Correlation functions of the integrable $SU(n)$ spin chains

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June 18, 2018

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Outline

- Introduction- summary of previous results
- Integrable $SU(n)$ spin chains
- Trotter-Suzuki decomposition
- Density matrices
- Functional equations
- Two-sites and three-sites correlation for $SU(3)$
- Lack of factorization

History about correlations: $\langle S^z_j S^z_{j+n} \rangle$ for $SU(2)$ chain

- (1938) - Hulthén: from ground state energy $\langle S^z_j S^z_{j+1} \rangle = \frac{1}{12} - \frac{1}{3}\zeta_a(1)$.
- (1977) - Takahashi: strong coupling expansion of groud state of Hubbard model: $\langle S^z_j S^z_{j+2} \rangle = \frac{1}{12} - \frac{4}{3}\zeta_a(1) + \zeta_a(3)$.

where the alternating zeta function is $\zeta_a(s) = \sum_{n=1}^{\infty}(-1)^{n-1}/n^s$, which is related to Riemann zeta as $\zeta_a(s) = (1 - 2^{1-s})\zeta(s)$ and $\zeta_a(1) = \log 2$.

- (1992) - Kyoto group (Jimbo, Miwa,...): derived multiple integral representation via representation theory of quantum affine algebra.
- (1999) - Kitanine, Maillet, Terras: derived the same expression within the quantum inverse scattering method (ABA), but now for $h \geq 0$.
- (2001) - Boos, Korepin: evaluated those integrals for short-distance correlators (EFP $n = 2, 3, 4, 5$). Conjectured the answer given only in terms of zeta of odd argument.
- (2003) - Sakai, Shiroishi, Nishiyama, Takahashi: putted other correlators together by Boos-Korepin method and obtained $\langle S^z_j S^z_{j+3} \rangle = \frac{1}{12} - 3\zeta_a(1) + \frac{74}{9}\zeta_a(3) - \frac{56}{9}\zeta_a(1)\zeta_a(3) - \frac{8}{3}\zeta_a(3)^2 - \frac{50}{9}\zeta_a(5) + \frac{80}{9}\zeta_a(1)\zeta_a(5)$.

continue...

- (2003) - Boos, Korepin, Smirnov: introduced inhomegenious density matrix which should fulfill the qKZ on level -4. More powerfull method which allowed for correlators at longer distances $n \leq 6$.
- (2005) - Sato, Shiroishi,Takahashi: Pushed this up to $n = 8$. Very long expressions.
- (2004,2005) - Wuppertal group - Klümper, Göhmann,...: finite temperature, magnetic field and finite size short-distance correlations via NLIE.
- (2008) - Kitanine, Kozlowski, Maillet, Slavnov, Terras: long-distance asymptotic behaviour of correlators ($n \to \infty$), which at leading order confirm Luttinger liquid and CFT predictions.

High-spin chains: $SU(2)$ spin-s, $s > 1/2$

- (1993) Idzumi; (1994) Bougourzi, Weston; (1994) Konno; (2001) Kitanine: multiple integral representation of correlation function
- (2010) Göhmann, Seel, Suzuki: finite temperature generalization via NLIE, however no explicit evaluation of the integrals.
- (2013) Nawrath, Klümper, Suzuki: first explicit results for spin-1 (two and three-sites correlation).

$$\langle S_i^z S_{i+1}^z \rangle = -\frac{17}{9} + \frac{16}{9}\zeta_a(2)$$

- (2016) GAPR, Klümper, explicit results for spin-3/2.

$$\langle S_i^z S_{i+1}^z \rangle = \left(-\frac{105}{8} + \frac{26}{10}\zeta_a(2)\right) + \left(15 - \frac{16}{5}\zeta_a(2)\right)\zeta_a(1)$$

High-srank spin chains

How about higher-rank quantum spin chains?

- One still lacks a better understanding of the correlation properties of models based on high rank algebras. In the $SU(n)$ case for $n > 2$, there were no results for correlation functions and this stayed as a longstanding problem for decades.
- We developed a general framework to tackle with this case which successfully provide explicit results. That's the subject of this talk.

Integrable $SU(n)$ quantum chain: high-rank spin chain

The Hamiltonian of the integrable $SU(n)$ spin chain (Uimin,Sutherland),

$$H^{(n)} = \sum_{j=1}^{L} P_{j,j+1},$$

For instance, in the case of $SU(3)$ spin chains the Hamiltonian can be written in terms of spin-1 matrices as follows,

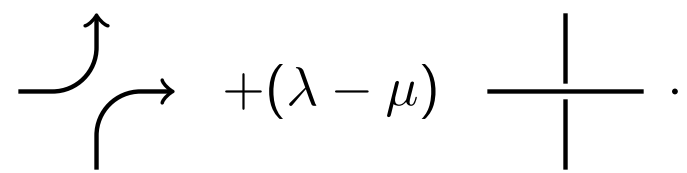
$$H^{(3)} = \sum_{j=1}^{L} [\vec{S}_j \cdot \vec{S}_{j+1} + (\vec{S}_j \cdot \vec{S}_{j+1})^2].$$

The Hamiltonian is the logarithmic derivative of the row-to-row transfer matrix

$$T^{(n)}(\lambda) = \mathrm{Tr}_{\mathcal{A}} \left[R^{(n,n)}_{\mathcal{A},L}(\lambda) \dots R^{(n,n)}_{\mathcal{A},1}(\lambda) \right],$$

where the R-matrix $R^{(n,n)}_{12}(\lambda - \mu) = P_{12} \check{R}^{(n,n)}_{12}(\lambda - \mu)$ in the fundamental representation of the $SU(n)$ denoted here by $[n]$.

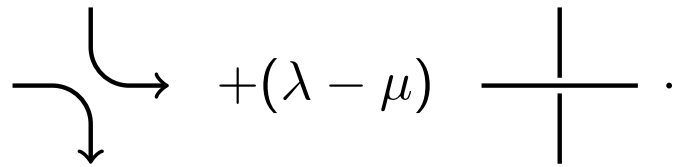
$$\check{R}^{(n,n)}(\lambda - \mu) = I + (\lambda - \mu) P =$$



Integrable structure

The R-matrix with mixed representations of fundamental $[n]$ and anti-fundamental $[\bar{n}]$,

$$\check{R}^{(n,\bar{n})}(\lambda-\mu) = E + (\lambda-\mu)P =$$



where E is the Temperley-Lieb operator.
We have six Yang-Baxter equations of the type

$$\check{R}_{12}^{(r_1,r_2)}(\lambda-\mu)\check{R}_{23}^{(r_1,r_3)}(\lambda)\check{R}_{12}^{(r_2,r_3)}(\mu) = \check{R}_{23}^{(r_2,r_3)}(\mu)\check{R}_{12}^{(r_1,r_3)}(\lambda)\check{R}_{23}^{(r_1,r_2)}(\lambda-\mu),$$

where $r_i \in \{n, \bar{n}\}$ for $i = 1, 2, 3$.
We also have two special Yang-Baxter eq. of the type,

$$\check{R}_{12}^{(n,\bar{n})}(\lambda-\mu+n)\check{R}_{23}^{(n,n)}(\lambda)\check{R}_{12}^{(\bar{n},n)}(\mu) = \check{R}_{23}^{(\bar{n},n)}(\mu)\check{R}_{12}^{(n,n)}(\lambda)\check{R}_{23}^{(n,\bar{n})}(\lambda-\mu+n).$$

The other is obtained by exchanging the representations $[n]$ and $[\bar{n}]$.

Integrable structure

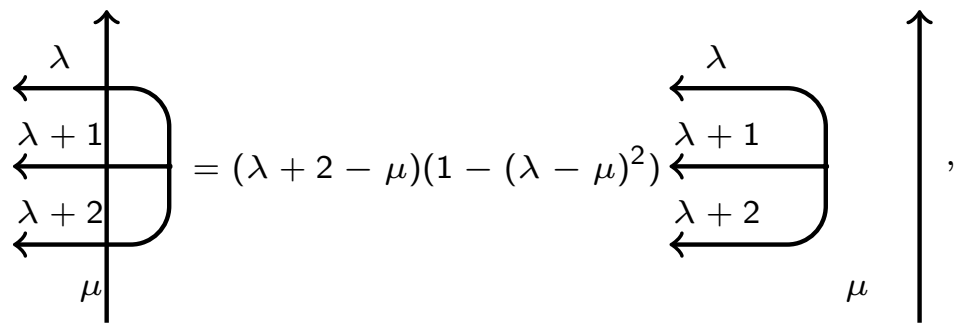
The fundamental R-matrix has important properties,

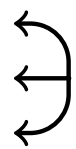
$$\check{R}_{12}^{(n,n)}(0) = I, \qquad \text{initial condition,} \tag{1}$$

$$\check{R}_{12}^{(n,n)}(\lambda-\mu)\check{R}_{21}^{(n,n)}(\mu-\lambda) = (1-(\lambda-\mu)^2)I, \qquad \text{standard unitarity,} \tag{2}$$

$$\begin{aligned}
\check{R}_{12}^{(n,\bar{n})}(\lambda-\mu+n)\check{R}_{21}^{(\bar{n},n)}(\mu-\lambda) &= (\mu-\lambda)(\lambda-\mu+n)I, \\
\check{R}_{12}^{(n,\bar{n})}(\lambda-\mu)\check{R}_{21}^{(\bar{n},n)}(\mu-\lambda+n) &= (\lambda-\mu)(\mu-\lambda+n)I.
\end{aligned}$$

