# Heat transport in a superconducting quantum chain

Antônio M. S. Macêdo and Oscar Bohórquez

DEPARTAMENTO DE FSICA
UNIVERSIDADE FEDERAL DE PERNAMBUCO

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# The typical setup

Consider a typical problem of charge, heat or spin transfer through a phase coherent device. There are universal relations, integrability conditions and very powerful mathematical structures that can be explored to find exact analytical solutions to certain problems.

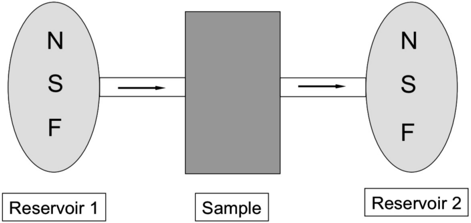
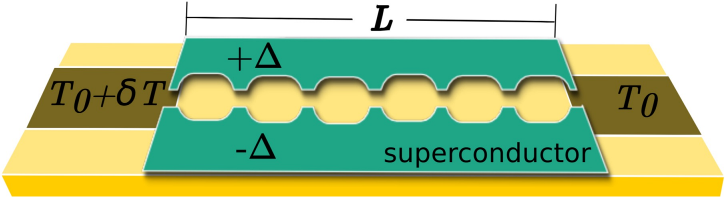



Fig. 1. A pictorial representation of a typical two-probe quantum transport setup. The letters have the following meaning: N = normal metal, S = superconductor, F = ferromagnet.

# Our model system

We consider a chain of Andreev quantum billiards connected ideally to two metal electrodes at different temperatures. A sign change of the superconducting pair potential is assumed to close the excitation gap inside the billiard without breaking TR symmetry.



# The ten-fold way

In the 1960's Freeman Dyson proposed a general classification of complex quantum systems in terms of certain symmetry properties of the Hamiltonian. It became known as the "three-fold way". Later, additional symmetries and constraints extended Dyson's classification to ten symmetry classes

1. Wigner-Dyson Class (WD)
2. Bogoliubov-de Gennes Class (BdG)
3. Chiral Class (Chiral)

| Class | TR | SR |
|---|---|---|
| WD | Yes | Yes |
| | No | Yes/No |
| | Yes | No |
| Chiral | Yes | Yes |
| | No | Yes/No |
| | Yes | No |
| BdG | Yes | Yes |
| | No | Yes |
| | Yes | No |
| | No | No |

# Another three-fold way

There is another three-fold way related to three different, but equivalent approaches

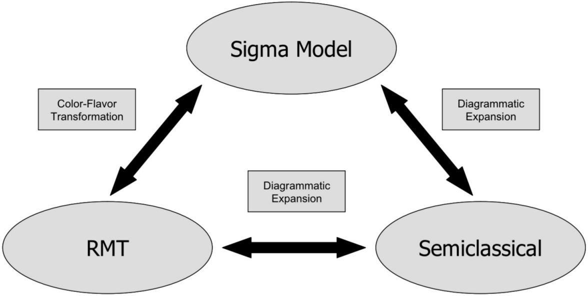



Fig. 2. Different approaches to quantum transport and their mutual relationships: The nonlinear $\sigma$-model, random matrix theory (RMT) and the trajectory-based semiclassical approach.

# How to use the classification?

Suppose you want to calculate the conductance distribution of a disordered quantum wire with $N$ open scattering channels, then

$$\mathcal{P}(g,t) = \int \prod_{i=1}^{N} d\tau_i \delta(g - \sum_{i=1}^{N} \tau_i) P_{\alpha\beta\gamma}(\{\tau\}, t), \qquad (1)$$

where $t$ is the length of the wire. Now parametrize $\tau_i = \mathrm{sech}^2(2q_i)$, then

$$\frac{\partial P_{\alpha\beta\gamma}}{\partial t} = \sum_{i=1}^{N} \left( -\frac{\partial}{\partial q_i} \frac{\partial \ln J_{\alpha\beta\gamma}}{\partial q_i} + \frac{\partial^2}{\partial q_i^2} \right) P_{\alpha\beta\gamma}, \qquad (2)$$

with $J_{\alpha\beta\gamma}$ given by

$$J_{\alpha\beta\gamma}(\{q\}) = \prod_{j=1}^{N} |\sinh^{\alpha}(2q_j)| \prod_{1\leq i<j\leq N} \left| \sinh^{\beta}(q_i - q_j) \sinh^{\gamma}(q_i + q_j) \right| . \qquad (3)$$

