Integrable chiral Potts and tau2 model: Yang-Baxter and Onsager integrability, cyclic representations and parafermions

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Abstract

In this talk we first introduce the integrable chiral Potts model defined by a higher-genus solution of the star-triangle (Yang-Baxter) equation. The R-matrix of this model connects with the asymmetric six-vertex model via a tau2 model as a cyclic representation in a quantum-group construction. We clarify, using some yet unpublished work, why the celebrated construction of Bazhanov and Stroganov fails for even roots-of-unity,† and how to go around it. After that we discuss some aspects of the Onsager algebra and parafermions for related quantum chains.

† Why Bazhanov and Stroganov, Jimbo, de Concini and Kac, Grosjean, Maillet and Niccoli, etc., only treat odd $N$ and how to resolve the problem for even $N$. 
Part 1: Remarks on sl($m,n$) vertex model
R-Matrix and Yang–Baxter Equation

Boltzmann Weights and corresponding Yang–Baxter Equation

\[ \omega_{b_1 b_2}(p, q) \]

\[ (= R\text{-matrix}) \]

with rapidities \( p, q, r \). Edge states \( a_2, b_2, c_2 \) are summed over.
Nonzero $sl(m,n)$ weights in fundamental representation

\[
\omega^{aa}_{aa}(p, q) = \mathcal{N} \sinh \left( \eta + \varepsilon_a (p_0 - q_0) \right) \frac{p + a q - a}{q + a p - a},
\]
\[(a = 1, \cdots, N \equiv m + n);\]

\[
\omega^{ab}_{ba}(p, q) = \mathcal{N} G_{ab} \sinh (p_0 - q_0) \frac{p + a q - b}{q + b p - a},
\]
\[(a \neq b, \ a, b = 1, \cdots, N);\]

\[
\omega^{ba}_{ab}(p, q) = \mathcal{N} e^{(p_0 - q_0) \text{sign}(a - b)} \sinh (\eta) \frac{p + b q - a}{q + b p - a},
\]
\[(a \neq b, \ a, b = 1, \cdots, N).\]

$(2N+1)$-component rapidities: $p = (p_{-N}, \cdots, p_{+N})$, $q = (q_{-N}, \cdots, q_{+N})$;

$\varepsilon_a = +1 \ (a = 1, \cdots m)$, $\varepsilon_a = -1 \ (a = m + 1, \cdots m + n)$, $G_{ab} G_{ba} = 1.$
Changing the additive rapidities \( p_0 \) and \( q_0 \) to multiplicative rapidities \( x \) and \( y \),

\[
q \equiv e^{2\eta}, \quad x = e^{2q_0}, \quad y = e^{2p_0}, \quad \mathcal{N} \frac{q^{1/2}}{2} \left( \frac{y}{x} \right)^{1/2} \equiv 1, \quad p_{\pm a} = q_{\pm a} \equiv 1, \quad (a \neq 0),
\]

we get

\[
\omega^{aa}_{aa}(p, q) = \begin{cases} 1 - q^{-1} \frac{x}{y}, & \text{if } \varepsilon_a = +1, \text{ for } m \text{ different } a\text{-values}, \\ \frac{x}{y} - q^{-1}, & \text{if } \varepsilon_a = -1, \text{ for } n \text{ different } a\text{-values}, \end{cases}
\]

\[
\omega^{ab}_{ba}(p, q) = G_{ab} q^{-1/2} \left( 1 - \frac{x}{y} \right), \quad \implies \begin{cases} 1 - \frac{x}{y}, & \text{if } a > b, \\ q^{-1} \left( 1 - \frac{x}{y} \right), & \text{if } a < b, \end{cases}
\]

\[
\omega^{ba}_{ba}(p, q) = \begin{cases} (1 - q^{-1}) \frac{x}{y}, & \text{if } a > b, \\ 1 - q^{-1}, & \text{if } a < b. \end{cases}
\]

If \( \eta = n\pi i/N \), then \( q \equiv e^{2\eta} = e^{2n\pi i/N} \), the root-of-unity case, one may try to find cyclic representations of quantum groups. The standard choice \( G_{ab} \equiv 1 \) leads to complications that can be resolved choosing \( G_{ab} = q^{\pm \text{sign}(a-b)/2}, \ (G_{ab}G_{ba} = 1) \), instead. Then any \( \omega^{cd}_{ab}(p, q) \) is a linear combination of \( 1, q^{-1}, \frac{x}{y}, q^{-1} \frac{x}{y} \) only!
Part 2: Integrable chiral Potts model
Integrable chiral Potts model Boltzmann weights

\[ p = (a_p, b_p, c_p, d_p), \]
\[ q = (a_q, b_q, c_q, d_q). \]