In addition, we need additional properties which hold for $SU(n)$. For instance, for the $SU(3)$, we have





Trotter-Suzuki decomposition

To compute

$$Z_L = \mathrm{Tr}\left[e^{-\beta H^{(n)}}\right],$$

in the thermodynamic limit ($L \to \infty$).

$$\begin{aligned} T^{(n)}(\lambda) &= e^{\mathrm{i}P + \lambda H^{(n)} + O(\lambda^2)} \\ \bar{T}^{(n)}(\lambda) &= e^{-\mathrm{i}P + \lambda H^{(n)} + O(\lambda^2)}, \end{aligned}$$ where $e^{\pm \mathrm{i}P}$ is the right/left shift operator

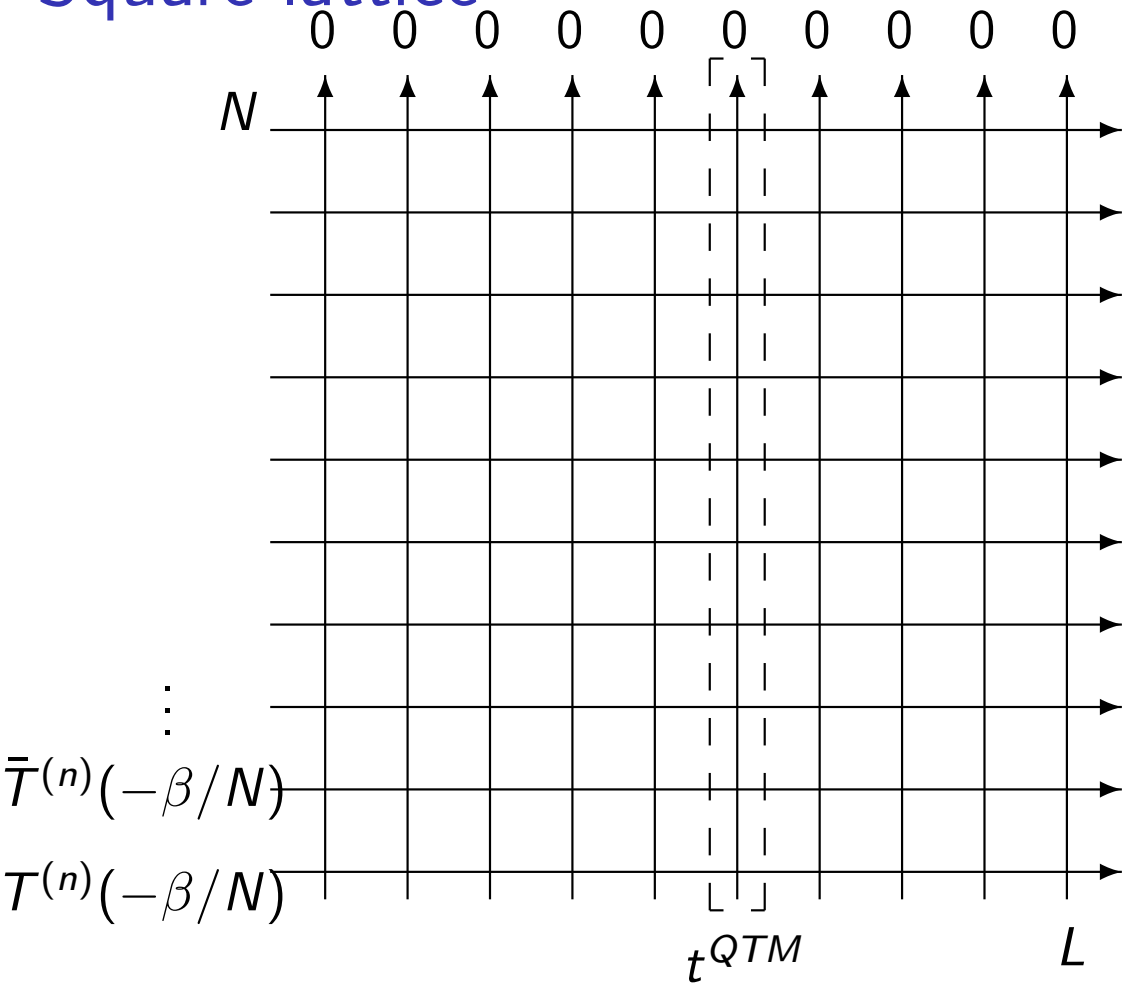
$$T^{(n)}(\lambda)\bar{T}^{(n)}(\lambda) = e^{2\lambda H^{(n)} + O(\lambda^2)}.$$

In order to get rid of the high-order terms we set $\lambda = -\beta/N$ and take the limit $N \to \infty$. In addition we raise this product to the $N/2$ power preserving the Hamiltonian term.

$$\rho_L = e^{-\beta H^{(n)}} = \lim_{N\to\infty} \rho_{N,L}, \qquad \rho_{N,L} = \left[T^{(n)}(-\beta/N)\,\bar{T}^{(n)}(-\beta/N)\right]^{N/2},$$

$$Z_L = \mathrm{Tr}_{1\cdots L}\left[e^{-\beta H^{(n)}}\right] = \lim_{N\to\infty} \mathrm{Tr}_{1\cdots L}\left[\rho_{N,L}\right]$$

Square lattice



$$Z_L = \lim_{N\to\infty} \mathrm{Tr}\left(t^{QTM}(0)\right)^L$$

The quantum transfer matrix

$$t^{QTM}(\lambda) = \mathrm{Tr}_i\left[\mathcal{T}_i^{(n)}(\lambda)\right],$$

$$\rho_{N,L} = \mathrm{Tr}_{1\cdots N}\left[\mathcal{T}_1^{(n)}(0)\cdots\mathcal{T}_L^{(n)}(0)\right],$$

$$\mathcal{T}_i^{(n)}(\lambda) = R_{i,N}^{(n,n)}(\lambda+\mathrm{i}u)R_{N-1,i}^{(n,n)t_{N-1}}(\mathrm{i}u-\lambda)\cdots R_{i,2}^{(n,n)}(\lambda+\mathrm{i}u)R_{1,i}^{(n,n)t_1}(\mathrm{i}u-\lambda),$$

where $u = -\beta/N$.

Free-energy

$$
\begin{aligned}
f &= -\frac{1}{\beta} \lim_{L\to\infty} \frac{1}{L} \ln Z_L \\
&= -\frac{1}{\beta} \lim_{L\to\infty} \lim_{N\to\infty} \frac{1}{L} \ln \left[\mathrm{Tr} \left(t^{QTM}(0) \right)^L \right]
\end{aligned}
$$

Assuming we can diagonalize $t^{QTM}(0) = M^{-1} D M$, we obtain

$$
\begin{aligned}
f &= -\frac{1}{\beta} \lim_{L\to\infty} \lim_{N\to\infty} \frac{1}{L} \ln \left[\Lambda_1^L + \Lambda_2^L + \cdots + \Lambda_{\#^L}^L \right] \\
&= -\frac{1}{\beta} \lim_{N\to\infty} \ln \Lambda_{max}(0)
\end{aligned}
$$

Now the remaining problem is how to obtain the largest eigenvalues of the $t^{QTM}(0)$ for finite Trotter number N. Afterwards the limit $N \to \infty$ should be performed.

- How about correlation functions?

Reduced density matrix

The density matrix for the thermal correlation functions of integrable Hamiltonians was introduced for the $SU(2)$ case by (Göhmann,Seel,Klümper 2004). It can be extended to $SU(n)$ as follows,

$$D_m^{(n)} = \lim_{L\to\infty} \frac{\mathrm{Tr}_{m+1\cdots L}\left[\rho_L\right]}{Z_L} = \lim_{N,L\to\infty} \frac{\mathrm{Tr}_{m+1\cdots L}\left[\rho_{N,L}\right]}{\mathrm{Tr}_{1\cdots L}\left[\rho_{N,L}\right]},$$

