Four-dimensional defect CFT and integrable boundary states

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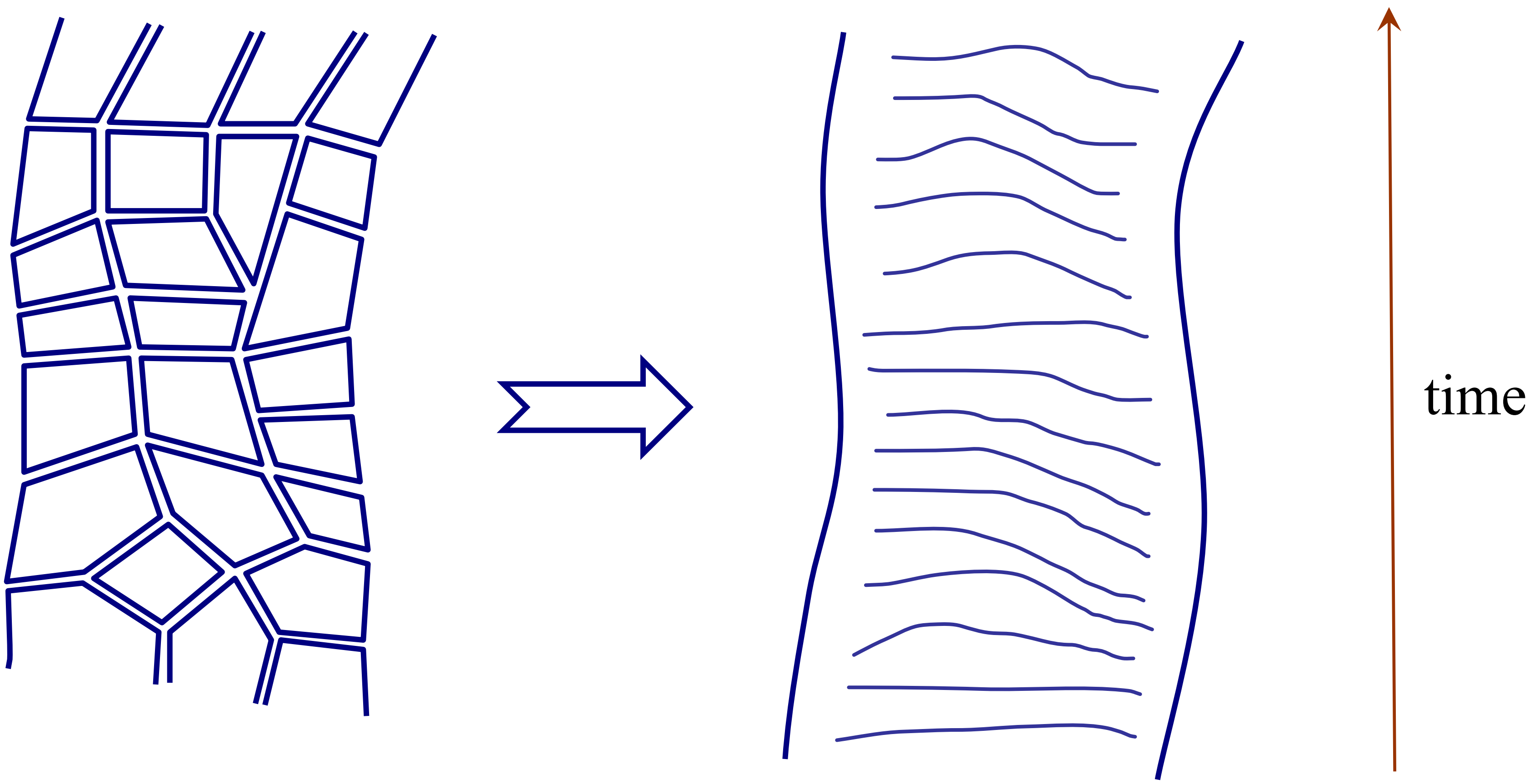
Exactly Solvable Quantum Chains, IIP, Natal, 26.6.18

Planar diagrams and strings

Large-N limit: $N \to \infty, \qquad \lambda - \text{fixed}$ 't Hooft'74



AdS/CFT correspondence

Yang-Mills theory with N=4 supersymmetry

Exact equivalence $\Updownarrow$

Maldacena'97
Gubser,Klebanov,Polyakov'98
Witten'98

String theory on $AdS_5 x S^5$ background

AdS/CFT correspondence

Maldacena'97

$\mathcal{N}=4$ SYM	Strings on $AdS_5 \times S^5$
't Hooft coupling: $\lambda = g_{YM}^2 N$	String tension: $T = \frac{\sqrt{\lambda}}{2\pi}$
Number of colors: N	String coupling: $g_s = \frac{\lambda}{4\pi N}$
Large-N limit	Free strings
Strong coupling	Classical strings
Local operators	String states
Scaling dimension: Δ	Energy: E