Boltzmann weights:

\[ \frac{W_{pq}(n)}{W_{pq}(0)} = \prod_{j=1}^{n} \frac{d_p b_q - a_p c_q \omega^j}{b_p d_q - c_p a_q \omega^j}, \]

\[ \frac{\overline{W}_{pq}(n)}{\overline{W}_{pq}(0)} = \prod_{j=1}^{n} \frac{\omega a_p d_q - d_p a_q \omega^j}{c_p b_q - b_p c_q \omega^j}. \]

Chiral Potts curve:

\[ a_p^N + k'b_p^N = k d_p^N, \]
\[ k'a_p^N + b_p^N = k c_p^N, \]
\[ k^2 + k'^2 = 1, \quad \omega = e^{2\pi i / N}. \]
Checkerboard Yang–Baxter Equation vs Star-Triangle Equation
The Diamond and the Star of Four Boltzmann Weights

The shading can now be forgotten.

Bazhanov and Stroganov used this map to relate chiral Potts with the six-vertex model for $N = \text{odd}$.  

\[ J. \text{ Stat. Phys.} \ 59, \ 799–817 \ (1990). \]

Baxter, Bazhanov and Perk used this instead to relate chiral Potts with the six-vertex model for all $N$.  

The $\tau_2$ model and six-vertex model differ from Bazhanov–Stroganov’s.  

\[ \text{Int. J. Mod. Phys. B} \ 4, \ 803–870 \ (1990). \]
The Succession of Four Yang–Baxter Equations

Single rapidity line: spin-$\frac{1}{2}$ representation of $U_q(\hat{\mathfrak{sl}}(2, \mathbb{C}))$, quantum affine SL(2).

Double rapidity line: Two chiral Potts rapidities $(p, p')$ represent a minimal cyclic representation of $U_q(\hat{\mathfrak{sl}}(2, \mathbb{C}))$, requiring $q$ to be a root-of-unity, say $q = \omega$. 
The three kinds of R-matrices of Boltzmann Weights to be Used

Here all $\sigma_i = 0, 1$, corresponding to the spin-$\frac{1}{2}$ representation.
All $n_i = 0, \cdots, N-1$, i.e. $n_i \in \mathbb{Z}_N$, corresponding to the cyclic representation.

The chiral-Potts star shown on the left is also an IRF model.
In this case: $n_1 = a-b$, $n_2 = d-c$, $n_3 = a-d$, $n_4 = b-c$, (mod $N$), using the old Wu–Kadanoff–Wegner mapping.
Part 3: The odd-even $N$ problem in chiral Potts
In the symmetric six-vertex model one has $a' = a$, $b' = b$, $c' = c$. This is not the best start: Korepanov found a $\tau_2$ model, but no chiral Potts. Different gauge choices lead to different $\tau_2$ models that have been connected with chiral Potts.
The weights of the symmetric six-vertex model can be parametrized as

\[ a = N \sin(\eta + (v - u)), \quad b = N \sin(v - u), \quad c = N \sin(\eta), \]

with additive rapidities \(u\) and \(v\). There is also a multiplicative parametrization:

\[ q \equiv e^{2i\eta}, \quad x = e^{2iu}, \quad y = e^{2iv}, \quad C = N \frac{q^{1/2}}{2i} \left( \frac{y}{x} \right)^{1/2}, \]

so that

\[ a = C \left( 1 - q^{-1} \frac{x}{y} \right), \quad b = C q^{-1/2} \left( 1 - \frac{x}{y} \right), \quad c = C \left( 1 - q^{-1} \right) \left( \frac{x}{y} \right)^{1/2}. \]

If \( \eta = n\pi/N \), then \( q \equiv e^{2i\eta} = e^{2n\pi i/N} \), the root-of-unity case, leading to cyclic representations of quantum groups.

