Integrable models with non-Hermitian Hamiltonians.

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1. Historical notes.

The first integrable model with a non-Hermitian Hamiltonian was treated by Alcaraz, Droz, Henkel, and Rittenberg in 1994 (Ann. Phys. 230).

Later, in 1997 P.P. Kulish and A.A. Stolin discussed a new development of the QISM.

2. Algebraic background.

Let H be a Hopf algebra with comultiplication Δ and antipode S, co-unit ε.

Definition. An invertible element $F \in H \otimes H$, $F = \sum_i f_i^{(1)} \otimes f_i^{(2)}$ is called a "quantum twist" $\Longleftrightarrow$

$$(QT): \; F^{12}(\Delta \otimes id)(F) = F^{23}(id \otimes \Delta) F.$$

Theorem. QT defines a new Hopf algebra structure on H:

1) $\Delta_F = F \Delta F^{-1}$;

2) $S_F = u S u^{-1}$ with $u = \sum_i f_i^{(1)} S(f_i^{(2)})$;

3) ε_F remains the same: $\varepsilon_F = \varepsilon$.

If H has an additional invertible element $R \in H \otimes H$ such that:

1) $\Delta^{op} = R \Delta R^{-1}$,

2) $(\Delta \otimes id)(R) = R_{13} R_{23}$,

3) $(id \otimes \Delta)(R) = R_{13} R_{12}$,

then R satisfies the QYBE and $R_F = F_{21} R F^{-1}$ satisfies the QYBE as well.

Moreover,

$1)'$ $\Delta_F^{op} = R_F \cdot \Delta_F \cdot R_F^{-1}$,

$2)'$ $(\Delta_F \otimes id)(R_F) = (R_F)_{13}(R_F)_{23}$,

$3)'$ $(id \otimes \Delta_F)(R_F) = (R_F)_{13}(R_F)_{12}$.

3. First examples.

$H = U(sl_2)$, $\Delta = \Delta_0$ defined

as $\Delta_0(a) = a\otimes 1 + 1 \otimes a$, $a \in sl_2$.

$$F = 1\otimes 1 + \zeta E \otimes H + \frac{\zeta^2}{2!} E^2 \otimes H(H+2) + \cdots$$

$$\zeta^n/n! \cdot E^n \otimes H(H+2) \ldots (H+2n-2) \ldots$$

(*) (Here, $\{E, F, H\}$ is the standard basis of sl_2).

Remarks. In 1983, Drinfeld explained that such an $F \in U(g) \otimes U(g)$ "quantizes" certain Poisson brackets on G $(\mathrm{Lie}(G)=\mathfrak{g})$. It was done in different terms because quantum groups did not exist in 1983.

Many such F were constructed in Leningrad in 90s by P.P. Kulish, V.D. Lyakhovsky and their students.

4'. From quantum twist to deformation of integrable models.

We take into account that

$U(sl_2)$ is a Hopf subalgebra

of $Y(sl_2)$. $\Longrightarrow$ The same F

can be used to deform (twist)

$Y(sl_2)$:

$$R_\xi(u) = F_{21}\left(1 + \frac{P}{u}\right)F^{-1} =$$

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ -\xi & 1 & 0 & 0 \\ \xi & 0 & 1 & 0 \\ \xi^2 & -\xi & \xi & 1 \end{pmatrix} + \frac{P}{u}$$

Now, we have all the ingridients at hands to apply the QISM:

$$H_\xi = H_{XXX} + \xi^2 \sigma_n^{\cancel{+}-} \sigma_{n+1}^- + \xi(\sigma_n^- - \sigma_{n+1}^-)$$

$$= \sum_n \Big(\sigma_n^x \sigma_{n+1}^x + \sigma_n^y \sigma_{n+1}^y + \sigma_n^z \cdot \sigma_{n+1}^z$$

$$+ \xi^2 (\sigma_n^- \sigma_{n+1}^-) + \xi(\sigma_n^- - \sigma_{n+1}^-)\Big).$$

Here, $\sigma^x, \sigma^y, \sigma^z$ are Pauli matrices,

$$\sigma^- = E = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}.$$

It turns out that H_ξ has the same spectrum as H_{XXX}.

We follow the QISM formalism

$$T(u) = \begin{pmatrix} A(u) & B(u) \\ C(u) & D(u) \end{pmatrix} \longrightarrow$$

$$R_\xi(u-v)\, T_1(u)\, T_2(v) = T_2(v)\, T_1(u)\, R_\xi(u-v).$$

5. What to do with this model?

Relations between A, B, C, D become so complicated that there is no idea to write them here.

We failed to solve this model.

However, in memory of P.P. Kulish I suggest to call it the Kulish model.

20 years ago, when we obtained

this result I met Vladimir Rittenberg at the Weizmann Inst. of Science. I told him that (under some choice of parameters) the ADHR* model degenerates to the Kulish model.

* ADHR = Alcaraz, Droz, Henkel, Rittenberg

Rittenberg said: We solved our model without any QISM.

Use your QISM to solve your model and we will compare results!

6. <u>Classification of quantum groups.</u>

Recently, in a series of papers "finite dimensional" quantum groups were classified by means of the so called Belavin-Drinfeld and Galois cohomologies.

"Finite dimensional" $\Leftrightarrow$ lim = finite dimensional simple Lie algebra over $\mathbb{C}$.

The theory says (in combinatorial terms) that there exists yet unknown quantum group of type $U_q(sl_2)$, let us denote it by $U_{unq}(sl_2)$.

Similar to that of

$U(sl_2) \subset Y(sl_2)$,

$U_q(sl_2) \subset U_q(\widehat{sl_2})$,

$U_{unq}(sl_2)$ can be embedded to a yet unknown "infinite dim" quantum group $U_{unq}(\widehat{sl_2})$.

"<u>A very weak conjecture</u>":

$U_{unq}(sl_2)$ can be used to "construct" QISM for the ADHR model.

7. Why it might be true?

Quantum groups have classical limits, which are Lie bialgebras.

Usually, Lie bialgebras are defined by classical r-matrices.

For instance, if we use $F = 1\otimes 1 + \xi E\otimes H + \dots$ we obtain in the classical limit

$r = E\otimes H - H\otimes E$.

The quantum group $U_{unq}(sl_2)$ is unknown. However, it is possible to prove that in the classical limit it will produce $r = E \otimes H - H \otimes E$. Recall that

ADHR $\xrightarrow{\text{degener.}}$ Kulish. Probably, $U_{unq}(\hat{sl}_2)$ can be degenerated to $Y(sl_2)$?

8. Faddeev and classification of quantum groups for sl_3.

It turns out that to classify quantum groups such that their classical limit is sl_3, one has to employ the so called cubic rings. Recently, M. Bhargava used Faddeev's results to obtain cubic analogues of Gauss composition law. Same results of Faddeev were used to classify quantum groups.

<u>Reference:</u> D. K. Faddeev,
On the theory of cubic $\mathbb{Z}$-rings.
Mat. Inst. Steklov, 1965, v. 80.