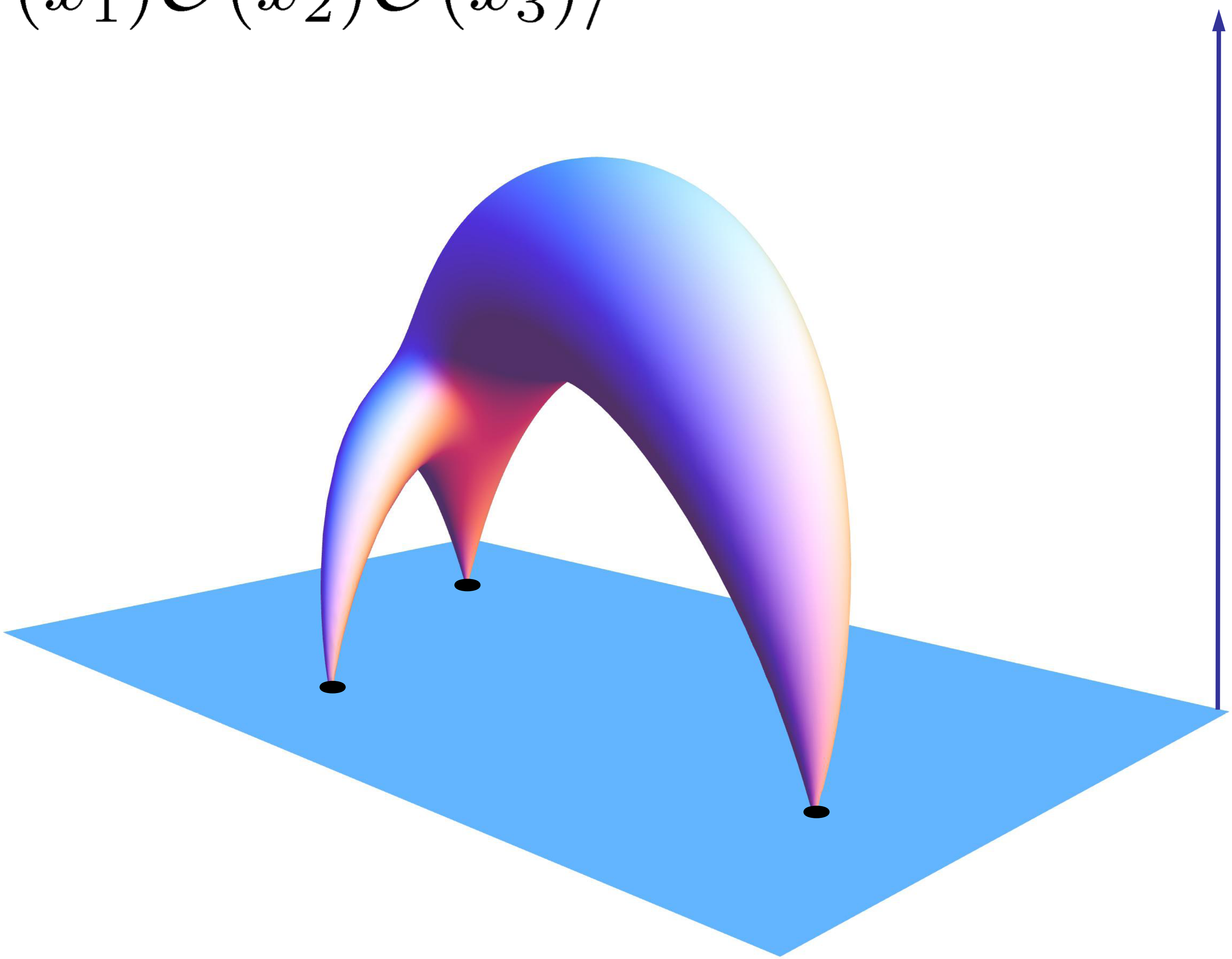
N=∞
in this talk

Gubser,Klebanov,Polyakov'98
Witten'98

Anti-de-Sitter space (AdS_5)

$$\boxed{ds^2 = \frac{dx^\mu dx_\mu + dz^2}{z^2}}$$

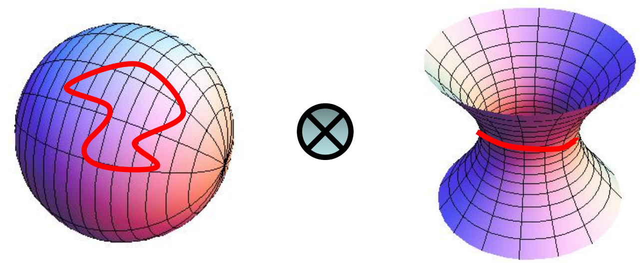
$$\langle \mathcal{O}(x_1)\mathcal{O}(x_2)\mathcal{O}(x_3)\rangle$$



String integrability

$$S^5 = SO(6)/SO(5)$$

$$AdS_5 = SO(4,2)/SO(4,1)$$



symmetric spaces

- σ – model on $AdS_5 x S^5$ is integrable

Eichenherr,Forger'79

Superstring:

$$\text{Super}(AdS_5 \times S^5) = PSU(2,2|4)/SO(5) \times SO(4,1)$$

- Green-Schwarz σ-model is also integrable

Bena,Polchinski,Roiban'03

Metsaev,Tseytlin'98

N=4 Supersymmetric Yang-Mills Theory

Gliozzi,Scherk,Olive'77
Brink,Schwarz,Scherk'77

Field content:

$A_\mu \qquad \Phi_I \qquad \Psi^A_\alpha$ all in the adjoint of $SU(N)$

$I = 1 \ldots 6 \qquad A = 1 \ldots 4$

Action:

$$S = \frac{1}{g_{YM}^2} \int d^4x \ \mathrm{tr} \left\{ \frac{1}{2} F_{\mu\nu}^2 + (D_\mu \Phi_I)^2 - \frac{1}{2} [\Phi_I, \Phi_J]^2 + i \bar{\Psi} \Gamma^\mu D_\mu \Psi - \bar{\Psi} \Gamma^I [\Phi_I, \Psi] \right\}$$