However, the symmetric gauge is not a good start for the fundamental representation of sl(2) quantum: The square root \( \sqrt{x/y} \) makes things ugly and it is commonly eliminated by a gauge transformation. Up to normalization \( C \):
\[ R_{\text{sym}}(x, y) = \begin{pmatrix}
1 - \frac{x}{y}q^{-1} & 0 & 0 & 0 \\
0 & (1 - \frac{x}{y})q^{-1/2} & (\frac{x}{y})^{1/2}(1 - q^{-1}) & 0 \\
0 & (\frac{x}{y})^{1/2}(1 - q^{-1}) & (1 - \frac{x}{y})q^{-1/2} & 0 \\
0 & 0 & 0 & 1 - \frac{x}{y}q^{-1}
\end{pmatrix} \]

The \((\frac{x}{y})^{1/2}\) and \(q^{-1/2}\) cause complications especially for \(N\) even.

\[ R_{\text{B&S}}(x, y) = \begin{pmatrix}
1 - \frac{x}{y}q^{-1} & 0 & 0 & 0 \\
0 & (1 - \frac{x}{y})q^{-1/2} & \frac{x}{y}(1 - q^{-1}) & 0 \\
0 & 1 - q^{-1} & (1 - \frac{x}{y})q^{-1/2} & 0 \\
0 & 0 & 0 & 1 - \frac{x}{y}q^{-1}
\end{pmatrix} \]

The \(q^{-1/2}\) causes complications for \(N\) even, as \((q^{-1/2})^N = -1 \neq 1\).

\[ R_{\text{BBP}}(x, y) = \begin{pmatrix}
1 - \frac{x}{y}q^{-1} & 0 & 0 & 0 \\
0 & 1 - \frac{x}{y} & \frac{x}{y}(1 - q^{-1}) & 0 \\
0 & 1 - q^{-1} & (1 - \frac{x}{y})q^{-1} & 0 \\
0 & 0 & 0 & 1 - \frac{x}{y}q^{-1}
\end{pmatrix} \]

Only 1, \(\frac{x}{y}\), \(q^{-1}\), and \(\frac{x}{y}q^{-1}\) show up: “smallest linear dimension”.
Gauge Changes of Six-Vertex Boltzmann Weights
(sl(2) case only, not sl(m, n))

\[ G = \begin{pmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{pmatrix} \]

A staggered gauge transform (a) with \( \lambda = q^{1/8} \), can be used to connect \( R_{\text{B&S}} \) and \( R_{\text{BBP}} \) in each of two different ways.

A uniform gauge transform (b) with \( \lambda = (x/y)^{1/8} \) connects \( R_{\text{sym}} \) and \( R_{\text{B&S}} \).

In the Baxter–Bazhanov–Perk approach there is no difficulty with even roots of unity. However, gauge transforms to the Bazhanov–Stroganov approach and then also to the Korepanov symmetric gauge, lead to complications: Two distinct \( \tau_2 \) matrices arise in the \( R_{6v} R_{\tau_2} R_{\tau_2} \) Yang–Baxter equation, as proposed before by Korepanov to solve the even root-of-unity problem.
The Three Different $\tau_2$ Versions

During 1986–1987 Korepanov solved the first line using $R_{\text{sym}}$, giving one $R_{\tau_2}$ for $N = \text{odd}$, while for $N = \text{even}$ his solution has two different $R_{\tau_2}$. But he did not address the second line, so that he did not find chiral Potts.


Bazhanov and Stroganov were the first to address the second line starting with $R_{\text{B&S}}$, the typical choice for the intertwiner of two fundamental representations of $U_q(\mathfrak{sl}(2, \mathbb{C}))$. 
However, to explicitly represent $R_{\tau_2}$ for $q = \omega \equiv e^{2\pi i/N}$, Bazhanov and Stroganov introduce

$$q_1 = q^{(N+1)/2}, \quad \text{satisfying} \quad q_1^N = 1, \quad q = q_1^{-2},$$

which can only be done for $N = \text{odd}$: For $N = \text{even}$ and $q = q_1^{\pm 2}$, have $q_1^N = -1$, or such $q_1$ is a $2N$th root of unity, leading to unresolved complications.

There is no such problem with $R_{BBP}$ and its $R_{\tau_2}$. The two approaches of B&S and BBP lead to different $q$-Pochhammer symbols,

$$[a; q_1]_n = \prod_{k=1}^{n} \left( a^{-1} q_1^{k-1} - a q_1^{1-k} \right) \quad \text{versus} \quad (a; q)_n = \prod_{k=1}^{n} (1 - a q^{k-1}),$$

and $q$-integers,

$$[q_1]_n = \frac{q_1^n - q_1^{-n}}{q_1 - q_1^{-1}} \quad \text{versus} \quad (q)_n = \frac{1 - q^n}{1 - q}.$$