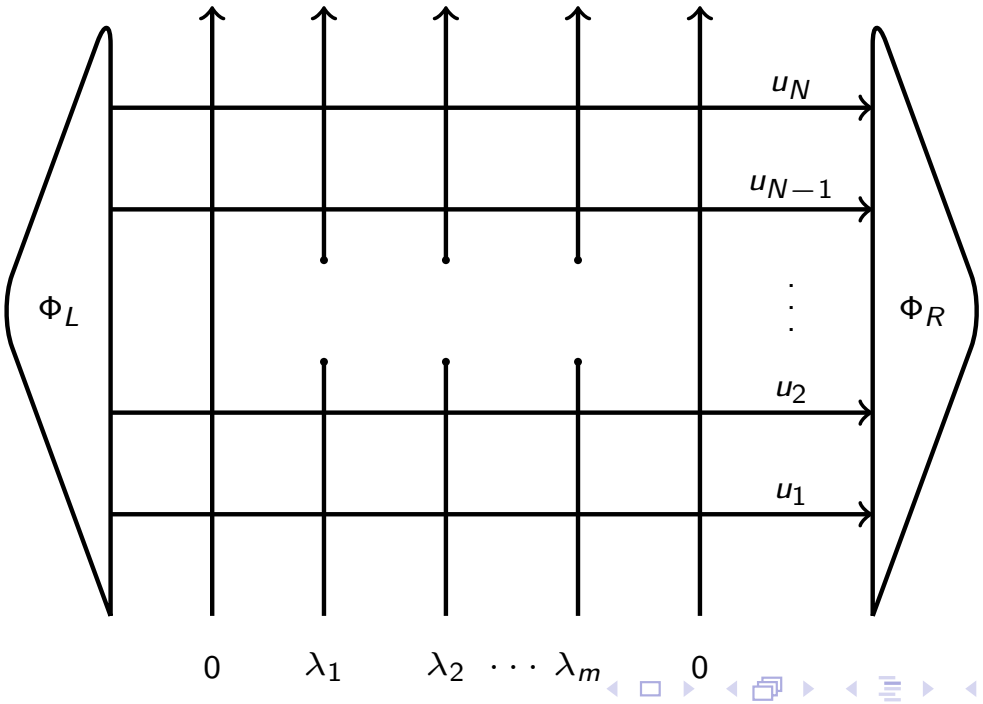
$$\begin{aligned}
D_m^{(n)} &= \lim_{N,L\to\infty} \frac{\mathrm{Tr}_{1\cdots N}\,\mathrm{Tr}_{m+1\cdots L}\left[\mathcal{T}_1^{(n)}(0)\cdots\mathcal{T}_L^{(n)}(0)\right]}{\mathrm{Tr}_{1\cdots N}\,\mathrm{Tr}_{1\cdots L}\left[\mathcal{T}_1^{(n)}(0)\cdots\mathcal{T}_L^{(n)}(0)\right]}, \\
&= \lim_{N,L\to\infty} \frac{\mathrm{Tr}_{1\cdots N}\left[\mathcal{T}_1^{(n)}(0)\cdots\mathcal{T}_m^{(n)}(0)(\,T^{(n)}(0))^{L-m}\right]}{\mathrm{Tr}_{1\cdots N}(\,T^{(n)}(0))^{L}}, \\
&= \lim_{N,L\to\infty} \frac{\sum_i(\Lambda_i^{(n)}(0))^{L-m}\left\langle\Phi_i\right|\mathcal{T}_1^{(n)}(0)\cdots\mathcal{T}_m^{(n)}(0)\left|\Phi_i\right\rangle}{\sum_i(\Lambda_i^{(n)}(0))^{L}}, \\
&= \lim_{N\to\infty} D_m^{(n)}, \qquad D_m^{(n)} := \frac{\left\langle\Phi_L\right|\mathcal{T}_1^{(n)}(0)\cdots\mathcal{T}_m^{(n)}(0)\left|\Phi_R\right\rangle}{(\Lambda_{max}^{(n)}(0))^m},
\end{aligned}$$

Inohomegeneous reduced density matrix: graphical depiction

$$D_m(\lambda_1, \cdots, \lambda_m) = \frac{\langle \Phi_L | \, \mathcal{T}_1^{(n)}(\lambda_1) \cdots \mathcal{T}_m^{(n)}(\lambda_m) \, | \Phi_R \rangle}{\Lambda_0^{(n)}(\lambda_1) \cdots \Lambda_0^{(n)}(\lambda_m)},$$

where $\mathcal{T}_j^{(n)}(\lambda_j)$ is the j-th monodromy matrix associated to the quantum transfer matrix, Φ_L and Φ_R represent the left and right leading eigenstates of the quantum transfer matrix and $\Lambda_0^{(n)}(\lambda_j)$ is the leading eigenvalue.

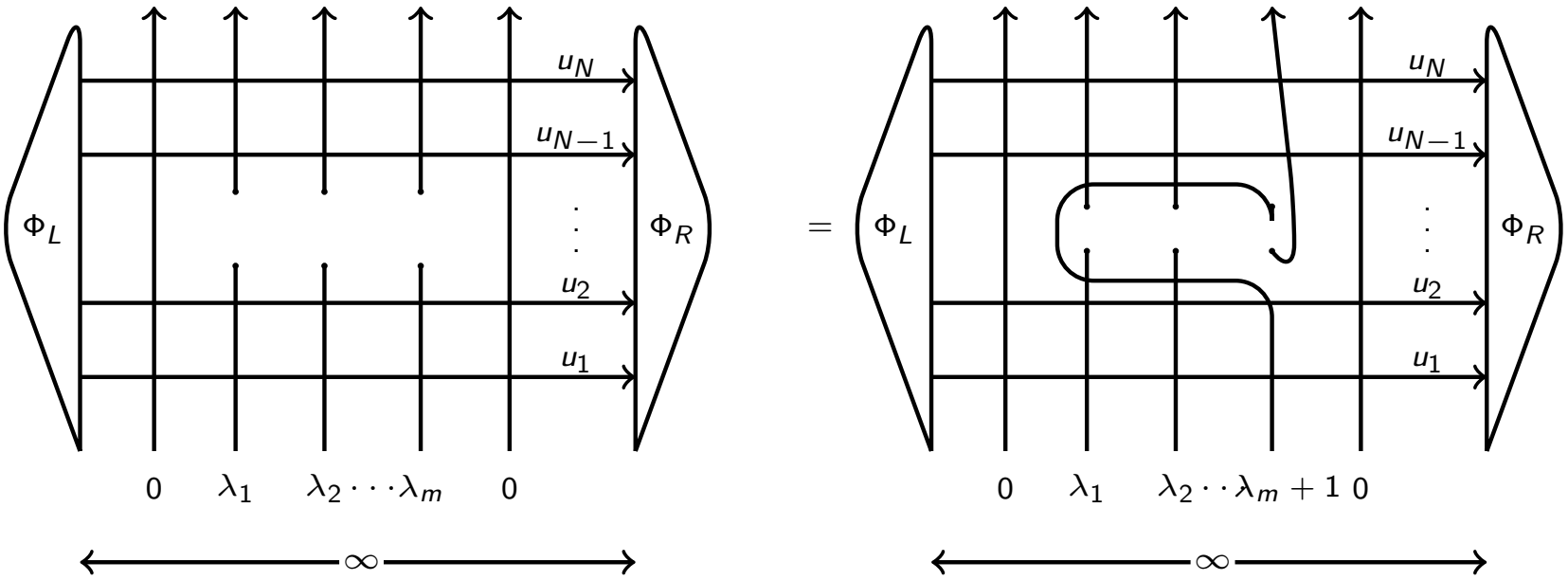
$D_m(\lambda_1, \ldots, \lambda_m) =$



Functional relations - Reduced density matrix

- Problem: How to fix the density matrix?

For the $SU(2)$: one functional equation is derived for the density matrix.



$$D_m^{(2)}(\lambda_1, \cdots, \lambda_m + 1) = A_m^{(2)}(\lambda_1, \cdots, \lambda_m)[D_m^{(2)}(\lambda_1, \cdots, \lambda_m)],$$

where $A_m^{(2)}$ is a linear operator (Aufgebauer, Klümper 2012). This equation holds for arbitrary Trotter number N and also in the limit $N \to \infty$ (zero temperature $T = 0$).

Closed functional relations for $SU(n)$

For the $SU(n)$ we developed two different frameworks.

- Approach 1: two functional equations can be derived for the density matrix and for an auxiliary density matrix (unpublished).

