partitions of skew shape λ/μ we have that

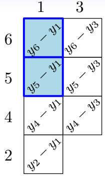
$$\sum_{\pi \in RPP(\lambda/\mu)} q^{|\pi|} = \sum_{S \in PD(\lambda/\mu)} \prod_{u \in S} \left[\frac{q^{h(u)}}{1 - q^{h(u)}} \right].$$

where $PD(\lambda/\mu) := \{S \subset [\lambda] : S \subset [\lambda] \setminus D, \text{ for some } D \in \mathcal{E}(\lambda/\mu)\}$ is the set of "pleasant diagrams".

Other recent proof by [M. Konvalinka]

Proof 1: factorial Schurs and Schubert polynomials





$v = 245613$, $w = 361245$

[Ikeda-Naruse, Kreiman]:
Let $w \preceq v$ be Grassmannian permutations whose unique descent is at position d with corresponding partitions $\mu \subseteq \lambda \subseteq d \times (n-d)$. Then the Schubert class X_w for w at point v is:

$$[X_w]\big|_v = \sum_{D \in \mathcal{E}(\lambda/\mu)} \prod_{(i,j) \in D} \left(y_{v(d+j)} - y_{v(d-i+1)} \right).$$

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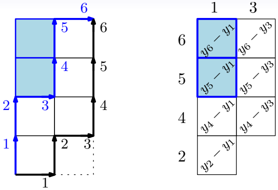

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Factorial Schur functions:

$$s_\mu^{(d)}(\mathbf{x}|\mathbf{a}) := \frac{\det\left[(x_j - a_1)\cdots(x_j - a_{\mu_i+d-i})\right]_{i,j=1}^{d}}{\prod_{1\le i<j\le d}(x_i - x_j)},$$

[Knutson-Tao, Lakshmibai–Raghavan–Sankaran] Schubert class at a point:

$$[X_w]\big|_v = (-1)^{\ell(w)} s_\mu^{(d)}\left(y_{v(1)}, \ldots, y_{v(d)} | y_1, \ldots, y_{n-1}\right).$$

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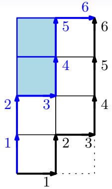

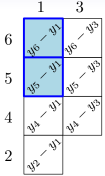

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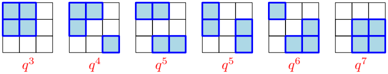
Evaluation at $y = 1, q, q^2, \ldots$, $v(d+1-i) = \lambda_i + d + 1 - i$, $x_i \to y_{v(i)} = q^{\lambda_i+d+1-i}$
$\to$ Jacobi-Trudi

$$s^{(d)}_\mu(q^{v(1)},\ldots|1,q,\ldots) = \frac{\det[\prod_{r=1}^{\mu_j+d-j}(q^{\lambda_i+d+1-i} - q^r)]_{i,j=1}^d}{\prod_{i<j}(q^{\lambda+d+1-i} - q^{\lambda_j+d+1-j})} = \ldots.$$

$$\ldots[simplifications]\ldots = \det[h_{\lambda_i-i-\mu_j+j}(1,q,\ldots)] \underbrace{=}^{\text{Jacobi-Trudi}} s_{\lambda/\mu}(1,q,\ldots)$$

Factorial Schur functions, multivariate lozenge tilings

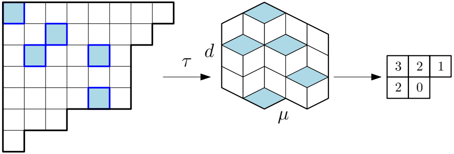


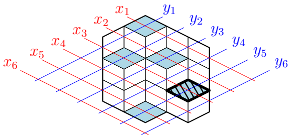
Theorem (Ikeda-Naruse Multivariate "Hook-Length Formula")

Let $\mu \subset \lambda \subset d \times (n-d)$. Let $v(n-d+1-i) = \lambda_i + (n-d+1-i)$ and $v(j) = d + j - \lambda'_j$. Then

$$s_\mu^{(d)}(y_{v(1)}, \ldots, y_{v(d)} | y_1, \ldots, y_{n-1}) = \sum_{D \in \mathcal{E}(\lambda/\mu)} \prod_{(i,j) \in D} (y_{v(d-i+1)} - y_{v(d+j)})$$

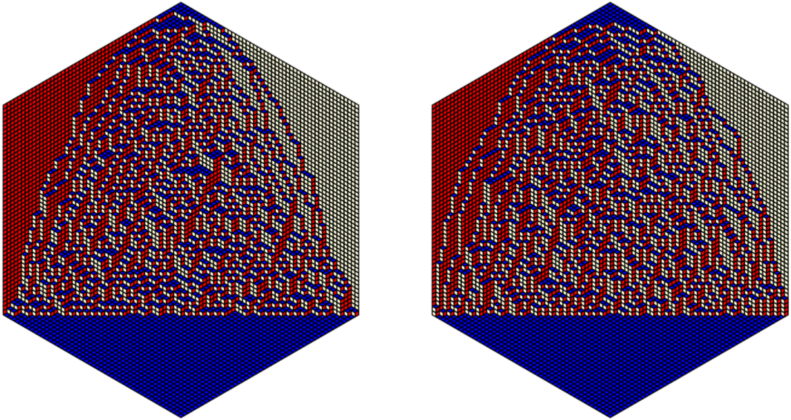
$$\Longrightarrow = \frac{\det\left[(y_{v(j)} - y_1) \cdots (y_{v(j)} - y_{\mu_i + d - i})\right]_{i,j=1}^{d}}{\prod_{1 \le i < j \le d} (y_{v(i)} - y_{v(j)})},$$





Simulation 2: base $= \delta_n$

Weights: "hook" weights $(4n - i - j)$ versus uniform (i.e. 1).



Corollaries and problems

Asymptotics of $f^{\lambda/\mu}$:
[Morales-Pak-P]:
1. If $\lambda^n, \mu^n \tilde{n}$ (linear growth, Thoma-Vershik-Kerov limit) $\log f^{\lambda^n/\mu^n} = cn + o(n)$, c – constant depending on the limit shapes of λ^n, μ^n.

2. "Thick shapes" $\frac{\lambda^n_{x\sqrt{n}}}{\sqrt{n}} \to \omega(x)$, then

$$\log f^{\lambda^n/\mu^n} = \frac{1}{2} n \log n + O(n)$$

3. If λ^n/μ^n – "thin" (ribbon) shaped, then

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[Morales-Pak-Tassy]: Using variational principle for the multivariate lozenge tilings, for "thick shapes":

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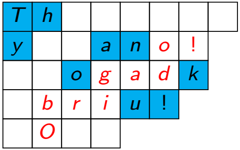
Problems:

?? Limit behavior of lozenge tilings with non-trapezoidal boundary conditions.

?? Asymptotics of

$$\lim_{n\to\infty} \frac{s_{\lambda^n/\mu^n}(x_1,\ldots,x_k,1^{n-k})}{s_{\lambda^n/\mu^n}(1^n)}$$


T h
y a n o !
o g a d k
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O