The propagator of the finite XXZ spin-$\frac{1}{2}$ chain

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- Recent development in the dynamics of integrable many body systems:
  - GGE paradigm: Unitary time evolution after a homogeneous quench:

    $$H_0|\Psi_0\rangle = E_0|\Psi_0\rangle \qquad t=0: \quad H_0 \to H: \quad |\Psi(t)\rangle = e^{-\mathrm{i}Ht}, \; |\Psi(0)\rangle = |\Psi_0\rangle$$

    It is widely believed, that the Generalized Gibbs Ensemble (GGE) gives the long time asymptotic of the behaviour. GGE is extended by local and quasi-local charges.
  - Generalized hydrodynamics (GHD): In the Euler-limit ($t \to \infty, x \to \infty, x/t$fixed) hydrodynamical equations describe the dynamics.
- what we want: description of real time dynamics (and non-confusing notation)

Definition of the model

- spin$-\frac{1}{2}$ XXZ spin chain, with periodic BC ($\sigma^{\alpha}_{j+L} = \sigma^{\alpha}_{j}, \ \alpha = x, y, z$):

$$H = \sum_{j=1}^{L} \left(\sigma_j^x \sigma_{j+1}^x + \sigma_j^y \sigma_{j+1}^y + \Delta \left(\sigma_j^z \sigma_{j+1}^z - 1 \right) \right)$$

- reference state (pseudovacuum):

$$|0\rangle = \otimes_{j=1}^{L} \begin{pmatrix} 0 \\ 1 \end{pmatrix}_{[j]} = | \downarrow\downarrow\downarrow \ldots \downarrow\rangle$$

(Differs from the usual convention for computational reasons.)

- spin basis (assume: $a_j < a_k \leftrightarrow j < k$):

$$|a_1, \ldots, a_n\rangle = \prod_{j=1}^{n} \sigma^{+}_{a_j} |0\rangle$$

Real space propagator

- Quantity to compute:

$$G_n(\{b\},\{a\},t) = \langle b_1,\ldots,b_n|e^{-iHt}|a_1,\ldots,a_n\rangle$$

- Properties of the propagator:

$$i\frac{d}{dt}G_n(\{b\},\{a\},t) = H_a G_n(\{b\},\{a\},t) = H_b G_n(\{b\},\{a\},t)$$

$$G_n(\{b\},\{a\},0) = \prod_{j=1}^{n} \delta_{a_j,b_j}$$

- Symmetries:

$$G_n(\{b\},\{a\},t) = G_n^*(\{a\},\{b\},-t)$$

but also (since H is symmetric in spin basis):

$$G_n(\{b\},\{a\},t) = G_n(\{a\},\{b\},t)$$

translation and space reflection symmetry:

$$G_n(\{b\},\{a\},t) = G_n(1+\{b\},1+\{a\},t)$$
$$G_n(\{b\},\{a\},t) = G_n(-\{b\},-\{a\},t)$$

Trotter decomposition

- R-matrix:

$$R(u) = \begin{pmatrix} 1 & & & \\ & b(u) & c(u) & \\ & c(u) & b(u) & \\ & & & 1 \end{pmatrix}, \quad b(u) = \frac{\sinh(u)}{\sinh(u+\eta)}, \quad c(u) = \frac{\sinh(\eta)}{\sinh(u+\eta)}$$

- Crossing relation:

$$\frac{\sinh(u-\eta)}{\sinh(u)} R^{-1}(u) = \sigma_1^y R^{t_1}(u-\eta)\sigma_1^y$$

- Monodromy and transfer matrix:

$$T(u) = R_{10}(u) \ldots R_{L0}(u) = \begin{pmatrix} A(u) & B(u) \\ C(u) & D(u) \end{pmatrix}$$

$$\tau(u) = \mathrm{Tr}_0\, T(u)$$

- Space reflected matrices (from the reflection symmetry of the R-matrix):

$$\tilde{T}(u) = R_{L0}(u) \ldots R_{10}(u) = \begin{pmatrix} \tilde{A}(u) & \tilde{B}(u) \\ \tilde{C}(u) & \tilde{D}(u) \end{pmatrix}$$

$$\tilde{\tau}(u) = \mathrm{Tr}_0\, \tilde{T}(u) = \left(\frac{\sinh(u)}{\sinh(u+\eta)}\right)^L \tau(-u-\eta)$$