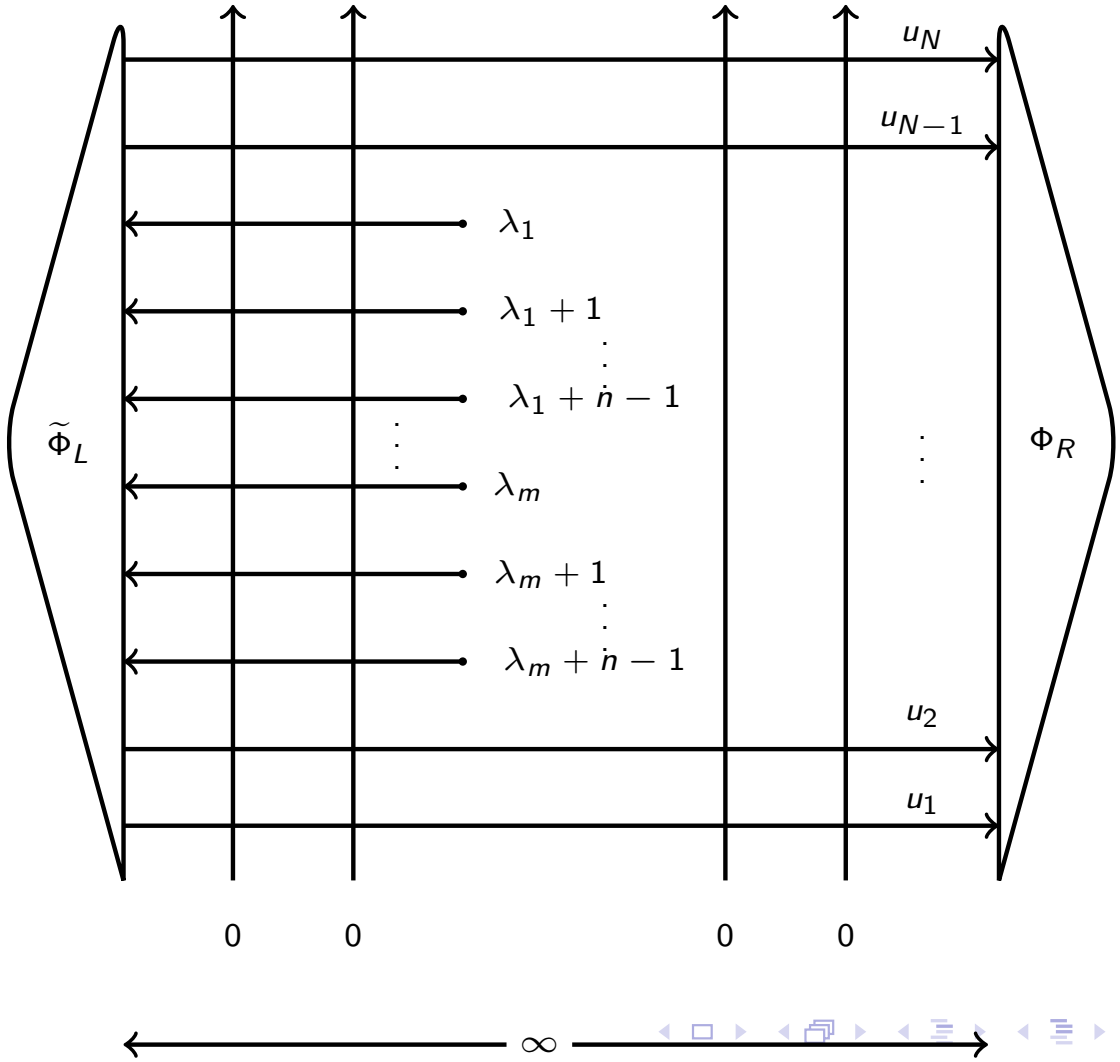
$$\begin{aligned} \bar{D}_m^{(n)}(\lambda_1, \cdots, \lambda_m) &= A_m^{(n)}(\lambda_1, \cdots, \lambda_m) D_m^{(n)}(\lambda_1, \cdots, \lambda_m), \\ D_m(\lambda_1, \cdots, \lambda_m + n) &= \bar{A}_m^{(n)}(\lambda_1, \cdots, \lambda_m) \bar{D}_m^{(n)}(\lambda_1, \cdots, \lambda_m), \end{aligned}$$

where $A_2^{(i)}(\lambda_1, \lambda_2)$ and $\bar{A}_2^{(i)}(\lambda_1, \lambda_2)$ are linear operators as before. However the proper derivation of the equations demands the limit $N \to \infty$ to be taken. **Therefore these equations do not hold for arbitrary Trotter number and consequently cannot describe finite temperature correlations. It would be good only for zero temperature case.**

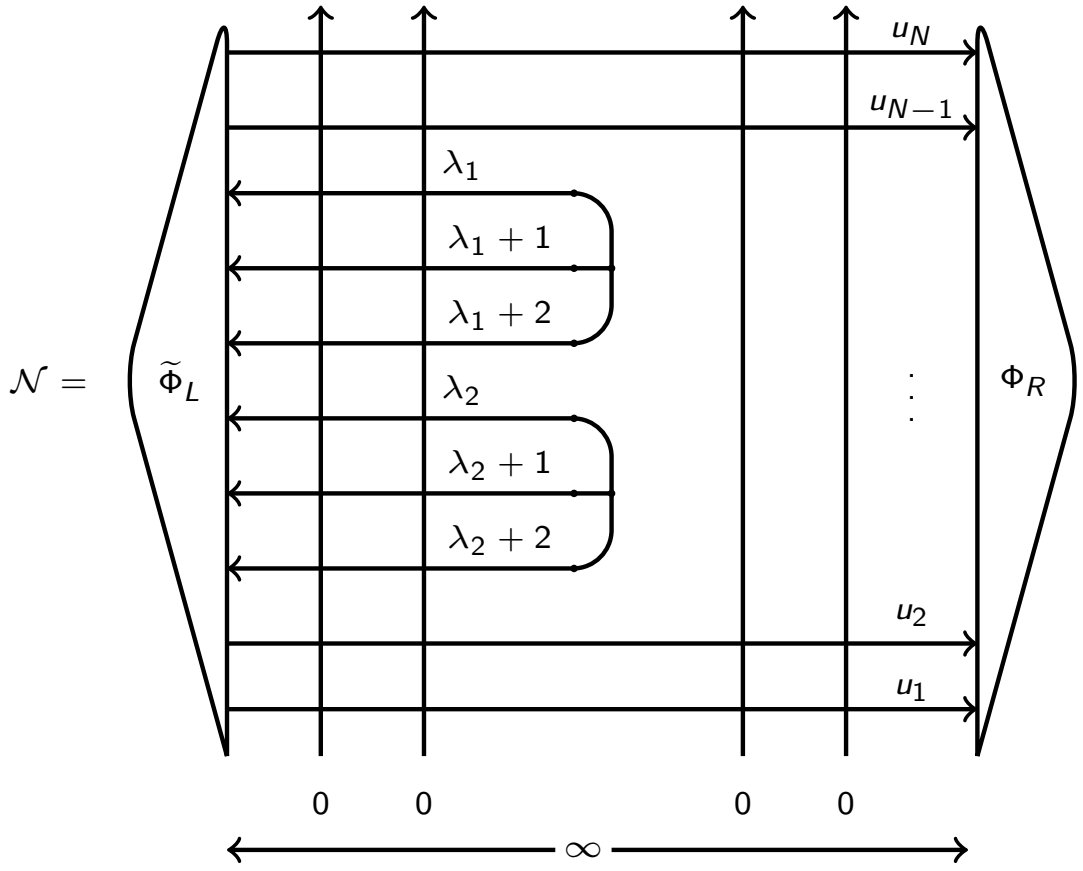
Closed functional relations for $SU(n)$

- Approach 2: single functional equations can be derived for a **generalized density matrix** (GAPR, A. Klümper, arXiv:1804.10169 [math-ph]).

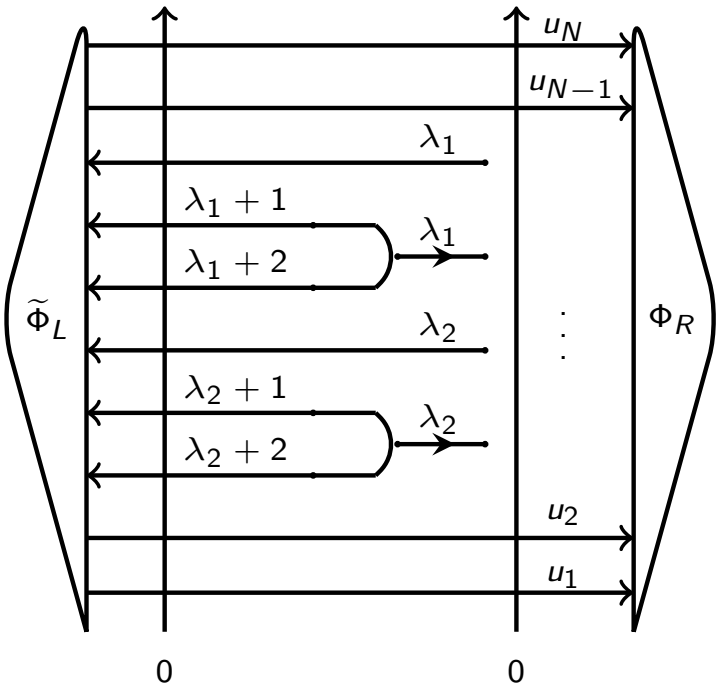
$\mathbb{D}_m(\lambda_1, \ldots, \lambda_m) =$



Properties: normalization ($n = 3, m = 2$)



Properties: reduction $(n = 3, m = 2)$



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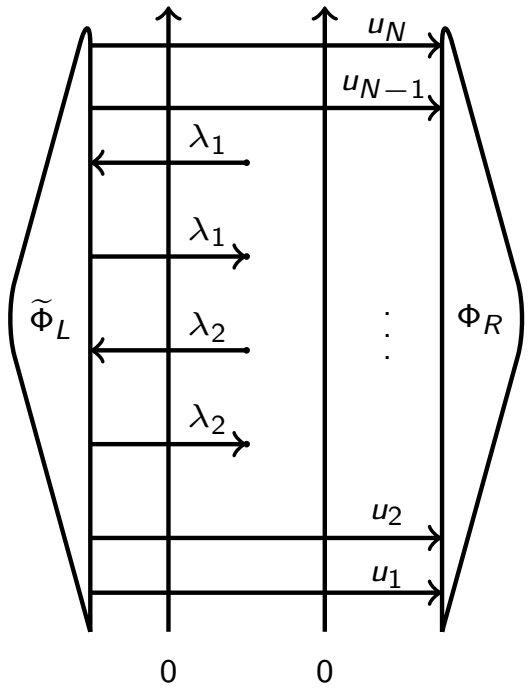