# The symmetry parameters

The symmetry parameters $\alpha$, $\beta$ and $\gamma$ can be read from the table

| Class | TR | SR | $\alpha$ | $\beta$ | $\gamma$ |
|---|---|---|---|---|---|
| | Yes | Yes | 1 | 1 | 1 |
| **WD** | No | Yes/No | 1 | 2 | 2 |
| | Yes | No | 1 | 4 | 4 |
| | Yes | Yes | 0 | 1 | 0 |
| **Chiral** | No | Yes/No | 0 | 2 | 0 |
| | Yes | No | 0 | 4 | 0 |
| | Yes | Yes | 2 | 2 | 2 |
| **BdG** | No | Yes | 2 | 2 | 2 |
| | Yes | No | 0 | 2 | 2 |
| | No | No | 0 | 1 | 1 |

Universality Classes and Symmetry Parameters

# An alternative classification

There is an alternative classification using matrix-valued Brownian motion [A.F. Macedo-Junior and AMSM, Nucl. Phys. B 752, 439 (2006)]. We rewrite the Fokker-Planck equation as

$$\frac{\partial P}{\partial t} = \sum_{i=1}^{N} \frac{\partial}{\partial x_i} \left( J_N w_N s(x_i) \frac{\partial}{\partial x_i} \frac{1}{J_N w_N} \right) P \qquad (4)$$

where

$$J_N(\{x\}) = \prod_{i<j} |x_i - x_j|^\beta, \quad w_N(\{x\}) = \prod_{i=1}^{N} w(x_i) \qquad (5)$$

The stationary solution

$$P_{st}(\{x\}) = C_N J_N(\{x\}) w_N(\{x\}) \qquad (6)$$

# An alternative classification

The functions $w(x)$, $s(x)$ and $r(x) \equiv \frac{1}{w(x)} \frac{d}{dx} w(x)s(x)$ are obtained from the table.

| **Ensemble** | $w(x)$ | $s(x)$ | $r(x)$ | *Interval* |
|---|---|---|---|---|
| Hermite | $e^{-x^2}$ | $1$ | $-2x$ | $(-\infty, \infty)$ |
| Laguerre | $x^\nu e^{-x}$ $(\nu > -1)$ | $x$ | $1 + \nu - x$ | $[0, \infty)$ |
| Jacobi | $(1-x)^\nu (1+x)^\mu$ $(\nu, \mu > -1)$ | $1 - x^2$ | $\mu - \nu - (2 + \mu + \nu)x$ | $[-1, 1]$ |
| SM-WD | $x^{\beta/2-1}$ | $x(1-x)$ | $\beta(1-x)/2 - x$ | $[0, 1]$ |
| SM-Chiral | $(1-x^2)^{\beta/2-1}$ | $1 - x^2$ | $-\beta x$ | $[-1, 1]$ |
| SM-BdG | $x^{\beta/2-1}(1-x)^{\gamma/2}$ $(\gamma = -1, 1)$ | $x(1-x)$ | $\beta(1-x)/2 - x(\gamma/2 + 1)$ | $[0, 1]$ |
| TM-WD | $1$ | $x^2 - 1$ | $2x$ | $[1, \infty)$ |
| TM-Chiral | $x^{[(1-N)\beta-2]/2}$ | $x^2$ | $[1 - \beta(N-1)/2]x$ | $[1, \infty)$ |
| TM-BdG | $(x^2-1)^{(\alpha-1)/2}$ $(\alpha = 0, 2)$ | $x^2 - 1$ | $(1+\alpha)x$ | $[1, \infty)$ |