- Trotter decomposition:

$$H = 2\sinh(\eta)\frac{d}{du}\log\tau(u)|_{u=0}$$

$$e^{-itH} = \lim_{N\to\infty}\left(1-\frac{itH}{N}\right)^N = \lim_{N\to\infty}\left(\tau(-i\beta/(2N))\tilde{\tau}(-i\beta/(2N))\right)^N =$$

$$= \lim_{N\to\infty}\left(\left(\frac{\sinh\left(-i\beta/(2N)\right)}{\sinh\left(-i\beta/(2N)+\eta\right)}\right)^L \tau\left(-i\beta/(2N)\right)\tau\left(i\beta/(2N)-\eta\right)\right)^N$$

$$\beta = \sinh(\eta)t$$

- For computational reasons, introduce inhomogeneous β's:

$$\beta_j = \beta + \mathcal{O}(1/N)$$

$$e^{-itH} = \lim_{N\to\infty}\prod_{j=1}^{N}\left(\left(\frac{\sinh\left(-i\beta_j/(2N)\right)}{\sinh\left(-i\beta_j/(2N)+\eta\right)}\right)^L \tau\left(-i\beta_j/(2N)\right)\tau\left(i\beta_j/(2N)-\eta\right)\right)$$

- R-matrix is invariant under the reflection along the North-West diagonal
- After mirroring, horizontally we have the QTM:

$$T^{QTM}(u) = R_{2N\,0}\left(u - \tfrac{i\beta}{2N}\right) R_{2N-1\,0}\left(u - \eta + \tfrac{i\beta}{2N}\right) \dots R_{20}\left(u - \tfrac{i\beta}{2N}\right) R_{10}\left(u - \eta + \tfrac{i\beta}{2N}\right)$$
$$= \begin{pmatrix} A^{QTM}(u) & B^{QTM}(u) \\ C^{QTM}(u) & D^{QTM}(u) \end{pmatrix}$$

- Auxiliary crossed quantum transfer matrix:

$$\tilde{T}^{QTM}(0) = R_{2N\,0}\left(\tfrac{i\beta}{2N} - \eta\right) R_{2N-1\,0}\left(-\tfrac{i\beta}{2N}\right) \dots R_{20}\left(\tfrac{i\beta}{2N} - \eta\right) R_{10}\left(-\tfrac{i\beta}{2N}\right) =$$
$$= \begin{pmatrix} \tilde{A}^{QTM}(0) & \tilde{B}^{QTM}(0) \\ \tilde{C}^{QTM}(0) & \tilde{D}^{QTM}(0) \end{pmatrix}$$

- Scalar factors pairwise cancel each other:

$$\tilde{T}^{QTM}(0) = S(T^{QTM}(0))^{t_0} S$$
$$S = \prod_{j=1}^{2N} \sigma_j^y \,, \qquad S^2 = 1$$

- the propagator with fixed $|a_1,\ldots,a_n\rangle$ and $|b_1,\ldots,b_n\rangle$ in and out states, at finite Trotter number N = 6V partition sum, on a segment of a cylinder, L in circumference, and $2N$ in height. Height corresponds to time and circumference to spatial extension. In and out states are the BCs on the two rims of the cylinder. Considering it in the quantum channel:

$$G_m(\{b\},\{a\},t)=\lim_{N\to\infty}\left[\prod_{j=1}^{N}\left(\frac{\sinh\left(-\mathrm{i}\beta_j/(2N)\right)}{\sinh\left(-\mathrm{i}\beta_j/(2N)+\eta\right)}\right)^{L}\times\mathrm{Tr}\prod_{j=1}^{L}T^{QTM}_{s_j^b,s_j^a}(0)\right]$$

where:

$$s_j^a=\begin{cases}1 & \text{if} \quad j\in\{a_1,\ldots,a_m\}\\ 2 & \text{if} \quad j\notin\{a_1,\ldots,a_m\},\end{cases}$$

- Since the relation between T and $\tilde{T}$, a nicer (later we see this) form:

$$G_m(\{b\},\{a\},t)=\lim_{N\to\infty}\left[\prod_{j=1}^{N}\left(\frac{\sinh\left(-\mathrm{i}\beta_j/(2N)\right)}{\sinh\left(-\mathrm{i}\beta_j/(2N)+\eta\right)}\right)^{L}\times\mathrm{Tr}\prod_{j=1}^{L}T^{QTM}_{s_j^a,s_j^b}(0)\right]$$

(in other words, G is symmetric for finite Trotter number too.)

