Rényi entropy of highly entangled spin chains

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Mainly based on

Bravyi et al, Phys. Rev. Lett. **118** (2012) 207202, arXiv:1203.5801
R. Movassagh and P. Shor, Proc. Natl. Acad. Sci. **113** (2016) 13278, arXiv:1408.1657

F.S. and V. Korepin, arXiv:1806.04049

Outline

Introduction 1

Quantum entanglement

- Most surprising feature of quantum mechanics,
  No analog in classical mechanics
- Crucial to quantum computation

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Quantum entanglement

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  No analog in classical mechanics
- Crucial to quantum computation
- From pure state of the full system S: $\rho = |\psi\rangle\langle\psi|$, reduced density matrix of a subsystem A: $\rho_A = \mathrm{Tr}_{S-A}\,\rho$ can become mixed states, and has nonzero entanglement entropy

$$S_A = -\mathrm{Tr}_A\left[\rho_A \ln \rho_A\right].$$

This is purely a quantum property.

Introduction 2

Area law of entanglement entropy

- Ground states of quantum many-body systems (with local interactions) typically exhibit the area law behavior of the entanglement entropy: $S_A \propto (\text{area of } A)$
- Gapped systems in 1D are proven to obey the area law.

  [Hastings 2007]
- For gapless case, $(1+1)$-dimensional CFT violates logarithmically: $S_A = \frac{c}{3} \ln(\text{volume of } A)$.

  [Holzhey, Larsen, Wilczek 1994], [Korepin 2004], [Calabrese, Cardy 2009]

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  - Beyond logarithmic violation: $S_A \propto \sqrt{(\text{volume of } A)}$

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- Recently, 1D solvable spin chain models which exhibit extensive entanglement entropy have been discussed.
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    Motzkin model (Shor-Movassagh model) [Movassagh, Shor 2014]

    Fredkin model [Salberger, Korepin 2016]

Introduction 3

Rényi entropy

[Rényi, 1970]

- Rényi entropy has further importance than the von Neumann entanglement entropy:

$$S_{A,\alpha} = \frac{1}{1-\alpha} \ln \mathrm{Tr}_A \, \rho_A^{\alpha} \qquad \text{with } \alpha > 0 \text{ and } \alpha \neq 1.$$

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In this lecture, I give a review of Motzkin spin chain and analytically compute its Rényi entropy of half-chain.

New phase transition found at $\alpha = 1$!

Motzkin spin model 1

[Bravyi et al 2012]

- 1D spin chain at sites $i \in S \equiv \{1, 2, \cdots, 2n\}$
- Spin-1 state at each site can be regarded as up, down and flat steps;

$$|u\rangle \Leftrightarrow \nearrow, \qquad |d\rangle \Leftrightarrow \searrow, \qquad |0\rangle \Leftrightarrow \longrightarrow$$

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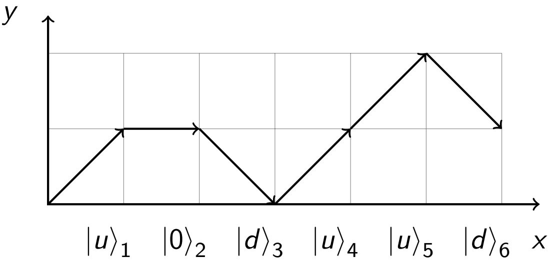
- Each spin configuration $\Leftrightarrow$ length-$2n$ walk in (x, y) plane
  Example)



Motzkin spin model 2

[Bravyi et al 2012]

Hamiltonian: $H_{Motzkin} = H_{bulk} + H_{bdy}$

- Bulk part: $H_{bulk} = \sum_{j=1}^{2n-1} \Pi_{j,j+1}$,

$$\Pi_{j,j+1} = |D\rangle_{j,j+1}\langle D| + |U\rangle_{j,j+1}\langle U| + |F\rangle_{j,j+1}\langle F|$$

(local interactions) with

$$|D\rangle \equiv \frac{1}{\sqrt{2}}\left(|0,\, d\rangle - |d,\, 0\rangle\right),$$

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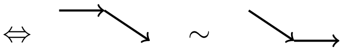

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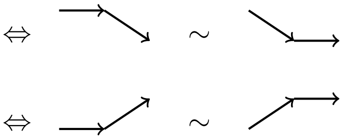

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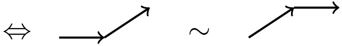

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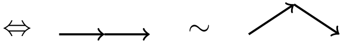

Motzkin spin model 3

[Bravyi et al 2012]

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- Boundary part: $H_{bdy} = |d\rangle_1\langle d| + |u\rangle_{2n}\langle u|$

$$\Downarrow$$

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- $H_{Motzkin}$ is the sum of projection operators.