# Calogero-Sutherland-Moser Hamiltonian

It is possible to derive an effective Schrödinger equation via the similarity transformation $P(\{x\}, it) = w_N J_N^{1/2} \Psi(\{x\}, t)$. The integrable effective Hamiltonian is of Calogero-Sutherland-Moser type

$$\mathcal{H} = \sum_{i=1}^{N} \frac{1}{w(x_i)} \frac{\partial}{\partial x_i} \left( w(x_i) s(x_i) \frac{\partial}{\partial x_i} \right) + \frac{\beta(\beta - 2)}{4} \sum_{i \neq j} \frac{s(x_i)}{(x_i - x_j)^2} + V_0 \quad (7)$$

The exact eigenfunctions and eigenvalues of $\mathcal{H}$ can be obtained via the transformation $\Psi = J_N^{1/2} \Phi$. For the classical random-matrix ensembles the function $\Phi$ yields Jack-type multivariate extensions of the classical orthogonal polynomials.

# Integral Transform and Dual FP

In order to calculate averages, it is useful to perform the integral transform

$$W(\{\nu\}, t) = \int d^N x \prod_{i=1}^{N} \frac{\prod_{k=1}^{n_0}(x_i - \nu_{0k})}{\prod_{l=1}^{n_1}(x_i - \nu_{1l})^{\beta/2}} P(\{x\}, t), \quad (8)$$

where $n_0$ and $n_1$ are positive integers satisfying $\beta n_1 = 2n_2$. The image functions satisfies a dual FP equation

$$\frac{\partial W}{\partial t} = \frac{1}{VB}\left[\sum_{l=1}^{n_1} \frac{\partial}{\partial \nu_{1l}} s(\nu_{1l}) VB \frac{\partial}{\partial \nu_{1l}} - \frac{\beta}{2}\sum_{k=1}^{n_0} \frac{\partial}{\partial \nu_{0k}} s(\nu_{0k}) VB \frac{\partial}{\partial \nu_{0k}}\right] W$$

$$V \equiv \prod_{k=1}^{n_0} w_0(\nu_{0k}) \prod_{l=1}^{n_1} w_1(\nu_{1l}); \quad w_0(\nu) = \frac{w^{2/\beta}(\nu)}{s^{1-2/\beta}(\nu)}; w_1(\nu) = \frac{s^{\beta/2-1}(\nu)}{w(\nu)}$$

$$B \equiv \prod_{k<k'} |\nu_{0k} - \nu_{0k'}|^{4/\beta} \prod_{l<l'} |\nu_{1l} - \nu_{1l'}|^{\beta} \prod_{k,l} |\nu_{0k} - \nu_{1l}|^{-2}$$

# The Dual Calogero-Sutherland-Moser Hamiltonian

It is possible to derive an effective Schrdinger equation for the dual FP equation via the similarity transformation $W(\{\nu\}, it) = B^{-1/2}\Psi(\{\nu\}, t)$. The integrable effective dual Hamiltonian is also of Calogero-Sutherland Moser type. We write $\mathcal{H} = \mathcal{H}_0 + \mathcal{V}$, where

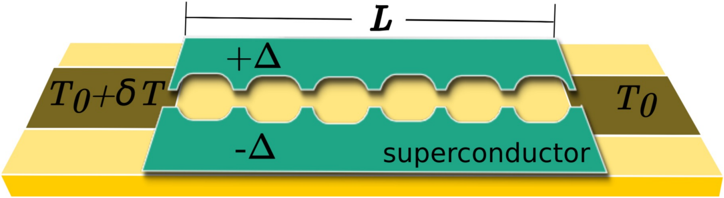
$$\mathcal{H}_0 = \sum_{l=1}^{n_1} \frac{1}{w_1(\nu_{1l})} \frac{\partial}{\partial \nu_{1l}} w_1(\nu_{1l}) s(\nu_{1l}) \frac{\partial}{\partial \nu_{1l}} - \frac{\beta}{2} \sum_{k=1}^{n_0} \frac{1}{w_0(\nu_{0k})} \frac{\partial}{\partial \nu_{0k}} w_0(\nu_{0k}) s(\nu_{0k}) \frac{\partial}{\partial \nu_{0k}}$$