Operators

- Protected

energy-momentum tensor

$$\mathrm{tr}\left(F_{\mu\lambda}F^{\lambda\nu} - \frac{1}{4}\,\delta^{\nu}_{\mu}F_{\lambda\rho}F^{\lambda\rho} + \text{scalars} + \text{fermions}\right): \qquad \Delta = 4$$

- Non-degenerate

$$\mathrm{tr}\,\Phi_I\Phi_I: \qquad \Delta = 2 + \frac{3\lambda}{4\pi^2} - \frac{3\lambda^2}{16\pi^4} + \frac{21\lambda^3}{256\pi^6} + \ldots$$

Konishi operator

- Degenerate

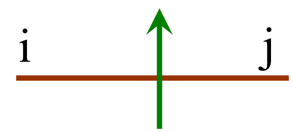
$$\mathrm{tr}\,\Phi_I\Phi_I\Phi_J\Phi_J + \text{loop corrections}: \qquad \Delta = 4 + \text{loop corrections}$$

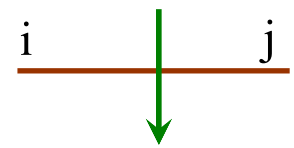
(mixes with $\mathrm{tr}\,\Phi_I\Phi_J\Phi_I\Phi_J$)

Local operators and spin chains

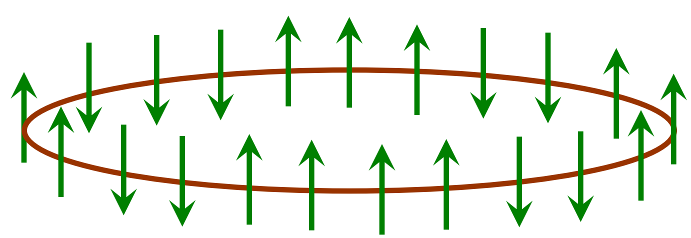
$$Z = \Phi_1 + i\Phi_2$$

$$W = \Phi_3 + i\Phi_4$$

Z_{ij} $\Longleftrightarrow$ 

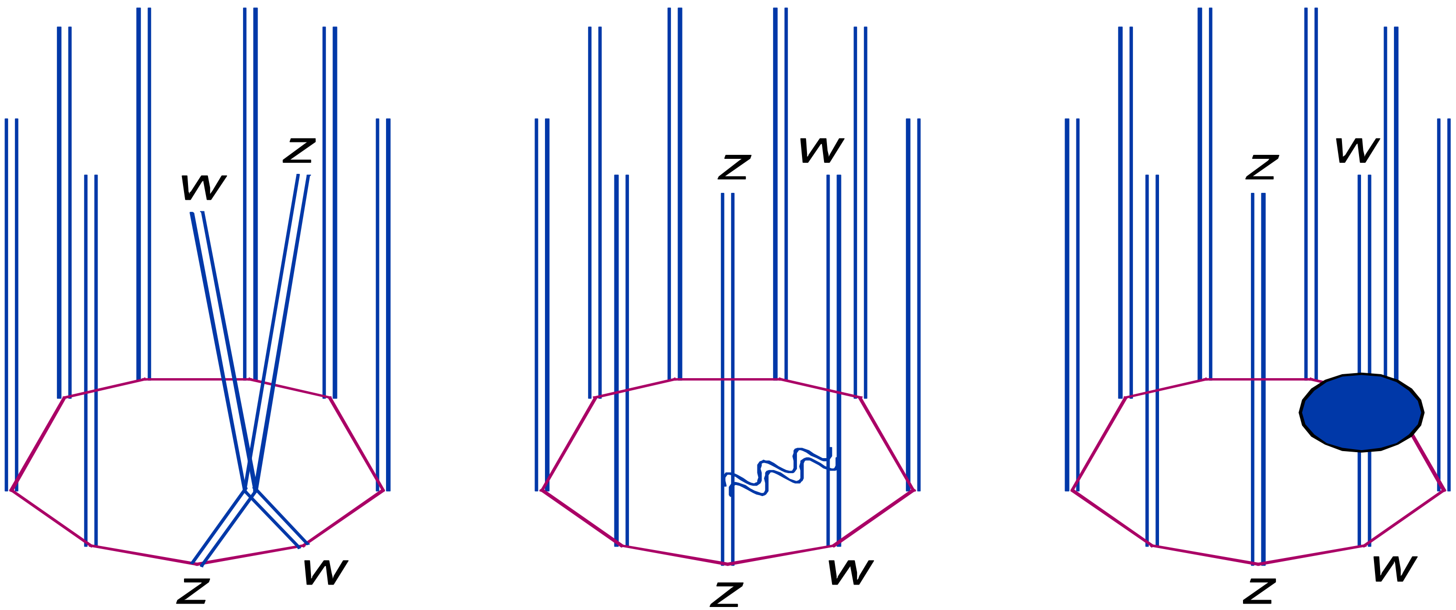
W_{ij} $\Longleftrightarrow$ 

$$\mathrm{tr}\, Z Z Z Z W W Z Z Z W W Z Z Z W W W Z Z W W$$



Tree level: $\Delta=L$ (huge degeneracy)