The second forms are the usual ones of basic hypergeometrics.
Some $N$-state Generalization of the Pauli Matrices

\[
X \equiv \begin{pmatrix}
0 & 0 & 0 & \ldots & 0 & 1 \\
1 & 0 & 0 & \ldots & 0 & 0 \\
0 & 1 & 0 & \ldots & 0 & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & 0 & \ldots & 0 & 0 \\
0 & 0 & 0 & \ldots & 1 & 0
\end{pmatrix}, \quad
Z \equiv \begin{pmatrix}
1 & 0 & 0 & \ldots & 0 & 0 \\
0 & \omega & 0 & \ldots & 0 & 0 \\
0 & 0 & \omega^2 & \ldots & 0 & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & 0 & \ldots & \omega^{N-2} & 0 \\
0 & 0 & 0 & \ldots & 0 & \omega^{N-1}
\end{pmatrix},
\]

\[
Y \equiv \begin{pmatrix}
0 & \omega^{1-N/2} & 0 & \ldots & 0 & 0 \\
0 & 0 & \omega^{3-N/2} & \ldots & 0 & 0 \\
0 & 0 & 0 & \ldots & 0 & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & 0 & \ldots & 0 & \omega^{N-3/2} \\
\omega^{N-1/2} & 0 & 0 & \ldots & 0 & 0
\end{pmatrix}, \quad \omega = e^{2\pi i/N}.
\]
These matrices—generating a generalized quaternion algebra—are all unitary and

\[ X^N = Y^N = Z^N = 1, \quad Y = \omega^{(N-1)/2} X^{-1} Z, \]

\[ ZX = \omega XZ, \quad YX = \omega XY, \quad YZ = \omega ZY. \]

This is called Weyl algebra, even though it was pioneered by Sylvester in his paper on quaternions, nonions, sedenions, etc.

When \( N = 2, \ \omega = -1, \) so that then \( X = \sigma^x, \quad Y = \sigma^y, \quad Z = \sigma^z. \)

We can assign a copy of these operators to a site in a chain:

\[ Z_j = 1 \otimes 1 \otimes \cdots 1 \otimes Z \otimes 1 \cdots \otimes 1, \quad X_j = 1 \otimes 1 \otimes \cdots 1 \otimes X \otimes 1 \cdots \otimes 1, \]

\[ Y_j = 1 \otimes 1 \otimes \cdots 1 \otimes Y \otimes 1 \cdots \otimes 1, \]

so that operators on different sites commute.

These operators are used to construct the cyclic representations, but:
Summarizing this part: Many authors end up working with “Pochhammers”

\[ a^{-1}Z_j^{-n/2} - aZ_j^{n/2} \text{ and } a^{-1}X_j^{-n/2} - aX_j^{n/2}, \]

starting from the Drinfeld–Jimbo choice of fundamental R-matrix. This leads to trouble, resolved for odd $N$ choosing

\[ Z_j^{1/2} = Z_j^{(N+1)/2} \text{ and } X_j^{1/2} = X_j^{(N+1)/2}. \]

With the more asymmetric R-matrix we only need

\[ 1 - a^2Z_j^n \text{ and } 1 - a^2X_j^n, \]

so that there is no odd-even problem. Also, if one sets up the quantum group starting with this modified R-matrix, one ends up with the usual Pochhammers in classical basic hypergeometric functions.
Part 4: Onsager algebra in quantum chain models
Cluster Ising and XY model hamiltonians, like

\[ \mathcal{H}^{(c)} = -\sum_{j=1}^{N} J_x \sigma^x_j \left( \prod_{k=j+1}^{j+n} \sigma^z_k \right) \sigma^x_{j+n+1} + J_y \sigma^y_j \left( \prod_{k=j+1}^{j+n} \sigma^z_k \right) \sigma^y_{j+n+1} + B \sigma^z_j, \]

should be compared with the Onsager algebra for the 2D Ising model,

\[ A_n = \sum_{j=1}^{N} \sigma^x_j \left( \prod_{k=j+1}^{j+n-1} \sigma^z_k \right) \sigma^x_{j+n}, \]

\[ G_n = \frac{1}{2} i \sum_{j=1}^{N} \left[ \sigma^x_j \left( \prod_{k=j+1}^{j+n-1} \sigma^z_k \right) \sigma^y_{j+n} + \sigma^y_j \left( \prod_{k=j+1}^{j+n-1} \sigma^z_k \right) \sigma^x_{j+n} \right]. \]

As periodicity \( \sigma^\alpha_{j+N} = \sigma^\alpha_j, \alpha = x, y, z \) is assumed, we have

\[ A_0 = -\sum_{j=1}^{N} \sigma^z_j, \quad A_{-n} = \sum_{j=1}^{N} \sigma^y_j \left( \prod_{k=j+1}^{j+n-1} \sigma^z_k \right) \sigma^y_{j+n}. \]