$$\mathcal{V} = \frac{2-\beta}{\beta} \sum_{k \neq k'} \frac{s(\nu_{0k})}{(\nu_{0k} - \nu_{0k'})^2} + \frac{\beta(2-\beta)}{4} \sum_{l \neq l'} \frac{s(\nu_{1l})}{(\nu_{1l} - \nu_{1l'})^2} + \frac{\beta-2}{2} \sum_{k,l} \frac{s(\nu_{0k}) + s(\nu_{1l})}{(\nu_{0k} - \nu_{1l'})^2} + \mathcal{V}_0$$

The construction of the eigenfunctions proceeds mutatis mutandis and yields, for the classical ensembles, deformed versions of the multivariate classical polynomials.

# The model system

We consider a chain of Andreev quantum billiards connected ideally to two metal electrodes at different temperatures. A sign change of the superconducting pair potential is assumed to close the excitation gap inside the billiard without breaking TR symmetry.



# The model system

The dimensionless heat conductance can be written as $g = \sum_i \tau_i$, where $\tau_i$ are the transmission eigenvalues, i.e. the eigenvalues of $tt^\dagger$, where $t$ is the transmission matrix. We want to calculate the first three moments of $g$, which are given by

$$\langle g^m \rangle = \int \prod_{i=1}^{N} d\tau_i (\sum_{i=1}^{N} \tau_i)^m P(\{\tau\}).$$

For a single Andreev quantum dot coupled to two reservoirs via ideal contacts with $N_1$ and $N_2$ open channels, we can use a maximum entropy principle to obtain

$$P_{dot}(\{\tau\}) = C_N \prod_{i<j} |\tau_i - \tau_j|^\beta \prod_i \tau_i^{\beta(\mu+1)/2-1} (1-\tau_i)^{\gamma/2},$$

where $\mu = |N_1 - N_2|$ and the symmetry parameters $\beta$ and $\gamma$ can be obtained from the classification tables.

# The continuum limit

Taking the continuum limit we obtain a FP initial value problem in the standard coordinates

$$\frac{\partial P}{\partial t} = \sum_{i}^{N} \frac{\partial}{\partial x_i} s(x_i) \omega_N J_\beta \frac{\partial}{\partial x_i} \frac{P}{\omega_N J_\beta} \quad (9)$$

$$P(\{x\}, t = 0) = P_{dot}(\{x\})$$

| RMT | Random variable | $\omega(x)$ | $s(x)$ | a | b |
|---|---|---|---|---|---|
| SM | $x_i = \tau_i$ | $x^{\beta(\mu+1)/2-1}(1-x)^{\gamma/2}$ | $x(1-x)$ | 0 | 1 |
| TM | $x_i = 2/\tau_i - 1$ | $(x^2-1)^{(\alpha-1)/2}$ | $x^2-1$ | 1 | $\infty$ |

# The integral transform

We may now perform the integral transform

$$W(\{\vartheta\}, t) = \int d^N x \prod_i^N \frac{x_i - \vartheta_{0,1}}{x_i - \vartheta_{1,1}} P(\{x\}, t) \tag{10}$$