  $\Rightarrow$ Positive semi-definite spectrum

- We find the unique zero-energy ground state.

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- $H_{Motzkin}$ is the sum of projection operators.

  $\Rightarrow$ Positive semi-definite spectrum
- We find the unique zero-energy ground state.
  - Each projector in $H_{Motzkin}$ annihilates the ground state.

    $\Rightarrow$ Frustration free
- The ground state corresponds to randoms walks starting at $(0, 0)$ and ending at $(2n, 0)$ restricted to the region $y \geq 0$ (Motzkin Walks (MWs)).

Motzkin spin model 4

[Bravyi et al 2012]

In terms of $S=1$ spin matrices

$$S_z = \begin{pmatrix} 1 & & \\ & 0 & \\ & & -1 \end{pmatrix}, \quad S_\pm \equiv \frac{1}{\sqrt{2}}(S_x \pm i S_y) = \begin{pmatrix} & 1 & \\ & & 1 \\ & & \end{pmatrix}, \begin{pmatrix} & & \\ 1 & & \\ & 1 & \end{pmatrix},$$

$$H_{bulk} = \frac{1}{2}\sum_{j=1}^{2n-1}\Big[1_j 1_{j+1} - \frac{1}{4}S_{z\,j}S_{z\,j+1} - \frac{1}{4}S_{z\,j}^2 S_{z\,j+1} + \frac{1}{4}S_{z\,j}S_{z\,j+1}^2$$

$$-\frac{3}{4}S_{z\,j}^2 S_{z\,j+1}^2 + S_{+\,j}\left(S_z S_-\right)_{j+1} + S_{-\,j}\left(S_+ S_z\right)_{j+1} - \left(S_- S_z\right)_j S_{+\,j+1}$$

$$-\left(S_z S_+\right)_j S_{-\,j+1} - \left(S_- S_z\right)_j \left(S_+ S_z\right)_{j+1} - \left(S_z S_+\right)_j \left(S_z S_-\right)_{j+1}\Big],$$

$$H_{bdy} = \frac{1}{2}\left(S_z^2 - S_z\right)_1 + \frac{1}{2}\left(S_z^2 + S_z\right)_{2n}$$

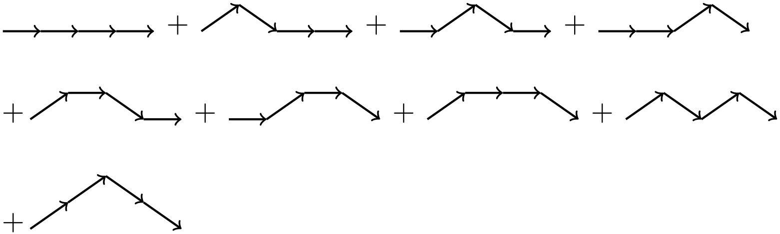
Quartic spin interactions

Motzkin spin model 5

[Bravyi et al 2012]

Example) $2n = 4$ case,

MWs:



$$\Updownarrow$$

Ground state:

$$
\begin{aligned}
|P_4\rangle \quad = \quad & \frac{1}{\sqrt{9}} \left[|0000\rangle + |ud00\rangle + |0ud0\rangle + |00ud\rangle \right. \\
& + |u0d0\rangle + |0u0d\rangle + |u00d\rangle + |udud\rangle \\
& \left. + |uudd\rangle \right].
\end{aligned}
$$

Motzkin spin model 6

[Bravyi et al 2012]