The propagator as a 6V partition function

- In state: $-$, Out state $-$: $D^{QTM}(0)$
- In state: $+$, Out state $-$: $B^{QTM}(0)$
- In state: $-$, Out state $+$: $C^{QTM}(0)$
- In state: $+$, Out state $+$: $A^{QTM}(0)$
- From now on, we leave the QTM superscript from the monodromy matrix elements
- E.g.: $L = 6$, $\langle 1,3|e^{-iHt}|2,3\rangle \sim \mathrm{Tr}\ C(0)B(0)A(0)D(0)D(0)D(0)$
- Is it computable? E.g.:

$$B_{1\dots n}(u) = \sum_{i=1}^{n} \sigma_i^- \Omega_i + \sum_{\substack{i,j,k \\ i\neq j\neq k}} \sigma_i^- \sigma_j^- \sigma_k^+ \Omega_{ijk} + \text{higher terms},$$

  where Ω_i, Ω_{ijk}, ... are diagonal on all sites, but i, i,j,k, etc. matrices. Seems computationally challenging.
- BUT!

- J.M.Maillet, J.Sanchez de Santos: Drinfel'd Twists and Algebraic Bethe Ansatz (Translations of the American Mathematical Society-Series 2, vol 201, 137-178 (2000)) arXiv: 9612012
- For $\pi \in S_n$ define R^π (uniquely defined because of YBE and unitarity):

$$R^{\pi}_{1\ldots n}(\xi_1,\ldots,\xi_n)T_{1\ldots n}(\xi_1,\ldots,\xi_n) = T_{\pi(1)\ldots\pi(n)}(\xi_{\pi(1)},\ldots,\xi_{\pi(n)})R^{\pi}_{1\ldots n}(\xi_1,\ldots,\xi_n)$$

- (factorizing) F-matrix def: Invertible matrix, s.t. for $\forall\pi \in S_n$:

$$F_{\pi(1)\ldots\pi(n)}(\xi_{\pi(1)},\ldots,\xi_{\pi(n)})R^{\pi}_{1\ldots n}(\xi_1,\ldots\xi_n) = F_{1\ldots n}(\xi_1,\ldots,\xi_n)$$

- Basistransformation by F:

$$\tilde{T}(u,;\xi_1,\ldots,\xi_n) = \begin{pmatrix} \tilde{A}(u) & \tilde{B}(u) \\ \tilde{C}(u) & \tilde{D}(u) \end{pmatrix} \equiv$$

$$\equiv F_{1\ldots n}(\xi_1,\ldots,\xi_n) \begin{pmatrix} A(u) & B(u) \\ C(u) & D(u) \end{pmatrix} F^{-1}_{1\ldots n}(\xi_1,\ldots,\xi_n)$$

Monodromy matrix elements in F-basis

$$\tilde{D}_{1\dots n}(u;\xi_1,\dots,\xi_n)=\otimes_{i=1}^{n}\begin{pmatrix} b(u,\xi_i) & 0 \\ 0 & 1 \end{pmatrix}_{[i]}$$

$$\tilde{B}_{1\dots n}(u;\xi_1,\dots,\xi_n)=\sum_{i\,=\,1}^{n}\sigma_i^-\;c\,(u,\;\xi_i)\;\otimes_{j\neq i}\begin{pmatrix} b\,(u,\;\xi_j) & 0 \\ 0 & b^{-1}\,(\xi_j,\;\xi_i) \end{pmatrix}_{[j]}$$

$$\tilde{C}_{1\dots n}(u;\xi_1,\dots,\xi_n)=\sum_{i\,=\,1}^{n}\sigma_i^+\;c\,(u,\;\xi_i)\;\otimes_{j\neq i}\begin{pmatrix} b\,(u,\;\xi_j)\;b^{-1}\,(\xi_i,\;\xi_j) & 0 \\ 0 & 1 \end{pmatrix}_{[j]}$$

$$\tilde{A}_{1\dots n}(u;\xi_1,\dots,\xi_n)=\otimes_{i=1}^{n}\begin{pmatrix} b(u,\xi_i)b^{-1}(u+\eta,\xi_i) & 0 \\ 0 & b(u,\xi_i) \end{pmatrix}_{[i]}+$$