The dual FP equation is

$$\frac{\partial W}{\partial t} = \frac{1}{VB} \sum_{i=0}^{1} (-1)^{1+i} \frac{\partial}{\partial \vartheta_{i,1}} \left( s(\vartheta_{i,1}) VB \frac{\partial}{\partial \vartheta_{i,1}} \right) W \tag{11}$$

where

$$V = \frac{\omega(\vartheta_{0,1})}{\omega(\vartheta_{1,1})} \quad \text{and} \quad B = \frac{1}{(\vartheta_{0,1} - \vartheta_{1,1})^2} \tag{12}$$

$$W(\{\vartheta\}, t = 0) = \int d^N x \prod_i^N \frac{x_i - \vartheta_{0,1}}{x_i - \vartheta_{1,1}} P_{dot}(\{x\}) \tag{13}$$

# Hamiltonian Formulation

We may now map the FP equation onto an effective Schrödinger equation

$$W(\{\vartheta\}, t) = 1 + \omega(\vartheta_{1,1}) B^{-1/2} \Psi(\{\vartheta\}, t) \tag{14}$$

$$\frac{\partial \Psi}{\partial t} + \mathcal{H}\Psi = 0; \quad \mathcal{H} = \sum_{i=0}^{1} (-1)^i \frac{1}{\omega(\vartheta_i)} \frac{\partial}{\partial \vartheta_i} \left( \omega(\vartheta_i) s(\vartheta_i) \frac{\partial}{\partial \vartheta_i} \right) \tag{15}$$

The complete set of eigenfunctions are given by

$$\varphi_{nk}(\vartheta_0, \vartheta_1) = \frac{A_k^{(-\nu)}}{(h_n^{(\nu)})^{1/2}} P_n^{(\nu)}(\vartheta_0) \frac{F_k^{(-\nu)}(\vartheta_1)}{\omega(\vartheta_i)}; \quad n = 0, \dots; \; k \geq 0, \tag{16}$$

where $F_k^{(\nu)}(\vartheta_1) = F[\nu + \frac{1}{2} + ik, \nu + \frac{1}{2} - ik; \nu + 1; \frac{1-\vartheta_1}{2}]$ and

$$h_n^{(\nu)} = \frac{2^{2\nu+1}(\Gamma(n+\nu+1))^2}{n!(2n+2\nu+1)\Gamma(n+2\nu+1)}; \quad (A_k^{(\nu)})^2 = \frac{|\Gamma(\nu+1/2+ik)|^2}{2^{2\nu}(\Gamma(\nu+1))^2|\Gamma(ik)|^2}$$

# Initial condition

The initial condition can be written in the dual space as follows

$$W(\vartheta_0, \vartheta_1) = 1 + (\vartheta_0 - \vartheta_1) \sum_{l=0}^{N-1} \frac{(1-\vartheta_0)^l}{(1-\vartheta_1)^{l+1}} (f_{N-l-1}(\vartheta_0) g_{N-l-1}(\vartheta_1) - 1) \tag{17}$$

where

$$\begin{aligned} f_n(\vartheta_0) &= F[-n, -n-\mu; -2n-\mu-\frac{\gamma}{2}; \frac{1-\vartheta_0}{2}] \\ g_n(\vartheta_1) &= F[n+1, n+1+\mu; 2n+\mu+\frac{\gamma}{2}+2; \frac{1-\vartheta_1}{2}] \end{aligned} \tag{18}$$

and $F[a, b; c; d]$ is the Gauss hypergeometric function.

# Green's Function

With the eigenfunctions we can construct the propagator or Green's function

$$G(\{\vartheta\},\{\vartheta'\},t) = (1-\vartheta_0'^2)^\nu(\vartheta_1'^2-1)^\nu \sum_{n=0}^{\infty}\int_0^\infty dk \varphi_{nk}(\vartheta_0,\vartheta_1)\varphi_{nk}(\vartheta_0',\vartheta_1')e^{-\varepsilon_{nk}t}$$

where $\varepsilon_{nk} = k^2 + (n+\nu+1/2)^2$ are the eigenvalues. By construction

$$G(\{\vartheta\},\{\vartheta'\},0) = \delta(\vartheta_0-\vartheta_0')\delta(\vartheta_1-\vartheta_1') \qquad (19)$$