One loop:



One loop dilatation operator:

$$D = L + \frac{\lambda}{16\pi^2} \sum_{l=1}^{L} (1 - \boldsymbol{\sigma}_l \cdot \boldsymbol{\sigma}_{l+1}) + O(\lambda^2)$$

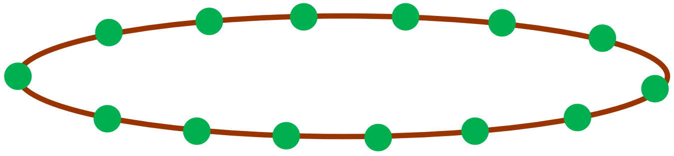
$$D\,|\mathcal{O}_n\rangle = \Delta_n\,|\mathcal{O}_n\rangle$$

$$\langle \mathcal{O}_n(x)\,\mathcal{O}_m(y)\rangle = \frac{\delta_{nm}}{|x-y|^{2\Delta_n}}$$

Heisenberg Hamiltonian

Integrability!

Generic scalar operators :



$$\mathcal{O} = \Psi^{i_1 \ldots i_L} \, \mathrm{tr}\, \Phi_{i_1} \ldots \Phi_{i_L}$$

Bethe wavefunction of <u>integrable</u> SO(6) spin chain

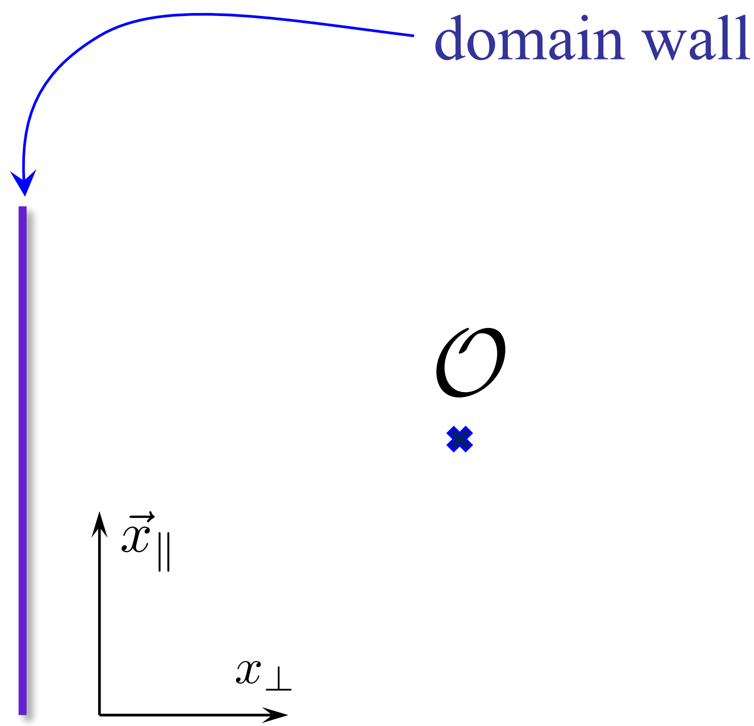
$$H = L + \frac{\lambda}{16\pi^2} \sum_{l=1}^{L} h_{l,l+1} \qquad h_{i_1 i_2}^{j_1 j_2} = 2\delta_{i_1}^{j_1}\delta_{i_2}^{j_2} - 2\delta_{i_1}^{j_2}\delta_{i_2}^{j_1} + \delta_{i_1 i_2}\delta^{j_1 j_2}$$