$$+\sum_{i=1}^{n}c^2(u,\xi_i)\sigma_i^+\sigma_i^-\;\otimes_{j,j\neq i}\begin{pmatrix} b(u,\xi_j)b^{-1}(\xi_i,\xi_j) & 0 \\ 0 & b^{-1}(\xi_j,\xi_i) \end{pmatrix}_{[j]}+$$

$$+\sum_{i,j,i\neq j}c(u,\xi_i)c(u,\xi_j)b^{-1}(\xi_i,\xi_j)\sigma_i^+\otimes\sigma_j^-\;\otimes_{k\neq i,j}\begin{pmatrix} b(u,\xi_k)b^{-1}(\xi_i,\xi_k) & 0 \\ 0 & b^{-1}(\xi_k,\xi_j) \end{pmatrix}_{[i}$$

- We chose the $|\downarrow\downarrow\downarrow\ldots\downarrow\rangle$ as the reference state, because $\tilde{D}$ is diagonal
- diagonal and almost diagonal matrices, consequently the trace will be easily computable
- $\tilde{D}$ is manifestly symmetric in the simultaneous permutation of ξ's and associated spaces
- from now on the following notation:

$$\tilde{A}(u) \to A(u) \qquad \tilde{B}(u) \to B(u)$$
$$\tilde{C}(u) \to C(u) \qquad \tilde{D}(u) \to D(u)$$

- The particle moves by ℓ sites with $l = 1 \dots L-1$. In this case we are dealing with a trace of the form

$$G_1(\ell, 0, t) \sim \text{Tr } D^{L-1-\ell}(0) B(0) D^{\ell-1}(0) C(0)$$

  The simplest case is when $\ell = 1$, whereas the generic case is $\ell = 2 \dots L-1$.

- The particle stays at its position. In this case we are dealing with a trace of the form

$$G_1(0, 0, t) \sim \text{Tr } D^{L-1}(0) A(0)$$

- Introduce simplified notation:

$$b_i \equiv b(-\xi_i) = \frac{\sinh(u - \xi_i)}{\sinh(u - \xi_i + \eta)}$$

$$b_{ij}^{-1} \equiv b^{-1}(\xi_i - \xi_j) = \frac{\sinh(\xi_i - \xi_j + \eta)}{\sinh(\xi_i - \xi_j)}$$

$$c_i \equiv c(-\xi_i) = \frac{\sinh(\eta)}{\sinh(u - \xi_i + \eta)}$$

- One particle propagate by ℓ sites:

$$\mathrm{Tr}\ D^{L-1-\ell} B D^{\ell-1} C = \sum_{i=1}^{n} c_i^2 b_i^{\ell-1} \prod_{j, j\neq i} \left(b_j^L b_{ij}^{-1} + b_{ji}^{-1} \right)$$

- Substitution of ξ_i's:

$$\xi_j = \begin{cases} \mathrm{i}\beta_j/(2N) & j = 1, \ldots, N \\ -\mathrm{i}\beta_{j-N}/(2N) + \eta & j = N+1, \ldots, 2N \end{cases}$$

- The trace is singular in b_{ij}^{-1} if both i, j are "small" or big:

$$b_{ij}^{-1} = \frac{\sinh(\xi_i - \xi_j + \eta)}{\sinh(\xi_i - \xi_j)}$$

- Solution: Express the sum as a contour integral: f is free of poles inside $\mathcal{C}$, $g_i(z)$ has pole at ξ_j inside $\mathcal{C}$:

$$\oint_{\mathcal{C}} \frac{du}{2\pi i} f(u) \prod_j g_j(u) = \sum_{\xi_j} f(\xi_j) \mathrm{Res}_{z=\xi_j} g_j(z) \prod_{k\neq j} g_k(\xi_j)$$

$$\mathcal{C} = \mathcal{C}_0 \cup \mathcal{C}_\eta$$

$$\oint_{\mathcal{C}} \frac{du}{2\pi i} f(u) \prod_j g_j(u) = \sum_{\xi_j} f(\xi_j) \mathrm{Res}_{z=\xi_j} g_j(z) \prod_{k \neq j} g_k(\xi_j)$$

$$\sum_{j=1}^{n} c_j^2 b_j^{\ell-1} \prod_{k, k\neq j} \left(b_k^L b_{jk}^{-1} + b_{kj}^{-1} \right)$$