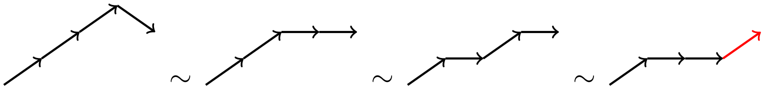
Note

Forbidden paths for the ground state

1. Path entering $y < 0$ region

Forbidden by H_{bdy}

2. Path ending at nonzero height



Forbidden by H_{bdy}

Motzkin spin model 7

[Bravyi et al 2012]

Entanglement entropy of a subsystem $A = \{1, 2, \cdots, n\}$:

- Normalization factor of the ground state $|P_{2n}\rangle$ is given by the number of MWs of length $2n$: $M_{2n} = \sum_{k=0}^{n} C_k \binom{2n}{2k}$.

$C_k = \frac{1}{k+1}\binom{2k}{k}$: Catalan number

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- Consider to trace out the density matrix $\rho = |P_{2n}\rangle\langle P_{2n}|$ w.r.t. the complement subsystem $B = S - A = \{n+1, \cdots, 2n\}$. Schmidt decomposition:

$$|P_{2n}\rangle = \sum_{h \geq 0} \sqrt{p_{n,n}^{(h)}} \left| P_n^{(0 \to h)} \right\rangle \otimes \left| P_n^{(h \to 0)} \right\rangle$$

with $p_{n,n}^{(h)} \equiv \frac{\left(M_n^{(h)}\right)^2}{M_{2n}}$.

↑ Paths from $(0, 0)$ to (n, h)

Motzkin spin model 8

[Bravyi et al 2012]

- $M_n^{(h)}$ is the number of paths in $P_n^{(0\to h)}$.
  For $n \to \infty$,

  Gaussian distribution

$$p_{n,n}^{(h)} \sim \frac{3\sqrt{6}}{\sqrt{\pi}} \frac{(h+1)^2}{n^{3/2}} e^{-\frac{3}{2}\frac{(h+1)^2}{n}} \times [1 + O(1/n)] \, .$$

- Reduced density matrix

$$\rho_A = \mathrm{Tr}_B \, \rho = \sum_{h\geq 0} p_{n,n}^{(h)} \left| P_n^{(0\to h)} \right\rangle \left\langle P_n^{(0\to h)} \right|$$

- Entanglement entropy

$$\begin{aligned} S_A &= -\sum_{h\geq 0} p_{n,n}^{(h)} \ln p_{n,n}^{(h)} \\ &= \frac{1}{2}\ln n + \frac{1}{2}\ln\frac{2\pi}{3} + \gamma - \frac{1}{2} \end{aligned}$$

(γ: Euler constant)

up to terms vanishing as $n \to \infty$.

Motzkin spin model 9

[Bravyi et al 2012]

Notes

- The system is critical (gapless).
  S_A is similar to the $(1+1)$-dimensional CFT with $c = 3/2$.

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  The system cannot be described by relativistic CFT.

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- Correlation functions [Movassagh 2017]

$$\langle S_{z\,j} \rangle \sim \frac{2}{\sqrt{3\pi}} \frac{1 - j/n}{j(1 - j/(2n))}, \qquad \langle S_{x\,j} \rangle = \langle S_{y\,j} \rangle = \langle S_{z\,j} S_{z\,k} \rangle = 0$$

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- Excitations have not been much investigated.

Colored Motzkin spin model 1

[Movassagh, Shor 2014]

- Introducing color d.o.f. $k = 1, 2, \cdots, s$ to up and down spins as

$$\left|u^k\right\rangle \Leftrightarrow \overset{k}{\nearrow}, \qquad \left|d^k\right\rangle \Leftrightarrow \overset{k}{\searrow}, \qquad |0\rangle \Leftrightarrow \longrightarrow$$

Color d.o.f. decorated to Motzkin Walks

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Color d.o.f. decorated to Motzkin Walks