Minahan, Z.'02

Arbitrary operators ☛ PSU(2,2|4) spin chain

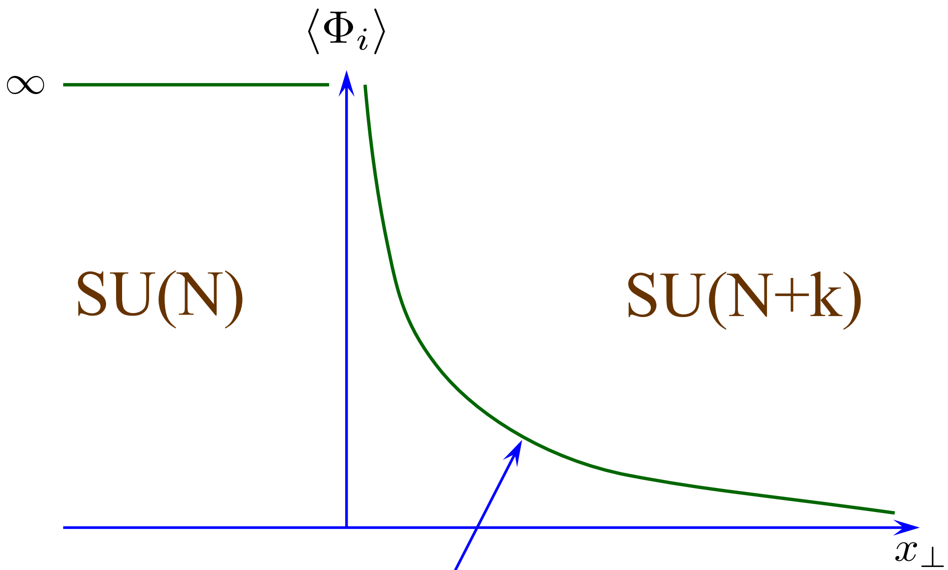
Beisert, Staudacher'03

Defect CFT & 1pt functions



$$\langle \mathcal{O}(x) \rangle = \frac{C}{x_{\perp}^{\Delta}}$$

Domain walls in N=4 SYM



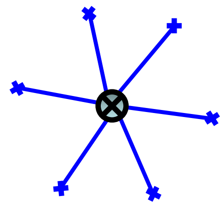
$$\Phi_i^{\mathrm{cl}} = \frac{1}{x_\perp} \begin{array}{cc} \begin{array}{cc} \mathrm{k} & \mathrm{N} \end{array} & \\ \begin{pmatrix} t_i & 0 \\ 0 & 0 \end{pmatrix} & \begin{array}{c} \mathrm{k} \\ \mathrm{N} \end{array} \end{array}$$

Eqs. of motion:

$[t_i, t_j] = i\varepsilon_{ijk} t_k$ ☛ k-dim. rep. of SU(2)

1pt functions

$$\mathcal{O} = \Psi^{i_1 \dots i_L} \operatorname{tr} \Phi_{i_1} \dots \Phi_{i_L} \qquad \langle \mathcal{O} \rangle = \Psi^{i_1 \dots i_L} \operatorname{tr} \Phi^{\mathrm{cl}}_{i_1} \dots \Phi^{\mathrm{cl}}_{i_L} = \frac{C}{x_\perp^L}$$



$$C = \left(\frac{8\pi^2}{\lambda} \right)^{\frac{L}{2}} L^{-\frac{1}{2}} \frac{\langle \mathrm{MPS} | \Psi \rangle}{\langle \Psi | \Psi \rangle^{\frac{1}{2}}}$$

Matrix product state: $\mathrm{MPS}_{i_1 \dots i_L} = \operatorname{tr} t_{i_1} \dots t_{i_L}$

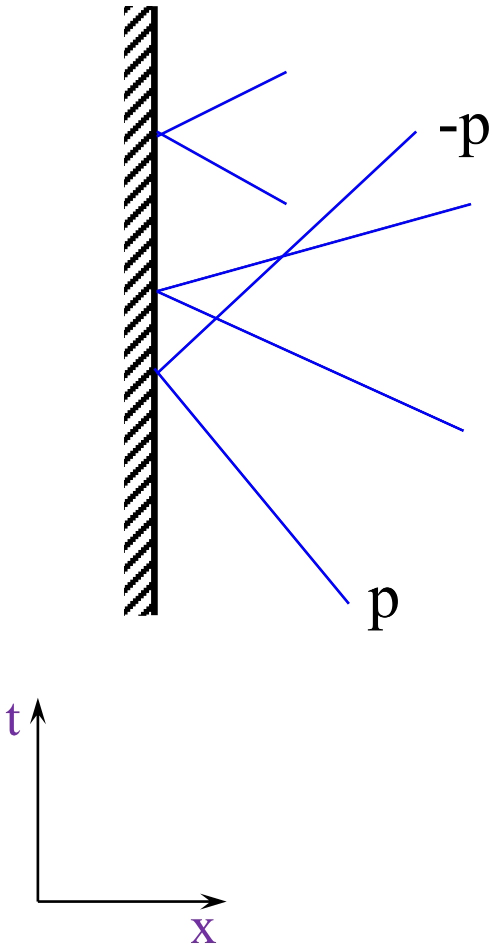
1pt function in BCFT $\Leftrightarrow$ Overlap in spin chain

de Leeuw, Kristjansen, Z.'15

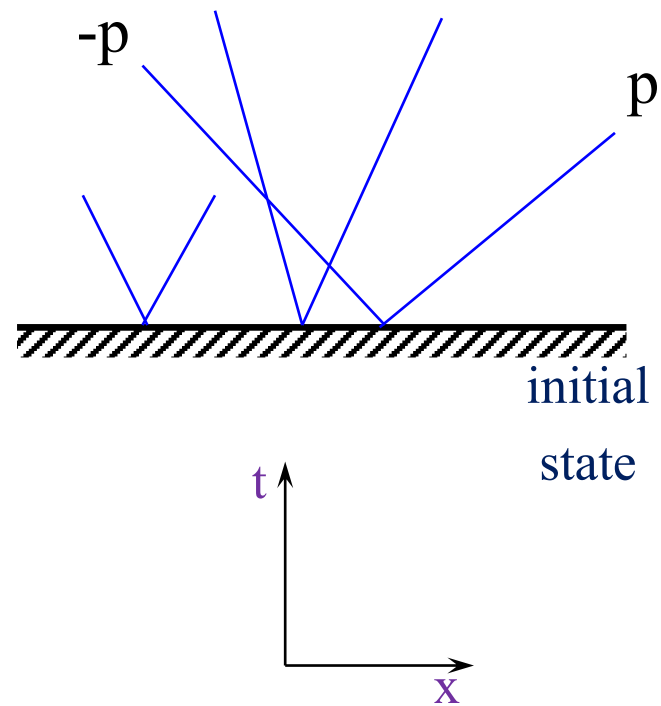
Integrable boundary states

Ghoshal, Zamolodchikov'93
Piroli, Pozsgay, Vernier'17

boundary conditions



Wick rotation

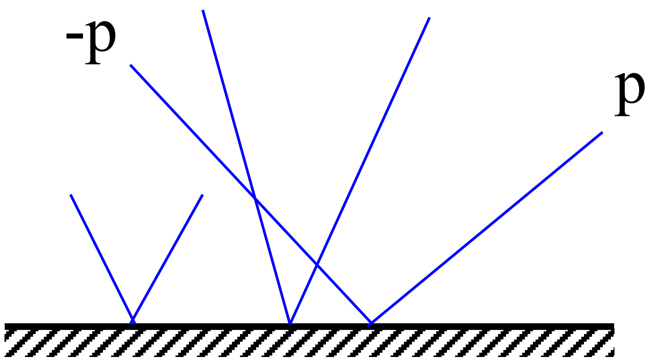


pure reflection (no particle production)