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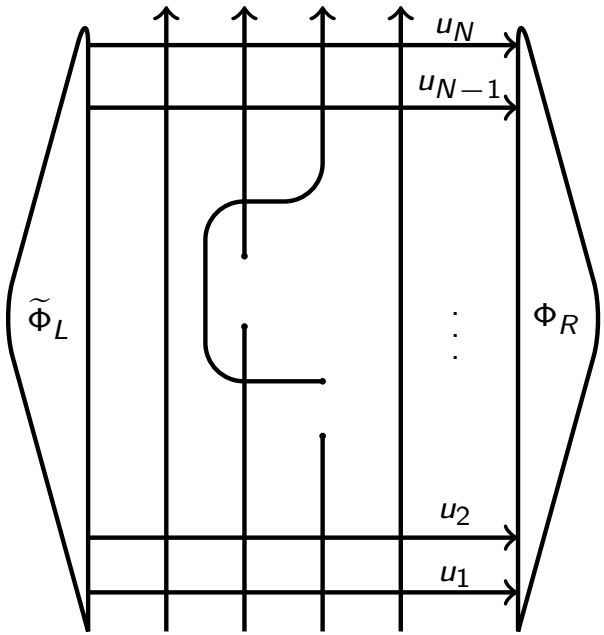


$= R(\lambda_1 - \lambda_2) D_2(\lambda_1, \lambda_2) R(\lambda_2 - \lambda_1).$

Reduced density matrix

The usual (normalized) two-point density operator for the fundamental-fundamental and also for antifundamental-fundamental representations can be written as follows,

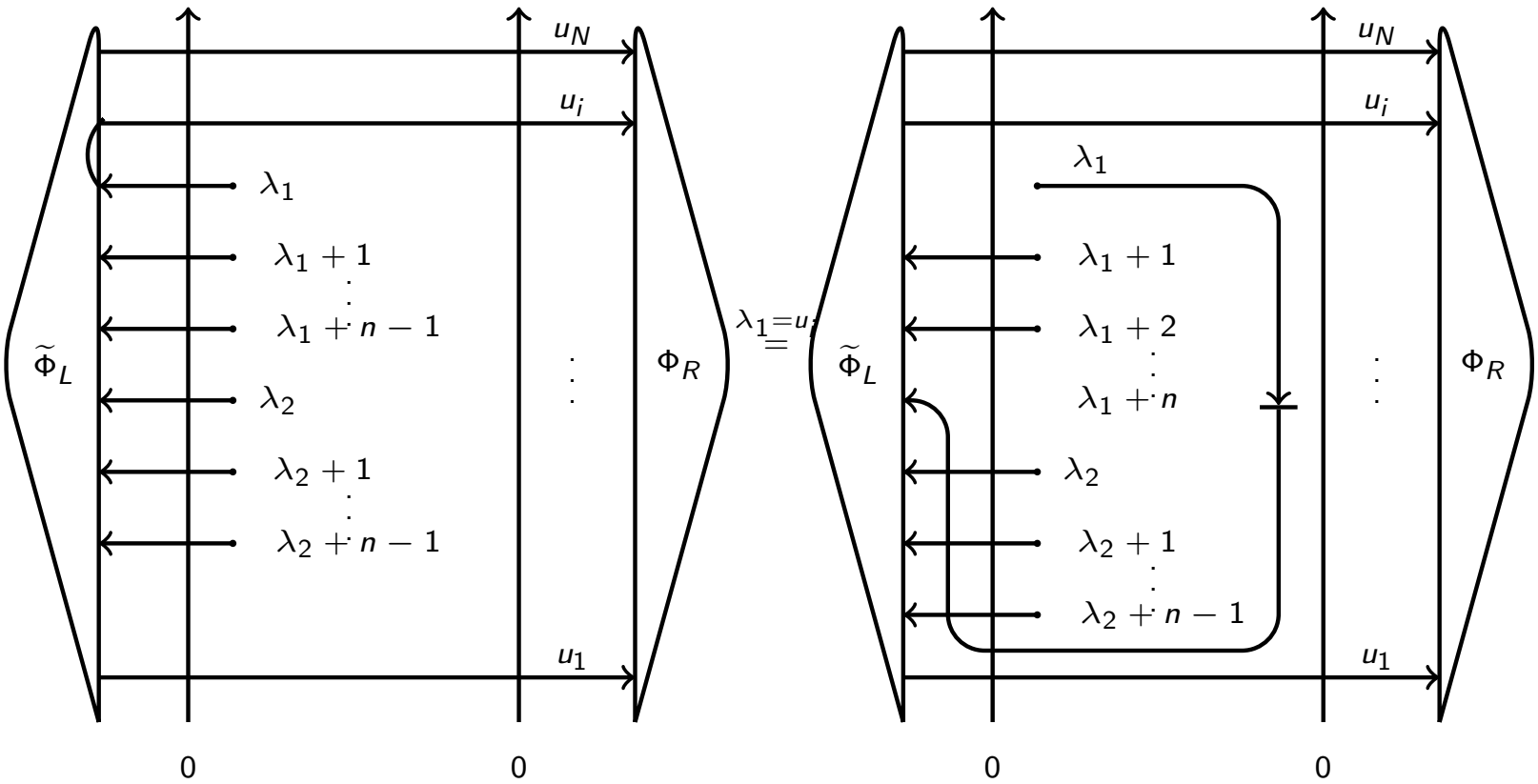
$$
\begin{aligned}
D_2^{(nn)}(\lambda_1,\lambda_2) &= \left(\frac{1}{n^2}-\frac{\alpha_{nn}(\lambda_1,\lambda_2)}{n}\right) I + \alpha_{nn}(\lambda_1,\lambda_2) P, \\
D_2^{(\bar{n}n)}(\lambda_1,\lambda_2) &= \left(\frac{1}{n^2}-\frac{\alpha_{\bar{n}n}(\lambda_1,\lambda_2)}{n}\right) I + \alpha_{\bar{n}n}(\lambda_1,\lambda_2) E.
\end{aligned}
$$

It is convenient to define some simple two-point correlation functions to work with,

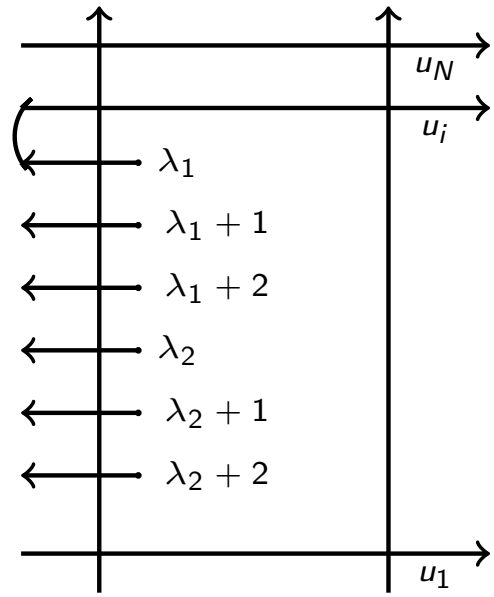
$$
\begin{aligned}
\omega_{nn}(\lambda_1,\lambda_2) &= \mathrm{Tr}[P D_2^{(nn)}] = \frac{1}{n} + (n^2-1)\alpha_{nn}(\lambda_1,\lambda_2), \\
\omega_{\bar{n}n}(\lambda_1,\lambda_2) &= \mathrm{Tr}[E D_2^{(\bar{n}n)}] = \frac{1}{n} + (n^2-1)\alpha_{\bar{n}n}(\lambda_1,\lambda_2).
\end{aligned}
$$

Approach 2: single functional relation for generalized density matrix

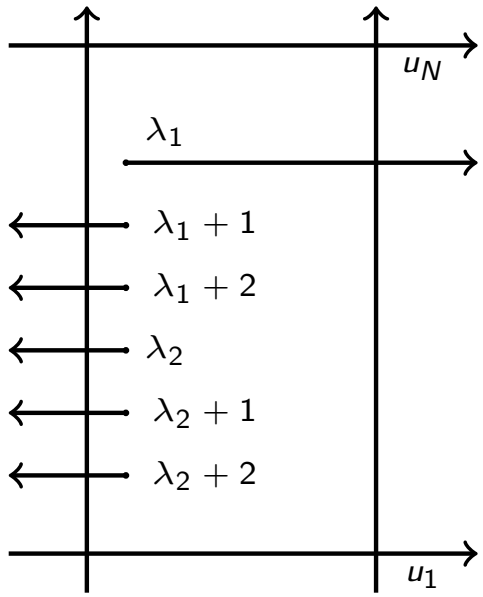
$$\mathbb{A}_m(\lambda_1,\ldots,\lambda_m)[\mathbb{D}_m(\lambda_1,\lambda_2,\ldots,\lambda_m)] = \mathbb{D}_m(\lambda_1{+}1,\lambda_2,\ldots,\lambda_m), \qquad \lambda_1 = u_i,$$