The identification follows:

$$f(u) = \frac{c^2(-u) b^{\ell-1}(-u)}{\sinh(\eta)(b^L(-u) - 1)} = \frac{\sinh(\eta)}{\sinh^2(-u+\eta)} \frac{\sinh^{\ell-1}(u)}{\sinh^{\ell-1}(u-\eta)} \frac{1}{\frac{\sinh^L(u)}{\sinh^L(u-\eta)} - 1}$$

$$g_j(u) = b^L(-\xi_j) b^{-1}(u - \xi_j) + b^{-1}(\xi_j - u) =$$

$$= \frac{\sinh^L(\xi_j)}{\sinh^L(\xi_j + \eta)} \frac{\sinh(u - \xi_j + \eta)}{\sinh(u - \xi_j)} + \frac{\sinh(\xi_j - u + \eta)}{\sinh(\xi_j - u)}$$

To take the Trotter limit ($n = 2N$):

$$G_1(\ell, 0, t) =$$

$$\lim_{N\to\infty} \left[\prod_{j=1}^{N} \left(\frac{\sinh(-i\beta_j/2N)}{\sinh(-i\beta_j/2N+\eta)} \right)^L \oint_{\mathcal{C}} \frac{du}{2\pi i} \frac{\sinh(\eta)}{\sinh^2(-u+\eta)} \frac{\sinh^{\ell-1}(u)}{\sinh^{\ell-1}(u-\eta)} \frac{1}{\frac{\sinh^L(u)}{\sinh^L(u-\eta)} - 1} \times \right.$$

$$\left. \times \prod_{j=1}^{2N} \frac{\sinh^L(\xi_j)}{\sinh^L(\xi_j+\eta)} \frac{\sinh(u-\xi_j+\eta)}{\sinh(u-\xi_j)} + \frac{\sinh(\xi_j-u+\eta)}{\sinh(\xi_j-u)} \right] =$$

$$= \ldots$$

$$= \oint_{\mathcal{C}} \frac{du}{2\pi \mathrm{i}} \frac{\sinh(\eta)}{\sinh^2(-u+\eta)} \frac{\sinh^{\ell-1}(u)}{\sinh^{\ell-1}(u-\eta)} \frac{\exp[i\,(\coth(u) - \coth(u-\eta))\,\beta]}{\frac{\sinh^L(u)}{\sinh^L(u-\eta)} - 1}$$

Other case: Particle not moving: $\sim \mathrm{Tr}\ D^{L-1}(0)A(0)$. Similar computation. Difference: corresponding f is not free of poles inside $\mathcal{C}$, the pole of f is cancelled with an additional term in the trace. Result:

$$G_1(0, 0, t) =$$

$$= \oint_{\mathcal{C}} \frac{du}{2\pi \mathrm{i}} \frac{\sinh(\eta)}{\sinh(u)\sinh(u-\eta)} \frac{b^L(-u)}{b^L(-u) - 1} \exp\left(\mathrm{i}(\coth(u) - \coth(u-\eta))\beta\right) \tag{1}$$

- The contour is fixed, independent of Bethe solution and string hypothesis of the model
- Rewrite the contour integral ($u \to u - \eta/2$, $\mathcal{C} = \mathcal{C}_{-\eta/2} \cup \mathcal{C}_{\eta/2}$ for this slide):

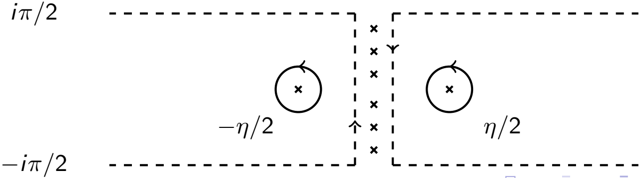
$$G_1(\ell, 0, t) = \oint_{\mathcal{C}} \frac{du}{2\pi \mathrm{i}} q(u) \frac{P^{\tilde{\ell}}(u)}{1 - P^L(u)} e^{-\mathrm{i}\varepsilon(u)t}$$

$$\tilde{\ell} = \begin{cases} L \text{ if } \ell = 0 \\ \ell \text{ if } \ell > 0 \end{cases}$$

$$q(u) = \frac{d}{du} \log(P(u)) = -\frac{\sinh(\eta)}{\sinh(u + \eta/2)\sinh(u - \eta/2)} = -\frac{\varepsilon(u)}{2\sinh(\eta)}$$

$$P(u) = \frac{\sinh(u + \eta/2)}{\sinh(u - \eta/2)}, \quad \varepsilon(u) = \frac{2\sinh^2(\eta)}{\sinh(u - \eta/2)\sinh(u + \eta/2)}$$