Therefore,

\[ \mathcal{H}^{(c)} = -J_x A_{n+1} - J_y A_{-n-1} + BA_0. \]
Onsager derived the following commutation rules:

$$[A_j, A_k] = 4G_{j-k}, \quad [G_m, A_l] = 2A_{l+m} - 2A_{l-m}, \quad [G_j, G_k] = 0.$$  

From these we also have “Dolan–Grady relations”

$$[A_j, [A_j, [A_j, A_k]]] = 16[A_j, A_k], \quad [A_j, [A_j, G_k]] = 16G_k.$$  

These relations also apply to the superintegrable chiral Potts chain discovered by von Gehlen and Rittenberg. However, Onsager’s lattice periodicity relations

$$A_{n\pm N} = -PA_n = -A_nP, \quad P \equiv \prod_{k=1}^{N} \sigma_k^\pm,$$

$$G_0 = 0, \quad G_{-n} = -G_n, \quad G_{n\pm N} = -PG_n = -G_nP,$$

$$A_{n\pm 2N} = A_n, \quad G_{n\pm 2N} = G_n,$$

only hold for the 2-state chiral Potts (Ising) case.
If we fermionize (following Kaufman, 1949):

\[ \Gamma_{2j-1} = \left( \prod_{k=1}^{j-1} \sigma_k^z \right) \sigma_j^x, \quad \Gamma_{2j} = \left( \prod_{k=1}^{j-1} \sigma_k^z \right) \sigma_j^y, \quad \sigma_j^z = -i\Gamma_{2j-1}\Gamma_{2j}, \]

satisfying

\[ \Gamma_k \Gamma_l + \Gamma_l \Gamma_k = 2\delta_{kl} 1, \]

\[ c_j = \frac{1}{2} (\Gamma_{2j-1} - i\Gamma_{2j}), \quad c_j^\dagger = \frac{1}{2} (\Gamma_{2j-1} + i\Gamma_{2j}), \]

the Hamiltonian becomes

\[ \mathcal{H}^{(c)} = i \sum_{j=1}^{N} \left[ J_x \Gamma_{2j} \Gamma_{2j+2n+1} - J_y \Gamma_{2j-1} \Gamma_{2j+2n+2} + B \Gamma_{2j-1}\Gamma_{2j} \right]. \]

As \( \Gamma_j \) is not periodic mod \( 2N \), but periodic mod \( 4N \),

\[ \Gamma_{j \pm 2N} = P \Gamma_j, \quad \Gamma_{j \pm 4N} = \Gamma_j, \quad P = \prod_{k=1}^{N} (-i\Gamma_{2k-1}\Gamma_{2k}), \]

the Hilbert space breaks up into two sectors, on which \( \mathcal{H}^{(c)} \) acts as either a cyclic or an anticyclic quadratic fermion operator (Kaufman, 1949).
Assuming \( N = (n + 1)N_1 \), we relabel the operators according to

\[
\Gamma_{2k+1}^{(p)} = \Gamma_{2p+2k(n+1)+1}^{(p)}, \quad \Gamma_{2k+2}^{(p)} = \Gamma_{2p+2k(n+1)+2}^{(p)},
\]

with

\[
p \equiv \left\lfloor (n + 1) \left\{ \frac{j - 1}{2(n + 1)} \right\} \right\rfloor = 0, \ldots, n,
\]

\[
k \equiv \left\lfloor \frac{j - 1}{2(n + 1)} \right\rfloor = 0, \ldots, N_1 - 1,
\]

where \( \lfloor x \rfloor \) = floor of \( x \), \( \{x\} \) = fractional part of \( x \), and

\[
\Gamma_{k}^{(p)} \Gamma_{l}^{(q)} + \Gamma_{l}^{(q)} \Gamma_{k}^{(p)} = 2\delta_{pq}\delta_{kl}1.
\]