Derivation of the functional equations: e.g $SU(3)$, $m = 2$



$\underset{=}{\lambda_1 = u_i}$



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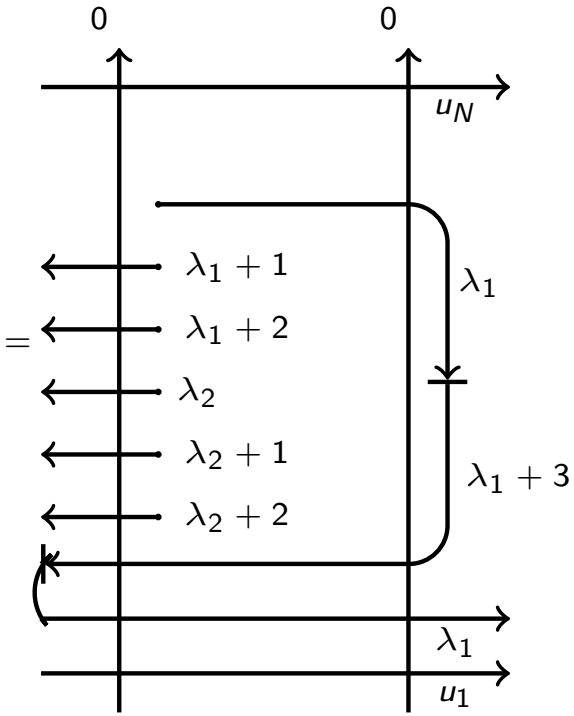
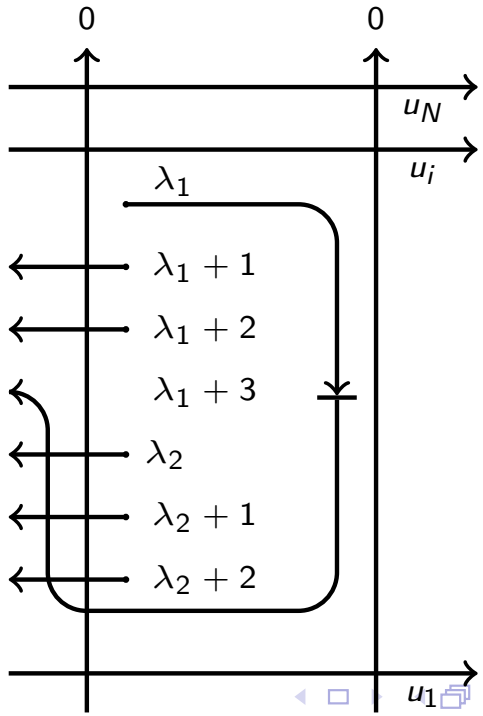


$\underset{=}{\lambda_1 = u_i}$

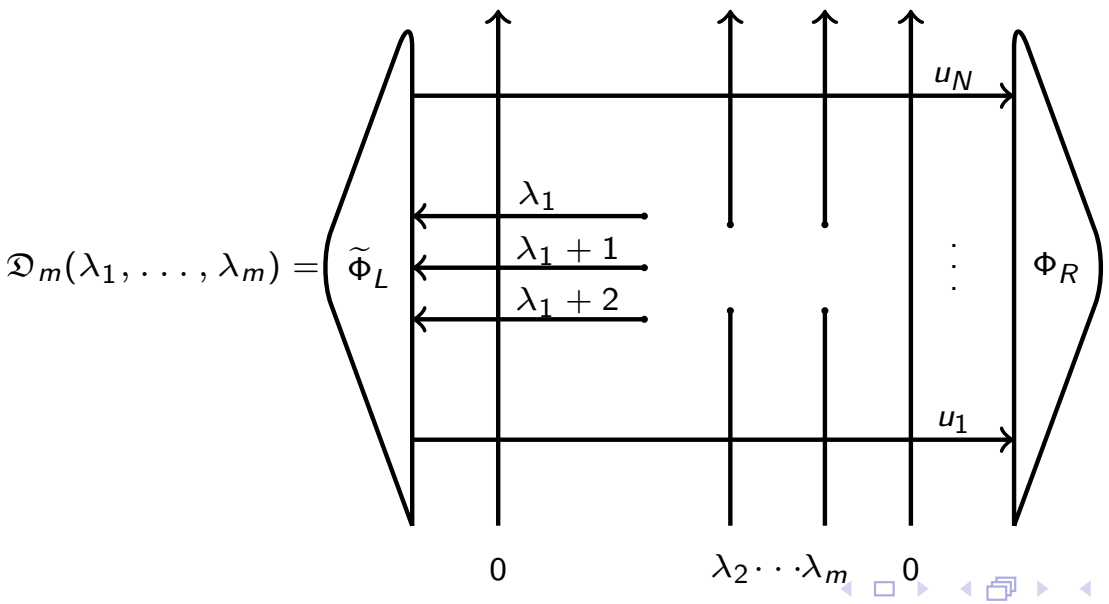


How to compute the density operator?

- Propose an ansatz as linearly independent operators (states) and solving the functional equations for the coefficients.

$\mathbb{D}_m(\lambda_1, \ldots, \lambda_m)$: the singlet space for $(m = 2)$ and three $(m = 3)$ sites density operator are 5- and 42-dimensional, respectively. Therefore the high number of required independent operators for three-sites makes the problem very hard to treat.

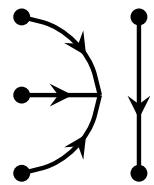
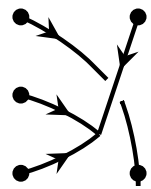
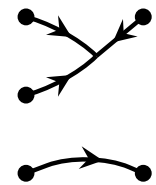
- To overcome that, we define a mixed density operator. Now for $(m = 2)$ and three $(m = 3)$ the singlet space are 3- and 11-dimensional.



Computation of the two-sites density operator

$$\mathfrak{D}_2(\lambda_1,\lambda_2) = \rho_1^{[2]}(\lambda_1,\lambda_2)P_1^{[2]} + \rho_2^{[2]}(\lambda_1,\lambda_2)P_2^{[2]} + \rho_3^{[2]}(\lambda_1,\lambda_2)P_3^{[2]},$$

where the operators $P_k^{[2]}$ are chosen as,

$P_1^{[2]} =$  , $P_2^{[2]} =$  , $P_3^{[2]} =$  ,

$$\begin{pmatrix} \rho_1^{[2]}(\lambda_1,\lambda_2) \\ \rho_2^{[2]}(\lambda_1,\lambda_2) \\ \rho_3^{[2]}(\lambda_1,\lambda_2) \end{pmatrix} = A^{[2]}(\lambda) \cdot \begin{pmatrix} \rho_1^{[2]}(\lambda_1+1,\lambda_2) \\ \rho_2^{[2]}(\lambda_1+1,\lambda_2) \\ \rho_3^{[2]}(\lambda_1+1,\lambda_2) \end{pmatrix}, \qquad \lambda_1 = u_i,$$

where $\lambda = \lambda_1 - \lambda_2$ and the matrix $A^{[2]}(\lambda)$ is given by,

$$A^{[2]}(\lambda) = \begin{pmatrix} \frac{(-1+3\lambda+\lambda^2)}{\lambda(\lambda+3)} & \frac{(-2+2\lambda+\lambda^2)}{\lambda(\lambda+3)} & \frac{1}{\lambda+3} \\ \frac{3}{\lambda(\lambda+3)} & \frac{-(-3+\lambda+\lambda^2)}{\lambda(\lambda+3)} & \frac{\lambda}{\lambda+3} \\ 0 & \frac{-(-1+\lambda)}{\lambda} & 0 \end{pmatrix}.$$

These equations can be disentangled by the transformation matrix

$$\begin{pmatrix} 1 \\ \omega_{33}(\lambda_1,\lambda_2) \\ \omega_{\bar{3}\bar{3}}(\lambda_1-1,\lambda_2) \end{pmatrix} = \begin{pmatrix} 18 & 6 & 6 \\ 6 & 18 & -6 \\ 6 & -6 & 18 \end{pmatrix} \cdot \begin{pmatrix} \rho_1^{[2]}(\lambda_1,\lambda_2) \\ \rho_2^{[2]}(\lambda_1,\lambda_2) \\ \rho_3^{[2]}(\lambda_1,\lambda_2) \end{pmatrix},$$

which expresses the normalization condition and the partial antisymmetrization in the lower and upper two semi-infinite lines of the density operator.
In terms of the above functions, the functional equation becomes,

$$\begin{aligned} \omega_{33}(\lambda_1,\lambda_2) &= \frac{(\lambda-1)(\lambda+1)}{\lambda(\lambda+3)}\omega_{\bar{3}\bar{3}}(\lambda_1,\lambda_2)+\frac{1}{\lambda}, \\ \omega_{\bar{3}\bar{3}}(\lambda_1-1,\lambda_2) &= -\frac{(\lambda-1)(\lambda+3)}{\lambda(\lambda+3)}\omega_{33}(\lambda_1+1,\lambda_2) \\ -\frac{(\lambda-1)(\lambda+2)}{\lambda(\lambda+3)}\omega_{\bar{3}\bar{3}}(\lambda_1,\lambda_2) &+ \frac{\lambda-1}{\lambda}, \end{aligned}$$

for $\lambda_1 = u_i$ and arbitrary λ_2.