- Rewriting as sum over Bethe states ($\Delta > 1, \ \eta \in \mathbb{R}$):



Two particle case

$$G_2(\{b_1, b_2\}, \{a_1, a_2\}, t) \equiv \langle b_1, b_2 | e^{-iHt} | a_1, a_2 \rangle$$

The trace depends on the spatial order of the in and out particles. (Up to cyclicality, all spectral param.s are 0):

- For $b_1 < a_1 < b_2 < a_2$ one needs to compute Tr $B^{(\ell_1)} C^{(\ell_2)} B^{(\ell_3)} C^{(\ell_4)}$
- For $b_1 < a_1 < a_2 < b_2$ one needs to compute Tr $B^{(\ell_1)} C^{(\ell_2)} C^{(\ell_3)} B^{(\ell_4)}$
- For $b_1 = a_1,\ a_2 < b_2$ one needs to compute Tr $A^{(\ell_1)} C^{(\ell_2)} B^{(\ell_3)}$
- For $b_1 = a_1,\ b_2 < a_2$ one needs to compute Tr $A^{(\ell_1)} B^{(\ell_2)} C^{(\ell_3)}$
- For $a_1 = b_1,\ a_2 = b_2$ one needs to compute Tr $A^{(\ell_1)} A^{(\ell_2)}$

where we introduced the generalized matrix elements:

$$X^{(\ell)} \equiv D^{\ell} X, \qquad X = A, B, C$$

Two particle case - partition function

- traces are easily computed, and differ case by case.
- E.g.:

$$\mathrm{Tr}\ B^{(\ell_1)}C^{(\ell_2)}B^{(\ell_3)}C^{(\ell_4)} = \sum_i c_i^4 b_i^{\ell_2+\ell_4} \prod_{j,j\neq i} \left(b_j^L b_{ij}^{-2} + b_{ji}^{-2} \right) +$$
$$+ \sum_{i,j,i\neq j} c_i^2 c_j^2\ b_{ij}^{-1} b_{ji}^{-1}\ \left(b_i^{\ell_2} b_j^{\ell_4} + b_i^{\ell_2+\ell_3+\ell_4+2} b_j^{\ell_1+\ell_2+\ell_4+2} \right)$$
$$\prod_{k,k\neq i,k\neq j} \left(b_k^L b_{ik}^{-1} b_{jk}^{-1} + b_{ki}^{-1} b_{kj}^{-1} \right)$$

- These expressions are also singular ($b_{ij}^{-2}, b_{ik}^{-1}, \ldots$). Solution is similar. Consider the following contour integral (f, g_j are different from the previous ones):

$$\oint\oint \frac{du_1 du_2}{(2\pi \mathrm{i})^2} f(u_1,u_2) \prod_j \frac{h_j(u_1,u_2)}{g_j(u_1)g_j(u_2)} =$$
$$= \sum_j f(\xi_j,\xi_j) h_j(\xi_j,\xi_j) \left(\mathrm{Res}_{u_1=\xi_j}\ \frac{1}{g_j(u_1)} \right) \left(\mathrm{Res}_{u_2=\xi_j}\ \frac{1}{g_j(u_2)} \right) \prod_{k,k\neq j} \frac{h_k(\xi_j,\xi_j)}{g_k(\xi_j)g_k(\xi_j)} +$$
$$+ \sum_{j,k,j\neq k} f(\xi_j,\xi_k) \frac{h_k(\xi_j,\xi_k)}{g_k(\xi_j)} \left(\mathrm{Res}_{u_2=\xi_k}\ \frac{1}{g_k(u_2)} \right) \frac{h_j(\xi_j,\xi_k)}{g_j(\xi_k)} \left(\mathrm{Res}_{u_1=\xi_j}\ \frac{1}{g_j(u_1)} \right) \prod_{l,l\neq j,k} \frac{h_l(\xi_j,\xi_k)}{g_l(\xi_j)g_l(\xi_k)}$$