We find

\[
\mathcal{H}^{(c)} = \sum_{p=0}^{n} \mathcal{H}^{(p)}, \quad \mathcal{H}^{(p)} = i \sum_{k=1}^{N_1} \left[ J_x \Gamma_{2k}^{(p)} \Gamma_{2k+1}^{(p)} - J_y \Gamma_{2k-1}^{(p)} \Gamma_{2k+2}^{(p)} + B \Gamma_{2k-1}^{(p)} \Gamma_{2k}^{(p)} \right].
\]
We can now define
\[
\sigma_{j}^{z(p)} = -i \Gamma_{2j-1}^{(p)} \Gamma_{2j}^{(p)},
\]
\[
\sigma_{j}^{x(p)} = \left( \prod_{k=1}^{j-1} \sigma_{k}^{z(p)} \right) \Gamma_{2j-1}^{(p)}, \quad \sigma_{j}^{y(p)} = \left( \prod_{k=1}^{j-1} \sigma_{k}^{z(p)} \right) \Gamma_{2j}^{(p)},
\]
so that
\[
\mathcal{H}^{(p)} = - \sum_{j=1}^{N_1} \left[ J_x \sigma_{j}^{x(p)} \sigma_{j+1}^{x(p)} + J_y \sigma_{j}^{y(p)} \sigma_{j+1}^{y(p)} + B \sigma_{j}^{z(p)} \right], \quad p = 0, \cdots, n.
\]

Thus \(\mathcal{H}^{(c)}\) is decoupled into \(n+1\) commuting XY chains, with identical couplings \(J_x, J_y\) and field \(B\), thus factorizing \(\exp(\beta \mathcal{H}^{(c)})\).\footnote{For the closed chain one has to deal with the odd and even fermion sectors as usual.} Thus the partition function and the spin correlations factorize (in the thermodynamic limit for the closed chain). Some factors may be zero or one.

If \(B=0\) one has \(2(n + 1)\) transverse-field Ising chain factors.

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\[\footnote{For the closed chain one has to deal with the odd and even fermion sectors as usual.} \]
For more details see arXiv:1710.03384. In this talk there is time only for the simplest case, the $N \to \infty$ equilibrium bulk correlation functions of $\mathcal{H}^{(c)}$,

$$Z^{(c)}(k) = \langle \sigma_j^z \sigma_{j+k}^z \rangle, \quad M_z^{(c)} = \langle \sigma_j^z \rangle,$$

in terms of the corresponding $Z(k)$ and $M_z$ for the standard XY chains $\mathcal{H}^{(p)}$. As

$$\sigma_j^z \sigma_{j+k}^z = \sigma_{k_1}^{z(p_1)} \sigma_{k_2}^{z(p_2)},$$

with $p_1 = p_2$ only if $k$ is a multiple of $n + 1$, we find

$$Z^{(c)}(k(n + 1)) = Z(k), \quad \text{but} \quad Z^{(c)}(m) = M_z^2, \quad \text{if} \ m \neq 0 \ \text{mod} \ n + 1.$$

Now we have only one or two factors remaining, as the other $n$ or $n - 1$ factors are trivially equal to one.
Part 5: Remarks on (free) parafermions
Chiral Potts Boltzmann Weights and Discrete Fourier

Here we forget some normalization factors $1/N$ or $1/\sqrt{N}$ with the discrete Fourier. States of internal vertices are summed over.

$$p = (a_p, b_p, c_p, d_p),$$
$$q = (a_q, b_q, c_q, d_q).$$

Boltzmann weights:

$$\frac{W_{pq}(n)}{W_{pq}(0)} = \prod_{j=1}^{n} \frac{d_p b_q - a_p c_q \omega^j}{b_p d_q - c_p a_q \omega^j},$$
$$\frac{\overline{W}_{pq}(n)}{\overline{W}_{pq}(0)} = \prod_{j=1}^{n} \frac{\omega a_p d_q - d_p a_q \omega^j}{c_p b_q - b_p c_q \omega^j}.$$ 

Chiral Potts curve:

$$a_p^N + k' b_p^N = k d_p^N,$$
$$k' a_p^N + b_p^N = k c_p^N,$$
$$k^2 + k'^2 = 1, \quad \omega = e^{2\pi i/N}.$$
\[
U_{pp'qq'}(a, b, c, d) = \sum_{n=0}^{N-1} V_{pqq'}(a, d; n)V_{p'q'q}(-c, -b; n),
\]
\[
V_{pqq'}(a, d; n) = \sum_{e=0}^{N-1} \omega^{ne} W_{pq}(a - e) W_{pq'}(e - d),
\]
\[
V_{p'q'q}(-c, -b; n) = \sum_{e'=0}^{N-1} \omega^{-ne'} \overline{W}_{p'q}(b - e') W_{p'q'}(e' - c).
\]
Note that $V_{pqq'}(a, d; n)$ appeared before in the Fourier-transformed star-triangle equation, (both in the discovery and in the proof of the chiral Potts solution),\(^\dagger\)

\[
W_{qq'}(a-d)V_{pqq'}(a, d; n) = R_{pqq'}^{-1}V_{pq}q(a, d; n)\widehat{W}_{qq'}(n), \quad \widehat{W}_{qq'}(n) \equiv \sum_{k=0}^{N-1} \omega^{nk}\widehat{W}_{qq'}(k).
\]