The complete solution is then given by

$$W(\{\vartheta\},t) = 1 + (\vartheta_0-\vartheta_1)\omega(\vartheta_1)\int_{-1}^{1} d\vartheta_0' \int_1^\infty d\vartheta_1' G(\{\vartheta\},\{\vartheta'\},t)\frac{W(\{\vartheta'\},0)}{(\vartheta_0'-\vartheta_1')\omega(\vartheta_1')}e^{-\varepsilon_{nk}t}$$

# The solution in dual space

The integrals over the Green's function can be performed by using identities from the theory of Meijer G functions. We find the explicit solution

$$W(\vartheta_0, \vartheta_1, t) = 1 + 2(\vartheta_0 - \vartheta_1) \sum_{n=0}^{N-1} \frac{P_n^{(\nu)}(\vartheta_0) P_n^{(\nu)}(1)}{h_n^{(\nu)}} \int_0^\infty d\mu_{nk} c_{nk}^{(\nu)}(N_1) c_{nk}^{(\nu)}(N_2) F_k^{(-\nu)}(\vartheta_1) e^{-\varepsilon_{nk} t}$$

where

$$d\mu_{nk} = dk |\Gamma(1/2 - \nu + ik)|^2 / (|\Gamma(ik)|^2 \varepsilon_{nk})$$

and

$$c_{nk}^{(\nu)}(N) = \frac{|\Gamma(N + \nu + 1/2 + ik)|^2}{(N - n - 1)! \Gamma(N + n + 2\nu + 1)} \tag{20}$$

# The conductance moments

The function $W$ can be interpreted as a generating function. Thus

$$\langle g \rangle = \frac{2\partial W}{\partial \vartheta_0}\bigg|_{\vartheta_0=1=\vartheta_1} \qquad \langle g^2 \rangle = -\frac{4\partial^2 W}{\partial \vartheta_0 \partial \vartheta_1}\bigg|_{\vartheta_0=1=\vartheta_1} \tag{21}$$

$$\langle g^3 \rangle = 4\left(\frac{\partial^3 W}{\partial \vartheta_0 \partial \vartheta_1^2} - \frac{\partial^3 W}{\partial \vartheta_1 \partial \vartheta_0^2}\right)\bigg|_{\vartheta_0=1=\vartheta_1}$$

$$\langle g^m \rangle = 4\sum_{n=0}^{N-1} \frac{(P_n^{(\nu)}(1))^2}{h_n^{(\nu)}} \int_0^\infty d\mu_{nk} g_{nk}^{(m)} c_{nk}^{(\nu)}(N_1) c_{nk}^{(\nu)}(N_2) e^{-\varepsilon_{nk} t} \tag{22}$$

where

$$g_{nk}^{(1)} = 1, \quad g_{nk}^{(2)} = \frac{(k^2 + (1/2-\nu)^2)}{(1-\nu)} + \frac{n(n+2\nu+1)}{(1+\nu)}, \tag{23}$$

$$g_{nk}^{(3)} = \frac{(k^2+(1/2-\nu)^2)(k^2+(3/2-\nu)^2)}{2(2-\nu)(1-\nu)} + \frac{2n(n+2\nu+1)(k^2+(1/2-\nu)^2)}{(1+\nu)(1-\nu)} + \frac{n(n-1)(n+2\nu+1)(n+2\nu+2)}{2(1+\nu)(2+\nu)}$$

# Asymptotics of conductance cumulants

It is useful to study the behavior of the conductance cumulants for a very long wire. We find for the classes DIII (TR and no SR) and CI (TR and SR) the following results