Two particle case - results

$$G_2(\{a,b\},\{c,d\},t) = \oint \oint_{\mathcal{C}} \frac{du_1 du_2}{(2\pi \mathrm{i})^2} q(u_1) q(u_2) \Psi_{\{a,b\},\{c,d\}}(u_1,u_2) \times$$
$$\times \frac{1}{1 - P^L(u_1) S(u_1 - u_2)} \frac{1}{1 - P^L(u_2) S(u_2 - u_1)} \times$$
$$\times \exp\left(-\mathrm{i}(\varepsilon(u_1) + \varepsilon(u_2)) t\right)$$

where (the u variable is shifted again, $\mathcal{C} = \mathcal{C}_{\eta/2} \cup \mathcal{C}_{\eta/2}$):

$$S(u) = \frac{\sinh(u-\eta)}{\sinh(u+\eta)}$$

$$\Psi_{\{a,b\},\{c,d\}}(u_1,u_2) =$$
$$= \begin{cases} P^{a-c}(u_1) P^{b-d}(u_2) + P^{b-c}(u_1) P^{L+a-d}(u_2) & \text{if } c < a < d < b \\ S(u_2 - u_1) P^{a-c}(u_1) P^{L+b-d}(u_2) + P^{b-c}(u_1) P^{L+a-d}(u_2) & \text{if } c < a < b < d \\ P^{L}(u_1) P^{L+b-d}(u_2) + P^{b-a}(u_1) P^{L+c-d}(u_2) & \text{if } a = c < b < d \\ S(u_1 - u_2) P^{L}(u_1) P^{b-d}(u_2) + P^{b-c}(u_1) P^{L+a-d}(u_2) & \text{if } a = c < d < b \\ P^{L}(u_1) P^{L}(u_2) + P^{b-c}(u_1) P^{L+a-d}(u_2) & \text{if } a = c < b = d \end{cases}$$

Easily generalize the equation for two particles:

$$G_m(\{b\},\{a\},t) =$$
$$= \oint \underset{\mathcal{C}}{\cdots} \oint \frac{\prod_{j=1}^{n} du_n}{(2\pi \mathrm{i})^m} \left(\prod_{j=1}^{m} q(u_j) \right) \frac{\Psi_{\{b\},\{a\}}(u_1,\ldots,u_m)\, e^{-\mathrm{i}\left(\sum_{j=1}^{m} \varepsilon(u_j)\right)t}}{\prod_{j=1}^{m} \left(1 - \prod_{k,k\neq j} P^L(u_j) S(u_j - u_k)\right)}$$

where:

$$\mathcal{C} = (\mathcal{C}_{-\eta/2} \cup \mathcal{C}_{\eta/2})^m$$
$$\Psi_{\{b\},\{a\}}(u_1,\ldots,u_m) = \sum_{\sigma \in S_m} \Psi^{\sigma}_{\{b\},\{a\}}(u_1,\ldots,u_m)$$

where $\Psi^{\sigma}_{\{b\},\{a\}}(u_1,\ldots,u_m)$ is the wavefunction amplitude corresponding to the σ permutation.

Proof: We have it, and it is very technical and long (and not yet written up).

Numerical check

Was done for one and two particle cases (with *Mathematica*):

- One particle: $L = 6, \Delta = 2, \ell = 1, t = 0.1$:
  Numerically exact: $0.14062275809462502` - 0.13657487584523947`i$
  Contour integral: $0.14062275809462524` - 0.13657487584523914`i$
- One particle not moving: $L = 6, \Delta = 2, \ell = 0, t = 0.1$:
  Numerically exact: $0.6691157650058658` + 0.6889473907802901`i$
  Contour integral: $0.6691157650058522` + 0.6889473907803108`i$
- Two particles: $L = 6, \Delta = 2, t = 0.1$ $\langle 2, 4|e^{-iHt}|1, 3\rangle$
  Numerically exact: $-0.003885840841277772` - 0.03754108661373063`i$
  Contour integral: $-0.003885840816969153` - 0.03754108660552163`4i$

Conclusion and outlook

- We found an p-folded contour integral expression for the real time propagator of the XXZ spin$-\frac{1}{2}$ chain (with PBC, for any Δ)
- The contours are fixed
- Proved the result for any p (very technical, not described here)
- Further possibilities:
  - derive $L \to \infty$ case (Yudson's result)
  - derive Lieb-Liniger propagator
  - apply to quench dynamics (?)

Reference: G.Z.Feher, B.Pozsgay: in preparation

Thank you for your attention! Questions?