Zero temperature solution: $m = 2$

At zero temperature the function $\omega_{33}(\lambda_1, \lambda_2)$ turns into a single-variable function $\omega_{33}(\lambda_1 - \lambda_2)$ depending only on the difference of the arguments,

$$\sigma(\lambda) = \frac{\omega_{33}(\lambda)}{(\lambda - 1)(\lambda + 1)}.$$

$$\sigma(\lambda + 1) + \sigma(\lambda) + \sigma(\lambda - 1) = \frac{\lambda^2 + 2}{(\lambda - 2)(\lambda - 1)(\lambda + 1)(\lambda + 2)},$$

$$\sigma(\lambda) = -\frac{d}{d\lambda} \log \left\{ \frac{\Gamma(1 + \frac{1}{3} + \frac{\lambda}{3}))\Gamma(1 - \frac{\lambda}{3})}{\Gamma(1 + \frac{1}{3} - \frac{\lambda}{3})\Gamma(1 + \frac{\lambda}{3})} \right\} - \frac{1}{\lambda^2 - 1}.$$

Having this solution we obtain $\omega_{33}(\lambda_1, \lambda_2)$. Taking the homogeneous limit ($\lambda_k \to 0$), we obtain

$$\omega_{33}(0, 0) = -\sigma(0) = 1 - \frac{\pi}{3\sqrt{3}} - \log 3 \approx -0.703212076746 18,$$

Therefore the α_{33} coefficient in the density operator is given by,

$$\alpha_{33}(0, 0) = \frac{1}{24} \left[2 - \frac{\pi}{\sqrt{3}} - 3 \log 3 \right] \approx -0.12956817625994.$$

Properties of the $\omega_{33}(\lambda)$ or $G(\lambda)$

We define a function $G(\lambda)$,

$$\begin{aligned} G(\lambda) &= \frac{\omega_{33}(\lambda)+1}{\lambda^2-1} \\ &= \frac{1}{3}\left[\psi_0(1-\frac{\lambda}{3})-\psi_0(1+\frac{1}{3}-\frac{\lambda}{3})+\psi_0(1+\frac{\lambda}{3})-\psi_0(1+\frac{1}{3}+\frac{\lambda}{3})\right], \end{aligned}$$

Expanding $G(\lambda)$ in a power series we obtain

$$3G(\lambda) = 2\sum_{k=0}^{\infty}\frac{1}{2k!}\left[\psi_{2k}(1)-\psi_{2k}(1+\frac{1}{3})\right]\left(\frac{\lambda}{3}\right)^{2k},$$

where now $\psi_m(\lambda)$ is the polygamma function.
We can use the fact that $\psi_m(z) = (-1)^{m+1}(m)!\zeta(m+1,z)$, where $\zeta(m,z)$ is the modified zeta function (Hurwitz zeta function) defined as

$$\zeta(m,a) = \sum_{k=0}^{\infty}\frac{1}{(k+a)^m},$$

and re-write expression (3) as

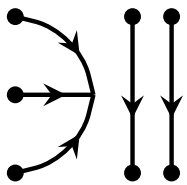
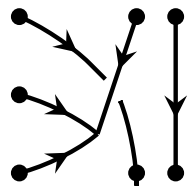
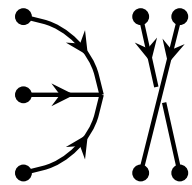
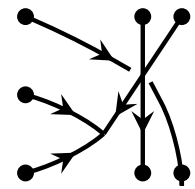
$$3G(\lambda) = 2\sum_{k=1}^{\infty}\left[\zeta(2k+1,1)-\zeta(2k+1,1+\frac{1}{3})\right]\left(\frac{\lambda}{3}\right)^{2k}+2\left[\psi_0(1)-\psi_0(1+\frac{1}{3})\right].$$

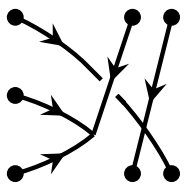
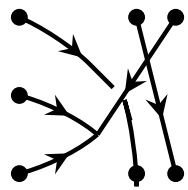
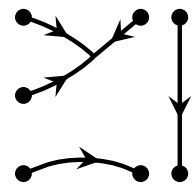
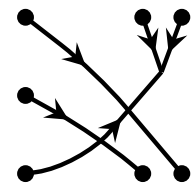
Computation of the three-sites density operator

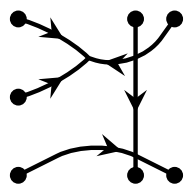
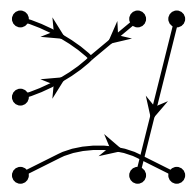
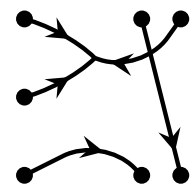
$m = 3$: singlet space 11 dimensional

$$\mathfrak{D}_3(\lambda_1, \lambda_2, \lambda_3) = \sum_{k=1}^{11} \rho_k^{[3]}(\lambda_1, \lambda_2, \lambda_3) P_k^{[3]}$$

where the operators $P_k^{[3]}$ are chosen as,

$P_1^{[3]} =$  , $P_2^{[3]} =$  , $P_3^{[3]} =$  , $P_4^{[3]} =$  ,

$P_5^{[3]} =$  , $P_6^{[3]} =$  , $P_7^{[3]} =$  , $P_8^{[3]} =$  ,

$P_9^{[3]} =$  , $P_{10}^{[3]} =$  , $P_{11}^{[3]} =$  ,

System of funcional equations

Inserting the above expansion of the density operator into equation (21) yields the set of functional equations

$$\vec{\rho}(\lambda_1, \lambda_2, \lambda_3) = A^{[3]}(\lambda_1, \lambda_2) \cdot \vec{\rho}(\lambda_1 + 1, \lambda_2, \lambda_3),$$

where $\vec{\rho}(\lambda_1, \lambda_2, \lambda_3)$ is a 11 dimensional vector whose entries are the expansion coefficients $\rho_k^{[3]}(\lambda_1, \lambda_2, \lambda_3)$ for $k = 1, \ldots, 11$ and the matrix $A^{[3]}(\lambda_1, \lambda_2, \lambda_3)$ is an 11×11 matrix.
Again, these equations can be suitably disentangled. **It results in the normalization, one trivial equation, three times the set of equations for two-points and finally three equations for three-points functions**,

$$\vec{f}(\lambda_1, \lambda_2, \lambda_3) = M \cdot \vec{\rho}(\lambda_1, \lambda_2, \lambda_3),$$

where the matrix M is given by

$$M = \begin{pmatrix}
54 & 18 & 18 & 18 & 6 & 6 & 18 & 6 & 18 & 6 & 6 \\
18 & 54 & 6 & 6 & 18 & 18 & -18 & -6 & 6 & -6 & -6 \\
18 & 6 & 54 & 6 & 18 & 18 & 6 & -6 & 6 & 18 & 18 \\
18 & 6 & 6 & 54 & 18 & 18 & 6 & 18 & -18 & -6 & -6 \\
6 & 18 & 18 & 18 & 54 & 6 & -6 & -18 & -6 & -18 & 6 \\
6 & 18 & 18 & 18 & 6 & 54 & -6 & 6 & -6 & 6 & -18 \\
18 & -18 & 6 & 6 & -6 & -6 & 54 & -6 & 6 & 18 & 18 \\
6 & -6 & -6 & 18 & -18 & 6 & -6 & 54 & -6 & 6 & 6 \\
18 & 6 & 6 & -18 & -6 & -6 & 6 & -6 & 54 & 18 & 18 \\
6 & -6 & 18 & -6 & -18 & 6 & 18 & 6 & 18 & 54 & 6 \\
6 & -6 & 18 & -6 & 6 & -18 & 18 & 6 & 18 & 6 & 54
\end{pmatrix}.$$

Suitable functions

$$1 = f_1,$$
$$\omega_{33}(\lambda_1, \lambda_2) = f_2,$$
$$\omega_{\bar{3}3}(\lambda_1 - 1, \lambda_2) = f_7,$$
$$\omega_{33}(\lambda_1, \lambda_3)(1 - y^2) = f_3 + yf_6 - yf_5 - y^2 f_4,$$
$$\omega_{\bar{3}3}(\lambda_1 - 1, \lambda_3)(1 - y)(2 + y) = f_7 - (y + 2)f_{10} + (y - 1)f_{11} - (y - 1)(y + 2)f_9,$$
$$\omega_{33}(\lambda_1, \lambda_2)(1 - x^2)(1 - (x - y)^2) = (1 - (x - y)^2)f_3 + x(x - 2y)f_4$$
$$+x(-1 + xy - y^2)f_5 + x(1 - xy + y^2)f_6 + x(x - y)(-2 + x^2 - xy)f_2,$$
$$\omega_{\bar{3}3}(\lambda_1 - 1, \lambda_2)(1 - x)(2 + x)(1 - (x - y)^2) =$$
$$(1 - (-1 + x)(x - y) - (2 + x)(x - y) + (-1 + x)(2 + x)(x - y)^2)f_7$$
$$+(2 - y - y^2)f_9 + (2 + y)(-1 + x^2 + y - x(1 + y))f_{10}$$
$$+(-1 + y)(1 - x^2 + x(-2 + y) + 2y)f_{11},$$
$$\omega_{33}(\lambda_2, \lambda_3) = f_3,$$
$$F_1(\lambda_1, \lambda_2, \lambda_3) = 2x(2 + x)y(2 + y)f_1 + 2x(2 + x)(2 + y)f_2$$
$$+2(2 + x)y(2 + y)f_4 + 2(2 + x)(2 + y)f_5,$$
$$F_2(\lambda_1, \lambda_2, \lambda_3) = 2(-2 - y - x(2 + y) + x(2 + x)y(2 + y))f_1$$
$$-2(-1 + x + x^2)(2 + y)f_2 + 2(1 + x)(2 + y)f_3$$
$$+2(1 + x + (2 + x)y - (2 + x)y(2 + y))f_4 - 2(1 + x + 2y + xy)f_5$$
$$+2(2 + y)f_6 - 2(-2 - y + xy + x^2(1 + y))f_7 + 2(1 + x - y)f_8$$
$$-2(1 + x)(-2 + y + y^2)f_9 - 2(1 + x)(2 + y)f_{10} - 2xf_{11},$$
$$F_3(\lambda_1, \lambda_2, \lambda_3) = 2(x^2 - 1)(y^2 - 1)f_1 + 2(x^2 - 1)(1 + y)f_7$$
$$+2(1 + x)(y^2 - 1)f_9 + 2(1 + x)(1 + y)f_{10},$$