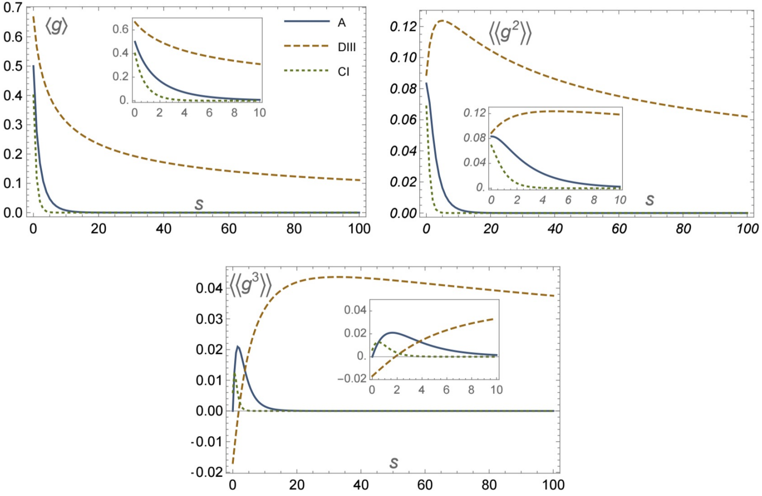
$$\langle g \rangle = \frac{3}{2}\langle\langle g^2 \rangle\rangle = \frac{15}{8}\langle\langle g^3 \rangle\rangle = \frac{2}{\sqrt{\pi t}}, \quad \text{(no localization)} \tag{24}$$

$$\langle g \rangle = t\langle\langle g^2 \rangle\rangle = 3t\langle\langle g^3 \rangle\rangle = 4c_{00}^{(1/2)}(N_1)c_{00}^{(1/2)}(N_2)\frac{e^{-t}}{\sqrt{\pi t}}, \tag{25}$$

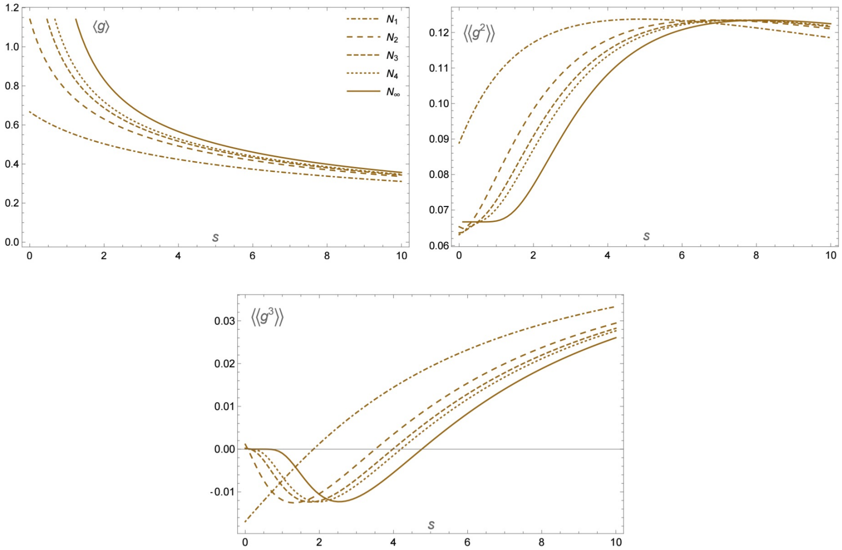
which should be contrasted with the corresponding results for a normal wire

$$\langle g \rangle = 4\langle\langle g^2 \rangle\rangle = \frac{64}{9}\langle\langle g^3 \rangle\rangle = 2c_{00}^{(0)}(N_1)c_{00}^{(0)}(N_2)\left(\frac{\pi}{t}\right)^{3/2} e^{-t/4} \tag{26}$$

# Conductance cumulants N=1 (Majorana mode)



# Conductance Cumulants



# Conclusions

1. We obtain exact expressions for the first three moments of the heat conductance of a quantum chain that crosses over from a superconducting quantum dot to a superconducting disordered quantum wire.
2. The striking effect of total suppression of the insulating regime in systems with broken spin-rotation invariance is observed at large length scales and for for a single mode topological superconductor it can be interpreted as a signature of Majorana modes.
3. The powerful mathematical structure of our solution may prove useful in establishing equivalence proofs between the various non-perturbative approaches to quantum transport problems.