If $q = (a_q, b_q, c_q, d_q)$, $q' = (b_q, \omega^2 a_q, d_q, c_q)$, (both on the chiral Potts curve !), then it is easily checked that

\[
W_{qq'}(n) = \widehat{W}_{qq'}(n) = 0, \text{ if } n \neq 0, 1 \text{ mod } N,
\]

so that one has the triangularity conditions\(^\ddagger\)

\[
\begin{align*}
V_{pqq'}(a, d; n) &= 0, & \text{if } a - d = 0 \text{ or } 1, \text{ but } n \neq 0, 1; \\
V_{pq}q(a, d; n) &= 0, & \text{if } n = 0 \text{ or } 1, \text{ but } a - d \neq 0, 1; \\
U_{pp'qq'}(a, b, c, d) &= 0, & \text{if } a - d = 0 \text{ or } 1, \text{ but } b - c \neq 0, 1.
\end{align*}
\]


\(^\ddagger\) R.J. Baxter, V.V. Bazhanov, J.H.H. Perk, Int. J. Mod. Phys. B 4, 803–870 (1990), eq. (2.26). (The chiral Potts curve makes it nontrivial. Also, one needs to use the explicit form of $R_{pqq'}$.)

31
If one assumes periodic boundary conditions in the horizontal direction, then the transfer matrix becomes block diagonal: In the first block each spin is 0 or 1 higher than the one above it, giving the \( \tau_2 \) model, while in the second block it is \( 2, \ldots, N - 1 \) mod \( N \) higher, resulting in a \( \tau_{N-2} \) model.

For the first block, one has the “IRF” transfer matrix

\[
\tau_2(t)_{\sigma, \sigma'} = \prod_{j=0}^{L} W_j(\sigma_j, \sigma_{j+1}, \sigma'_{j+1}, \sigma'_j)
\]

with \( (\sigma_{L+1} \equiv \sigma_0, \quad \sigma'_{L+1} \equiv \sigma'_0) \), and where leaving out some common factors of the weights at site \( j \), and with the \((q, q')\) collapsed to a single variable \( t \), Baxter found

\[
W_j(\sigma_j, \sigma_{j+1}, \sigma_{j+1}, \sigma_j) = b_{2j-1}b_{2j} - \omega^{\sigma_j - \sigma_{j+1} + 1}tc_{2j-1}c_{2j},
\]

\[
W_j(\sigma_j, \sigma_{j+1}, \sigma_{j+1}, \sigma_j - 1) = -\omega td_{2j-1}b_{2j} + \omega^{\sigma_j - \sigma_{j+1} + 1}ta_{2j-1}c_{2j},
\]

\[
W_j(\sigma_j, \sigma_{j+1}, \sigma_{j+1} - 1, \sigma_j) = b_{2j-1}d_{2j} - \omega^{\sigma_j - \sigma_{j+1} + 1}c_{2j-1}a_{2j},
\]

\[
W_j(\sigma_j, \sigma_{j+1}, \sigma_{j+1} - 1, \sigma_j - 1) = -\omega td_{2j-1}d_{2j} + \omega^{\sigma_j - \sigma_{j+1} + 1}a_{2j-1}a_{2j}.
\]

It is easily checked that these \( \tau_2(t) \) commute, even if the \( p_j = (a_j, b_j, c_j, d_j) \) do not lie on the chiral Potts curve. (But connecting with chiral Potts it is needed.)
Periodic and Open Transfer Matrices

Repeat this unit \(L+1\) times, with \(j = 0, \cdots, L\), to make the transfer matrix with periodic boundary conditions and column-dependent rapidities \(p_j\).

To get the \(\tau_2\) open boundary condition case, Baxter made a special choice for \(p_{2L}\) and \(p_{-1} \equiv p_{2L+1}\), which affects \(W_L\) and \(W_0\).