- Hamiltonian $H_{cMotzkin} = H_{bulk} + H_{bdy}$
  - Bulk part consisting of local interactions:

$$H_{bulk} = \sum_{j=1}^{2n-1} \left(\Pi_{j,j+1} + \Pi^{cross}_{j,j+1}\right),$$

$$\Pi_{j,j+1} = \sum_{k=1}^{s} \left[\left|D^k\right\rangle_{j,j+1}\left\langle D^k\right| + \left|U^k\right\rangle_{j,j+1}\left\langle U^k\right| + \left|F^k\right\rangle_{j,j+1}\left\langle F^k\right|\right]$$

with

Colored Motzkin spin model 2

[Movassagh, Shor 2014]

$$\left|D^k\right\rangle \equiv \frac{1}{\sqrt{2}}\left(\left|0,\, d^k\right\rangle - \left|d^k,\, 0\right\rangle\right),$$

$$\left|U^k\right\rangle \equiv \frac{1}{\sqrt{2}}\left(\left|0,\, u^k\right\rangle - \left|u^k,\, 0\right\rangle\right),$$

$$\left|F^k\right\rangle \equiv \frac{1}{\sqrt{2}}\left(\left|0,\, 0\right\rangle - \left|u^k,\, d^k\right\rangle\right),$$

and

$$\Pi^{cross}_{j,j+1} = \sum_{k\neq k'} \left|u^k,\, d^{k'}\right\rangle_{j,j+1}\left\langle u^k,\, d^{k'}\right|.$$

$\Rightarrow$ Colors should be matched in up and down pairs.

- Boundary part

$$H_{bdy} = \sum_{k=1}^{s}\left(\left|d^k\right\rangle_1\left\langle d^k\right| + \left|u^k\right\rangle_{2n}\left\langle u^k\right|\right).$$

Colored Motzkin spin model 3

[Movassagh, Shor 2014]

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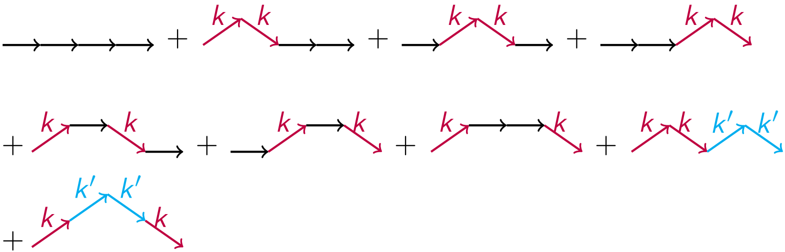
- Example) $2n = 4$ case,



$$|P_4\rangle = \frac{1}{\sqrt{1+6s+2s^2}}\left[|0000\rangle + \sum_{k=1}^{s}\left\{\left|u^k d^k 00\right\rangle + \cdots + \left|u^k 00 d^k\right\rangle\right\}\right.$$

$$\left. + \sum_{k,k'=1}^{s}\left\{\left|u^k d^k u^{k'} d^{k'}\right\rangle + \left|u^k u^{k'} d^{k'} d^k\right\rangle\right\}\right].$$

Colored Motzkin spin model 4

[Movassagh, Shor 2014]

Entanglement entropy

- Paths from $(0,0)$ to (n,h), $P_n^{(0\to h)}$, have h unmatched up steps.
  Let $\tilde{P}_n^{(0\to h)}(\{\kappa\})$ be paths with the colors of unmatched up steps frozen.
  (unmatched up from height $(m-1)$ to m) $\to u^{\kappa_m}$
- Similarly,

  $$P_n^{(h\to 0)} \to \tilde{P}_n^{(h\to 0)}(\{\kappa\}),$$

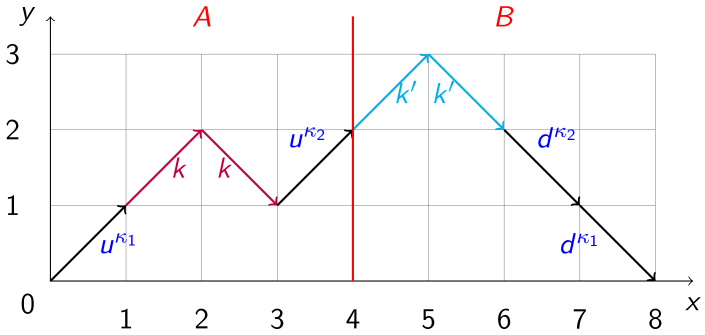
  (unmatched down from height m to $(m-1)$) $\to d^{\kappa_m}$.
- The numbers satisfy $M_n^{(h)} = s^h\, \tilde{M}_n^{(h)}$.