pair entanglement

Integrable boundary states (ctd)

Ghoshal, Zamolodchikov'93
Piroli, Pozsgay, Vernier'17



$$|B\rangle = \sum_N \int d^N p \, \Psi(p_i) \, |p_1, -p_1, \ldots, p_N, -p_N\rangle$$

Def. of integrable boundary state:

$$\boxed{Q_{2n+1} \, |B\rangle = 0}$$

$$Q_1 = P, \ldots$$

Piroli, Pozsgay, Vernier'17

all parity-odd charges

Is MPS integrable?

su(2) sector:

$$Z = \Phi_1 + i\Phi_2 \qquad \Longleftrightarrow \qquad \uparrow$$

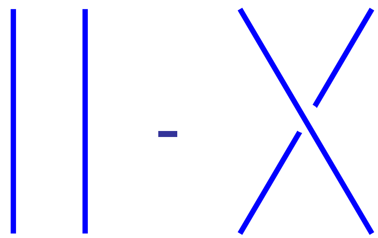
$$W = \Phi_3 + i\Phi_4 \qquad \Longleftrightarrow \qquad \downarrow$$

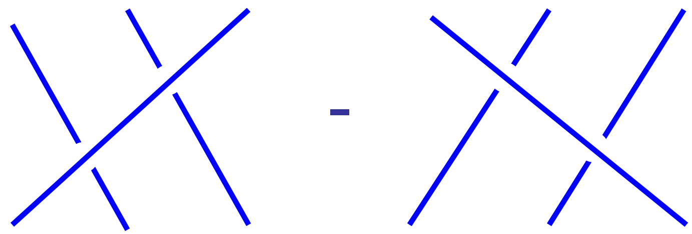
$$\langle MPS| = \frac{1}{2}\,\mathrm{tr}\prod_{l=1}^{L}\Big(\langle\uparrow_l|\otimes\sigma_1 + \langle\downarrow_l|\otimes\sigma_2\Big)$$

(for k=2)

$$\uparrow\ \uparrow\ \downarrow\ \downarrow\ \uparrow\ \uparrow\ \uparrow\ \downarrow\ \uparrow\ \downarrow$$

$$\frac{1}{2}\,\mathrm{tr}\,\sigma_1\sigma_1\sigma_2\sigma_2\sigma_1\sigma_1\sigma_1\sigma_2\sigma_1\sigma_2 \qquad = 1, -1, \text{ or } 0$$

H = 

Q_3 = 

$$Q_3 \cdot \mathrm{MPS}_{\ldots ijk \ldots} = \sigma_j \sigma_k \sigma_i - \sigma_k \sigma_i \sigma_j = 0$$

from $\{\sigma_i, \sigma_j\} = 2\delta_{ij}$

$\Rightarrow$ $\mathrm{MPS}_{k=2}$ is integrable de Leeuw, Kristjansen, Z.'15

Works for any Q_{2n-1}, k, and beyond su(2)

de Leeuw, Kristjansen, Linardopoulos'18

MPS and Néel

de Leeuw, Kristjansen, Z.'15
Piroli, Pozsgay, Vernier'17

Define unitary transformation:

$$
\begin{aligned}
|\uparrow\rangle &\rightarrow |\uparrow\rangle + i\,|\downarrow\rangle \\
|\downarrow\rangle &\rightarrow |\downarrow\rangle - i\,|\uparrow\rangle
\end{aligned}
$$

then:

$$
\begin{aligned}
\langle \mathrm{MPS}| &\rightarrow \frac{1}{2}\,\mathrm{tr}\prod_{l=1}^{L}\Big(\langle\uparrow_l|\otimes\sigma_- + \langle\downarrow_l|\otimes\sigma_+\Big) \\
&= \langle\uparrow\downarrow\uparrow\downarrow\ldots| + \langle\downarrow\uparrow\downarrow\uparrow\ldots| = \langle \mathrm{N\acute{e}el}|
\end{aligned}
$$

$$|\mathrm{MPS}\rangle = W\,|\mathrm{N\acute{e}el}\rangle$$

Piroli, Pozsgay, Vernier'17

global rotation by 90°

Néel state as integrable boundary state

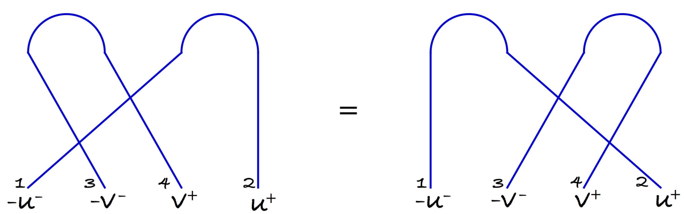
Generalized Néel states:

$$|\text{Néel}_M\rangle = \sum_{\substack{l_1<\ldots<l_M \\ |l_i-l_j|-\text{even}}} \Big|\uparrow\ldots\underset{l_1}{\uparrow\downarrow}\uparrow\ldots\underset{l_2}{\downarrow}\ldots\underset{l_M}{\downarrow}\ldots\uparrow\Big\rangle \qquad |\text{Néel}\rangle = \Big|\text{Néel}_{\frac{L}{2}}\Big\rangle$$