The remaining equations

$$\begin{pmatrix} G_1(\lambda_1,\lambda_2,\lambda_3) \\ G_2(\lambda_1,\lambda_2,\lambda_3) \\ G_3(\lambda_1,\lambda_2,\lambda_3) \end{pmatrix} = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \cdot \begin{pmatrix} G_1(\lambda_1+1,\lambda_2,\lambda_3) \\ G_2(\lambda_1+1,\lambda_2,\lambda_3) \\ G_3(\lambda_1+1,\lambda_2,\lambda_3) \end{pmatrix} + \begin{pmatrix} r(\lambda_1,\lambda_2,\lambda_3) \\ 0 \\ 0 \end{pmatrix},$$

where $\lambda_1 = u_i$, and we have introduced for convenience the G_k-functions as,

$$\begin{aligned} G_1(\lambda_1,\lambda_2,\lambda_3) &= \frac{xy}{(x^2-1)(y^2-1)(x+2)(y+2)} F_1(\lambda_1,\lambda_2,\lambda_3), \\ G_2(\lambda_1,\lambda_2,\lambda_3) &= \frac{(x+1)(y+1)}{(x^2-1)(y^2-1)(x+2)(y+2)} F_2(\lambda_1,\lambda_2,\lambda_3), \\ G_3(\lambda_1,\lambda_2,\lambda_3) &= \frac{1}{(x^2-1)(y^2-1)} F_3(\lambda_1,\lambda_2,\lambda_3), \end{aligned}$$

and

$$\begin{aligned} r(\lambda_1,\lambda_2,\lambda_3) &= \frac{2(-1+2x^2+2y^2)}{(x^2-1)(y^2-1)} + \frac{2(x+y)}{(x^2-1)(y^2-1)} \omega_{33}(\lambda_2,\lambda_3) \\ &+ \frac{2(-1+3x+x^2-3y-2xy+y^2-3xy^2+3y^3)}{x(x+3)(x-y)(y^2-1)} \omega_{\bar{3}3}(\lambda_1,\lambda_3) \\ &- \frac{2(-1-3x+x^2+3x^3+3y-2xy-3x^2y+y^2)}{(x^2-1)(x-y)y(y+3)} \omega_{\bar{3}\bar{3}}(\lambda_1,\lambda_2). \end{aligned}$$

$m = 3$: zero temperature solution

The problem can be significantly simplified by partially taking the homogeneous limit $\lambda_2 = \lambda_3 = 0$ and decoupling the equations by the following transformation

$$\begin{aligned} g_0(\lambda_1) &= G_1(\lambda_1, 0, 0) + G_2(\lambda_1, 0, 0) + G_3(\lambda_1, 0, 0), \\ g_1(\lambda_1) &= G_1(\lambda_1, 0, 0) + wG_2(\lambda_1, 0, 0) + w^2 G_3(\lambda_1, 0, 0), \\ g_{-1}(\lambda_1) &= G_1(\lambda_1, 0, 0) + w^{-1} G_2(\lambda_1, 0, 0) + w^{-2} G_3(\lambda_1, 0, 0), \end{aligned}$$

where $w = e^{\frac{2\pi \mathrm{i}}{3}}$.
Therefore, the resulting equations become (we now set $\lambda_1 = \lambda$)

$$g_l(\lambda) = w^l g_l(\lambda + 1) + \varphi(\lambda),$$

where $l = 0, 1, -1$ and $\varphi(\lambda) = \lim_{\lambda_2, \lambda_3 \to 0} r(\lambda, \lambda_2, \lambda_3)$,

$$\begin{aligned} \varphi(\lambda) = &-\frac{12}{(\lambda^2 - 1)} \omega_{33}(\lambda, 0) - \frac{2}{(\lambda^2 - 1)^2} \omega'_{33}(\lambda, 0) + \frac{4\lambda}{(\lambda^2 - 1)^2} \omega_{33}(0, 0) \\ &+ \frac{2(4\lambda^4 + 6\lambda^3 - \lambda^2 - 6\lambda - 1)}{\lambda^2(\lambda^2 - 1)^2}, \end{aligned}$$

and the prime denotes the derivative with respect to the argument λ.
We use analyticity in the variable λ and Fourier transform the above equations. The resulting equations are algebraically solved for the Fourier coefficients and yield product expressions. Then, we Fourier transform back and find integrals of convolution type

$$g_l(\lambda) = \int_{-\infty}^{\infty} h_l(\lambda - \mu) \varphi(\mu) \frac{d\mu}{2\pi},$$

where

$$h_l(z) = \int_{\mathbb{R} + \mathrm{i}0} \frac{e^{\mathrm{i}kz}}{1 - w^l e^k} dk.$$

Numerical evaluation

The integral expression can be evaluated numerically at the homogeneous point $\lambda = 0$. This allows us to obtain the functions $G_i(0,0,0)$ (and derivatives of G_i at $(0,0,0)$) from which we compute $F_i(0,0,0)$ directly in the homogeneous limit. The function $F_1(0,0,0)$ is related to a simple three point correlation function $F_1(0,0,0) = 8\langle P_{12}P_{23}\rangle$. Using the result of the numerical evaluation of the integral equation (3), we obtain

$$\langle P_{12}P_{23}\rangle = 0.191368820116674$$

Length	$\omega_{33}(0,0)$	$\langle P_{12}P_{23}\rangle$
$L = 3$	-1.000000000000000	1.000000000000000
$L = 6$	-0.767591879243998	0.309579305659537
$L = 9$	-0.731082881703061	0.239661721591669
$L \to \infty$	-0.703212076746182	0.191368820116674

Table: Comparison of numerical results from exact diagonalization for $L = 3, 6$ and Lanczos calculations for $L = 9$ sites with the analytical result in the thermodynamic limit.

$SU(3)$: investigation of factorization

- We propose an ansatz in a factorized form for the three-point functions at zero temperature (expecting the coefficients to be rational functions), but it does not work. The resulting equations for the coefficients are similar to the one satisfied by the ω. Therefore, it produces itself ω-like functions (in other words, it is not a rational functions, which contradict the initial assumption).

- Arbitrary correlation can be expressed in terms of $n = 2$ EFP and the $n = 3$ EFP

e.g

$$\begin{aligned}
&\mathrm{Tr}\left[D_3(\lambda_1,\lambda_2,\lambda_3)P_{singlet}\right] = \frac{Q_3^{(s)}(\lambda_1,\lambda_2,\lambda_3)}{\Lambda(\lambda_1)\Lambda(\lambda_2)\Lambda(\lambda_3)} \\
= \quad & 6 - 24\left[(1+\frac{1}{\lambda_{13}\lambda_{23}})P_2(\lambda_1,\lambda_2) + (1+\frac{1}{\lambda_{12}\lambda_{32}})P_2(\lambda_1,\lambda_3) + (1+\frac{1}{\lambda_{21}\lambda_{31}})P_2(\lambda_2,\lambda_3)\right] \\
+ \quad & 60P_3(\lambda_1,\lambda_2,\lambda_3),
\end{aligned}$$

where

$$\begin{aligned}
P_2(\lambda_1,\lambda_2) &= \frac{Q_2(\lambda_1,\lambda_2)}{\Lambda(\lambda_1)\Lambda(\lambda_2)}, \\
P_3(\lambda_1,\lambda_2,\lambda_3) &= \frac{Q_3(\lambda_1,\lambda_2,\lambda_3)}{\Lambda(\lambda_1)\Lambda(\lambda_2)\Lambda(\lambda_3)}
\end{aligned}$$

and $Q_n(\lambda_1,\cdots,\lambda_n)$ are polynomials of known degree in each variable and $\Lambda(\lambda)$ is the largest eigenvalues of the quantum transfer matrix.

- The $n = 4$ it also not factorized in terms of $n = 2, 3$ density matrix only, but it requires one four-point function (e.g EFP: $P_4(\lambda_1,\lambda_2,\lambda_3,\lambda_4)$).

Summary

- We developed a consistent approach to deal with short-distance correlation functions of $SU(n)$ spin chains at zero and finite temperature.
- For the special case of $SU(3)$ we computed the two and three-sites correlation function at zero temperature explicitly.
- We also investigated the factorization of the three-point function in terms of two-point correlation. We have strong indication for the lack of factorization.

Further perspectives

- Correlatiosn at longer distances.
- Evaluation of the correlations at finite temperature.
- Explicit calculation for $SU(4)$.

Reference

GAPR and A. Klümper, Correlation functions of the integrable $SU(n)$ spin chain, arXiv:1804.10169 [math-ph].

Acknowledgments

Brazilian agency São Paulo Research Foundation (FAPESP) for financial