Colored Motzkin spin model 5

[Movassagh, Shor 2014]

Example

$2n = 8$ case, $h = 2$



Colored Motzkin spin model 6

[Movassagh, Shor 2014]

- Schmidt decomposition

$$
\begin{aligned}
|P_{2n}\rangle &= \sum_{h\geq 0}\sum_{\kappa_1=1}^{s}\cdots\sum_{\kappa_h=1}^{s}\sqrt{p_{n,n}^{(h)}} \\
&\times \left|\tilde{P}_n^{(0\to h)}(\{\kappa\})\right\rangle \otimes \left|\tilde{P}_n^{(h\to 0)}(\{\kappa\})\right\rangle
\end{aligned}
$$

with

$$
p_{n,n}^{(h)} = \frac{\left(\tilde{M}_n^{(h)}\right)^2}{M_{2n}}.
$$

- Reduced density matrix

$$
\begin{aligned}
\rho_A &= \sum_{h\geq 0}\sum_{\kappa_1=1}^{s}\cdots\sum_{\kappa_h=1}^{s} p_{n,n}^{(h)} \\
&\times \left|\tilde{P}_n^{(0\to h)}(\{\kappa\})\right\rangle\left\langle\tilde{P}_n^{(0\to h)}(\{\kappa\})\right|.
\end{aligned}
$$

Colored Motzkin spin model 7

[Movassagh, Shor 2014]

- For $n \to \infty$,

$$p_{n,n}^{(h)} \sim \frac{\sqrt{2}\, s^{-h}}{\sqrt{\pi}\, (\sigma\, n)^{3/2}} (h+1)^2\, e^{-\frac{(h+1)^2}{2\sigma n}} \times [1 + O(1/n)]$$

with $\sigma \equiv \frac{\sqrt{s}}{2\sqrt{s}+1}$.

Note: Effectively $h \lesssim O(\sqrt{n})$.

$\swarrow$

- Entanglement entropy

$$S_A = -\sum_{h \geq 0} s^h\, p_{n,n}^{(h)} \ln p_{n,n}^{(h)}$$

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- Entanglement entropy

$$\begin{aligned} S_A &= -\sum_{h \geq 0} s^h\, p_{n,n}^{(h)} \ln p_{n,n}^{(h)} \\ &= (2 \ln s)\sqrt{\frac{2\sigma\, n}{\pi}} + \frac{1}{2} \ln n + \frac{1}{2} \ln(2\pi\sigma) + \gamma - \frac{1}{2} - \ln s \end{aligned}$$

up to terms vanishing as $n \to \infty$.

Grows as $\sqrt{n}$.

Colored Motzkin spin model 8

[Movassagh, Shor 2014]

Comments

- Matching color $\Rightarrow$ s^{-h} factor in $p_{n,n}^{(h)}$
  $\Rightarrow$ crucial to $O(\sqrt{n})$ behavior in S_A

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$$\langle S_{z,1} S_{z,2n} \rangle_{\text{connected}} \to -0.034\ldots \times \frac{s^3 - s}{6} \neq 0 \qquad (n \to \infty)$$

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- Deformation of models to achieve the volume law behavior ($S_A \propto n$)
  Weighted Motzkin/Dyck walks [Zhang et al, Salberger et al 2016]

Réyni entropy 1

[F.S., Korepin, 2018]

- What we compute is the asymptotic behavior of

$$S_{A,\alpha} = \frac{1}{1-\alpha} \ln \sum_{h=0}^{n} s^h \left(p_{n,n}^{(h)} \right)^{\alpha}.$$

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$$\begin{aligned} S_{A,\alpha} &= \frac{1}{2}\ln n + \frac{1}{1-\alpha}\ln\Gamma\left(\alpha+\frac{1}{2}\right) \\ &\quad - \frac{1}{2(1-\alpha)}\left\{(1+2\alpha)\ln\alpha + \alpha\ln\frac{\pi}{24} + \ln 6\right\} \end{aligned}$$

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  - Logarithmic growth
  - Reduces to S_A in the $\alpha \to 1$ limit.
  - Consistent with half-chain case in the result in [Movassagh, 2017]

Réyni entropy 2

[F.S., Korepin, 2018]

Colored case ($s > 1$)

- Before we saw

$$p_{n,n}^{(h)} \sim \frac{\sqrt{2}\, s^{-h}}{\sqrt{\pi}\,(\sigma n)^{3/2}}\,(h+1)^2\, e^{-\frac{(h+1)^2}{2\sigma n}} \times [1 + O(1/n)]$$

with $\sigma \equiv \frac{\sqrt{s}}{2\sqrt{s}+1}$.