Reflection matrix:

$$\langle K(u)| = \langle\uparrow\downarrow|\,(u^+ + \xi) + \langle\downarrow\uparrow|\,(u^+ - \xi) + \langle\uparrow\uparrow|\,\lambda u^+ \qquad u^\pm = u \pm \frac{i}{2}$$

Cherednik'84
de Vega, Gonzalez-Ruiz'93



(reflection equation)

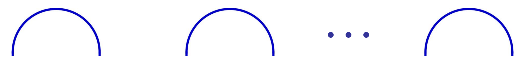
Sklyanin'88

$$\langle K(0)|^{\otimes\frac{L}{2}}\Big|_{\xi=\frac{i}{2}} + \Big|_{\xi=-\frac{i}{2}} = \sum_M i^M \left(\frac{i\lambda}{2}\right)^{\frac{L}{2}-M} \langle\text{Néel}_M|$$

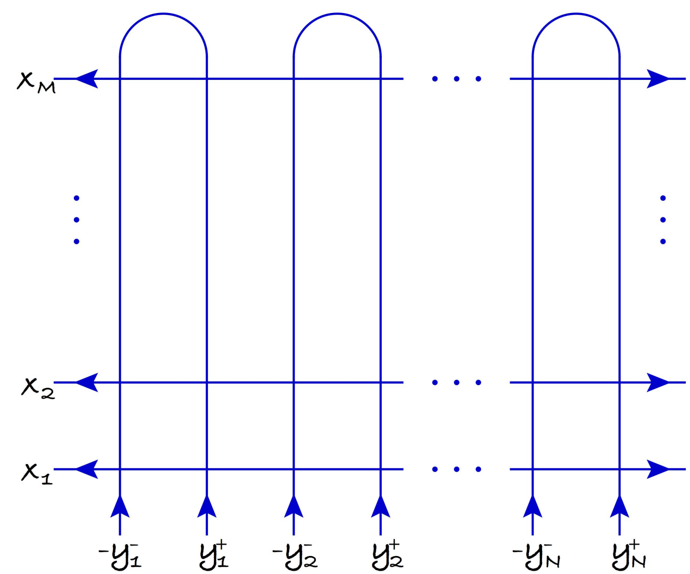
Overlaps

Interested in $\left\langle \mathrm{MPS} \middle| \Psi_{\{u_j\}} \right\rangle \propto \left\langle \mathrm{Néel}_M \middle| \Psi_{\{u_j\}} \right\rangle$

arbitrary Bethe state

$\langle K(u)|^{\otimes \frac{L}{2}} \quad =$ 

$\langle K(y_1)\ldots K(y_N)|\, B(x_1)\ldots B(x_M)\, |0\rangle \quad =$



Freezing trick ⇒ Recurrence relations ⇒ Determinant representation

Korepin'82; Izergin'87

Result:

$\left\langle \mathrm{MPS} \middle| \Psi_{\{u_j\}} \right\rangle$ **is non-zero only if** $\{u\} = \{u_j, -u_j\}_{j=1\ldots\frac{M}{2}}$

and is expressed in terms of Gaudin-like determinant

Tsuchiya'98
Pozsgay'13
Brockmann, DeNardis, Wouters, Caux'14
Foda, Z.'15

Gaudin norm:

$$\left\langle \Psi_{\{u_j\}} \middle| \Psi_{\{u_j\}} \right\rangle \propto \det G \qquad G_{ij} = \frac{\partial \ln \mathrm{Bethe}_i}{\partial u_j}$$

For paired states:

$$G = \begin{array}{cc} \begin{array}{cc} \{u_j\} & \{-u_j\} \end{array} & \\ \begin{pmatrix} A & B \\ B & A \end{pmatrix} & \begin{array}{c} \{u_j\} \\ \{-u_j\} \end{array} \end{array}$$

$$\det \begin{pmatrix} A & B \\ B & A \end{pmatrix} = \det(A + B)\det(A - B)$$

Factorization of Gaudin determinant

$$\det G = \det G^{+} \det G^{-}$$

$$K_{jk}^{\pm} = \frac{2}{1 + (u_j - u_k)^2} \pm \frac{2}{1 + (u_j + u_k)^2}$$

$$G_{jk}^{\pm} = \left(\frac{L}{u_j^2 + \frac{1}{4}} - \sum_n K_{jn}^{+} \right) \delta_{jk} + K_{jk}^{\pm}$$