Note: Valid for $h \leq O(\sqrt{n})$.

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  For $0 < \alpha < 1$, exponentially growing (colored case ($s > 1$)).
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  We need more careful treatment of asymptotics of $p_{n,n}^{(h)}$.

Réyni entropy 3

[F.S., Korepin, 2018]

Asymptotics of $p_{n,n}^{(h)}$

- Let us go back to the original expression

$$p_{n,n}^{(h)} = \frac{\left(\tilde{M}_n^{(h)}\right)^2}{M_{2n}},$$

where

$$\begin{aligned}
\tilde{M}_n^{(h)} &= (h+1) \sum_{\rho=0}^{n-h} \frac{1+(-1)^{n-\rho+h}}{2} \, C_{n,h,\rho}, \\
C_{n,h,\rho} &= \frac{n!\, s^{(n-\rho+h)/2}}{\rho! \left(\frac{n-\rho-h}{2}\right)! \, \left(\frac{n-\rho+h}{2}+1\right)!} \\
M_{2n} &= (h=0 \text{ and } n \to 2n \text{ in the above})
\end{aligned}$$

Réyni entropy 4

[F.S., Korepin, 2018]

- For $n, \rho, n-\rho \pm h \gg 1$, the sum can be evaluated by the saddle point method as

$$p_{n,n}^{(h)} \simeq \frac{s^{-h}}{\sqrt{\pi}\, s^{1/4}} \frac{(2n)^{3/2}}{\left(2\sqrt{s}+1\right)^{2n+\frac{3}{2}}} \frac{n^{2n+1}}{\rho_0^{2n+3}} \times \frac{(h+1)^2}{\left[4sn^2-(4s-1)h^2\right]^{1/2}} \left(\frac{n-\rho_0-h}{n-\rho_0+h}\right)^{h+1} \times \left[1+O(n^{-1})\right], \tag{1}$$

where the saddle point value of ρ is $\rho_0 + O(n^0)$ with

$$\rho_0 \equiv \frac{n}{4s-1}\left[-1+\sqrt{4s-(4s-1)\frac{h^2}{n^2}}\right].$$

Réyni entropy 5

[F.S., Korepin, 2018]

- When $h \leq O(\sqrt{n})$, the expression reduces to

$$p^{(h)}_{n,n} \simeq \frac{\sqrt{2}\, s^{-h}}{\sqrt{\pi}\,(\sigma\, n)^{3/2}}\,(h+1)^2\, e^{-\frac{(h+1)^2}{2\sigma n}} \times [1 + O(1/n)] \qquad (2)$$

Note:

$$\left(\frac{n}{\rho_0}\right)^{2n} = \left(2\sqrt{s}+1\right)^{2n}\, e^{\frac{2\sqrt{s}+1}{2\sqrt{s}}\frac{h^2}{n}} \times \left[1 + O(n^{-1})\right],$$

$$\left(\frac{n-\rho_0-h}{n-\rho_0+h}\right)^{h+1} = e^{-\frac{2\sqrt{s}+1}{\sqrt{s}}\frac{h(h+1)}{n}} \times \left[1 + O(n^{-1})\right].$$

Réyni entropy 6

[F.S., Korepin, 2018]

Rényi entropy for $0 < \alpha < 1$

- Compute $S_{A,\,\alpha} = \frac{1}{1-\alpha} \ln \sum_{h=0}^{n} s^h \left(p_{n,n}^{(h)}\right)^{\alpha}$ with use of (1).
- Saddle point analysis for the sum leads to

$$S_{A,\alpha} = n\frac{2\alpha}{1-\alpha} \ln\left[\sigma\left(s^{\frac{1-\alpha}{2\alpha}} + s^{-\frac{1-\alpha}{2\alpha}} + s^{-1/2}\right)\right] + \frac{1+\alpha}{2(1-\alpha)} \ln n + C(s,\alpha)$$

with $C(s,\alpha)$ being n-independent terms.

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- Linear growth in n.
- Some universal meaning of the subleading $\ln n$ term.

(Identical with the Fredkin case)

Réyni entropy 7

[F.S., Korepin, 2018]

- Explicit form of $C(s,\alpha)$:

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\begin{aligned}
C(s,\alpha) \equiv\; & \frac{1}{2}\ln\pi - \frac{1}{1-\alpha}\ln\left(s\sqrt{\alpha}\right) \\
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& + \frac{3\alpha}{2(1-\alpha)}\ln(2\sigma) + \frac{3\alpha-1}{1-\alpha}\ln\left(s^{\frac{1}{2\alpha}} + s^{1-\frac{1}{2\alpha}} + 1\right) \\
& - \frac{\alpha}{2(1-\alpha)}\ln\left[1 + 4\frac{\left(2s^{\frac{1}{2\alpha}}+1\right)\left(2s^{1-\frac{1}{2\alpha}}+1\right)}{\left(s^{\frac{1-\alpha}{2\alpha}} - s^{-\frac{1-\alpha}{2\alpha}}\right)^2}\right].
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- Singular term at $\alpha = 0$: $-\frac{1}{2(1-\alpha)}\ln\alpha$

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- For $\alpha > 1$, the factor $s^{(1-\alpha)h}$ in the summand $s^h \left(p_{n,n}^{(h)}\right)^\alpha$ exponentially decays.

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- Further simplification: $e^{-\frac{\alpha}{2\sigma n}(h+1)^2} = 1 + O(n^{-1})$

Réyni entropy 9

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- The result is expressed in terms of Lerch transcendent $\Phi(z, g, a) = \sum_{k=0}^{\infty} \frac{z^k}{(k+a)^g}$ as

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Phase transition

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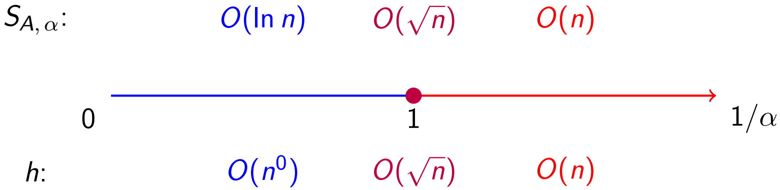
- The transition point $\alpha = 1$ itself forms the third phase.



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- We also have a similar result for the Fredkin spin chain.

[F.S., Korepin, 2018]

Summary and discussion 2

Rényi entropy of half-chain for the Fredkin model

- For $0 < \alpha < 1$,

$$S_{A,\alpha} = n\frac{2\alpha}{1-\alpha}\ \ln\cosh\frac{\theta}{2} + \frac{1+\alpha}{2(1-\alpha)}\ \ln n - \ln s + \frac{1}{2}\ln\frac{\pi}{4}$$
$$- \frac{1}{2(1-\alpha)}\ \ln\alpha - \frac{1}{1-\alpha}\ \ln\cosh\frac{\theta}{2} + \frac{2\alpha}{1-\alpha}\ \ln\sinh\theta$$

with $\theta \equiv \frac{1-\alpha}{\alpha}\ln s$.

- For $\alpha > 1$,

$$S_{A,\,\alpha} = \frac{3\alpha}{2(\alpha-1)}\ \ln n + \frac{\alpha}{2(\alpha-1)}\ \ln\frac{\pi}{32^2}$$
$$- \frac{1}{\alpha-1} \times \begin{cases} \ln\Phi\left(s^{-2(\alpha-1)}, -2\alpha, \frac{1}{2}\right) & (n\text{: even}) \\ \ln\Phi\left(s^{-2(\alpha-1)}, -2\alpha, 0\right) & (n\text{: odd}) \end{cases}$$

Summary and discussion 3

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Thank you very much for your attention!