Overlap:

$$\frac{\left\langle \mathrm{MPS} \middle| \Psi_{\{u_j\}} \right\rangle}{\left\langle \Psi_{\{u_j\}} \middle| \Psi_{\{u_j\}} \right\rangle^{\frac{1}{2}}} = 2^{1-\frac{L}{2}} \left(\frac{Q\left(\frac{i}{2}\right)}{Q(0)} \, \frac{\det G^{+}}{\det G^{-}} \right)^{\frac{1}{2}}$$

$$Q(u) = \prod_{j=1}^{M} (u - u_j)$$

Brockmann, DeNardis, Wouters, Caux'14

$$\langle \mathcal{O}(x) \rangle = \frac{1}{x_{\perp}^{L}} \left(\frac{8\pi^2}{\lambda} \right)^{\frac{L}{2}} L^{-\frac{1}{2}} \, \frac{\langle \mathrm{MPS} | \Psi \rangle}{\langle \Psi | \Psi \rangle^{\frac{1}{2}}}$$

Higher representations

$$\mathrm{MPS}_{i_1\ldots i_L} = \mathrm{tr}\, t_{i_1} \ldots t_{i_L}$$

k-dim. rep. of su(2)

$$\left\langle \mathrm{MPS}_k \middle| \Psi_{\{u_j\}} \right\rangle = \left\langle \mathrm{MPS}_2 \middle| \Psi_{\{u_j\}} \right\rangle \sum_{j=-\frac{k-1}{2}}^{\frac{k-1}{2}} j^L \, \frac{Q\left(0\right) Q\left(\frac{ik}{2}\right)}{Q\left(i\left(j+\frac{1}{2}\right)\right) Q\left(i\left(j-\frac{1}{2}\right)\right)}$$

Buhl-Mortensen, de Leeuw, Kristjansen, Z.'15

related to transfer matrix eigenvalue

Twisted spin chain

Widén'18

$$R_{al}(u) \to V_a^{-1} V_l^{-1} R_{al}(u) V_a V_l$$

V – arbitary 2x2 matrix. Usually: $V = \begin{pmatrix} e^{\frac{i\varphi}{2}} & \\ & e^{-\frac{i\varphi}{2}} \end{pmatrix}$

Only invariant reflection matrix can be twisted:

$$\langle K_{ab}(u)| V_a V_b = \langle K_{ab}(u)|$$

then $\langle K_{ab}(u)| V_a^2 V_b^{-2}$ solves reflection eqn.

$$\langle K(u)| = \langle \uparrow\downarrow| (u^+ + \xi) e^{i\varphi} + \langle \downarrow\uparrow| (u^+ - \xi) e^{-i\varphi} + \langle \uparrow\uparrow| \times 0$$

not allowed by twist

- (apparently) no determinant representation for $\langle \mathrm{MPS} | \Psi_{\{u_j\}} \rangle$

Beyond su(2)

Nested Bethe ansatz:

$$\{u_{k,a}\} = \{u_{j,a}, -u_{j,a}\}_{\substack{a=1\ldots \operatorname{rank} G \\ j=1\ldots \frac{M_a}{2}}}$$

$$G_{ai,bj} = \frac{\partial \mathrm{Bethe}_{ai}}{\partial u_{bj}}$$

$$\det G = \det G^{+} \det G^{-}$$

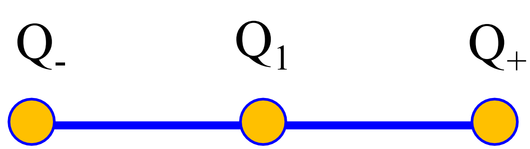
For SO(6) in vector rep:

$$C_k^{SO(6)} = \sqrt{\frac{Q_1(0)Q_1(\frac{i}{2})Q_1(\frac{ik}{2})Q_1(\frac{ik}{2})}{\bar{Q}_+(0)\bar{Q}_+(\frac{i}{2})\bar{Q}_-(0)\bar{Q}_-(\frac{i}{2})}} \cdot \mathbb{T}_{k-1}(0) \cdot \sqrt{\frac{\det G_+}{\det G_-}}$$

(conjectural)

$$\mathbb{T}_n(x) = \sum_{a=-\frac{n}{2}}^{\frac{n}{2}} (x+ia)^L \frac{Q_+(x+ia)Q_-(x+ia)}{Q_1(x+i(a+\frac{1}{2}))Q_1(x+i(a-\frac{1}{2}))}.$$

de Leeuw, Kristjansen, Linardopoulos’17
de Leeuw, Kristjansen, Mori’16



Group-theory interpretation of the prefactor?

For SL(2) spin chain:

$$\mathcal{O} = \sum_{s_1+\ldots+s_L=S} \frac{1}{s_1!\ldots s_L!} \Psi_{s_1\ldots s_L} \, \mathrm{tr}\, D_+^{s_1} Z \ldots D_+^{s_L} Z$$

$$\langle \mathcal{O} \rangle \propto \langle \mathrm{MPS} | \Psi \rangle \equiv \sum_{s_1+\ldots+s_L=S} \Psi_{s_1\ldots s_L}$$

- zero for any highest weight eigenstate

follows from a bCFT theorem
(tensor operators have trivial 1pt functions
in co-dimension 1 defect CFT),
no internal spin-chain proof

Liendo, Rastelli, van Rees'16

Open problems

- MPS for arbitrary symmetry group and general formula for nested BA
- Relation to Q-functions and Quantum Spectral Curve Gromov, Kazakov, Leurent, Volin'14
- Beyond 1-loop in SYM Buhl-Mortensen, de Leeuw, Ipsen, Kristjansen, Wilhelm'16'17
- Relation to string theory and σ-models
- What other boundary states are integrable?