Look at these two weights more carefully:

\[
W_j(\sigma_j, \sigma_{j+1}, \sigma_{j+1}, \sigma_j) = b_{2j-1} b_{2j} - \omega^{\sigma_j - \sigma_{j+1} + 1} t c_{2j-1} c_{2j},
\]

\[
W_j(\sigma_j, \sigma_{j+1}, \sigma_{j+1}, \sigma_j - 1) = -\omega t d_{2j-1} b_{2j} + \omega^{\sigma_j - \sigma_{j+1} + 1} t a_{2j-1} c_{2j},
\]

\[
W_j(\sigma_j, \sigma_{j+1}, \sigma_{j+1} - 1, \sigma_j) = b_{2j-1} d_{2j} - \omega^{\sigma_j - \sigma_{j+1} + 1} c_{2j-1} a_{2j},
\]

\[
W_j(\sigma_j, \sigma_{j+1}, \sigma_{j+1} - 1, \sigma_j - 1) = -\omega t d_{2j-1} d_{2j} + \omega^{\sigma_j - \sigma_{j+1} + 1} a_{2j-1} a_{2j}.
\]

We see that a lot disappears if we set \(p_{2L} = (0, b_{2L}, 0, 0)\) and \(p_{-1} = (0, b_{-1}, 0, 0)\).
More precisely, setting $a_{-1} = d_{-1} = c_{-1} = c_{2L} = a_{2L} = d_{2L} = 0$, one finds

$$W_0(\sigma_0, \sigma_1, \sigma_1, \sigma_0) = b_0,$$
$$W_0(\sigma_0, \sigma_1, \sigma_1 - 1, \sigma_0) = d_0,$$
$$W_L(\sigma_L, \sigma_0, \sigma_0, \sigma_L) = b_{2L-1},$$
$$W_L(\sigma_L, \sigma_0, \sigma_0, \sigma_L - 1) = -\omega t d_{2L-1},$$
$$W_0(\sigma_0, \sigma_1, \sigma_1, \sigma_0 - 1) = 0,$$
$$W_0(\sigma_0, \sigma_1, \sigma_1 - 1, \sigma_0 - 1) = 0,$$
$$W_L(\sigma_L, \sigma_0, \sigma_0 - 1, \sigma_L) = 0,$$
$$W_L(\sigma_L, \sigma_0, \sigma_0 - 1, \sigma_L - 1) = 0.$$

This means that $\sigma_0 = \sigma'_0$ and that no weight depends on the value of $\sigma_0$. Also, $\sigma_L$ and $\sigma_1$ are now uncorrelated: Free boundaries with boundary couplings.
From BBP, we have the functional equations

\[
\tau_{j+1}(t) = \tau_j(t)\tau_2(\omega^{j-1}t) - z(\omega^{j-1}t)\mathcal{X}\tau_{j-1},
\]
\[
\tau_{N+1} = z(\omega t)\mathcal{X}\tau_{N-1} + [\alpha(\lambda_q) + \alpha(1/\lambda_q)]\mathbf{1},
\]

with \(\mathcal{X}\) the spin shift operator, \(\mathcal{X}^N = \mathbf{1}\) and \(z(t) \equiv 0\) for the open case.\(^\dagger\) Next, as the weights are linear in \(t\) and \(W_0\) does not depend on \(t\) now, the transfer matrix \(\tau_2(t)\) is a polynomial of degree \(L\),

\[
\tau_2(t) = \sum_{m=0}^{L} (\omega t)^m \tau_{2,m}, \quad \tau_{2,0} = \tau_2(0) = A_0 \mathbf{1}, \quad A_0 \equiv \prod_{\ell=0}^{2L-1} b_\ell.
\]

Therefore, from the functional equations,

\[
\tau_2(t)\tau_2(\omega t) \cdots \tau_2(\omega^{N-1}t) = A_0^N \mathbf{1} \prod_{j=1}^{L} (1 - r_j^N t^N),
\]

which is a polynomial in \(t^N\), as this is invariant under \(t \rightarrow \omega t\). Also, \(\mathbf{1}\) is the unit matrix of dimension \(N^{L+1}\), or \(N^L\), as \(\sigma_0\) has become irrelevant.

The zeros of this polynomial are $\omega^k r_j$, $(k = 0, \cdots, N-1; \ j = 1, \cdots, L)$, satisfying
\[ s_0 r_j^{NL} + s_1 r_j^{N(L-1)} + s_2 r_j^{N(L-2)} + \cdots + s_L = 0. \]
Thus Baxter obtained all the eigenvalues of the $\tau_2(t)$ matrix, namely
\[ \tau_2(t) = A_0 \prod_{j=1}^{L} (1 - r_j \omega^{1+p_j t}), \quad 0 \leq p_j \leq N - 1, \quad 1 \leq j \leq L. \]
Assuming all $b_\ell \neq 0$, we can expand
\[ t \frac{d}{dt} \ln \tau_2(t) = \sum_{m=1}^{\infty} (\omega t)^m \mathcal{H}^{(m)}, \quad \tau_2(t) = A_0 \exp \left( \sum_{m=1}^{\infty} \frac{(\omega t)^m}{m} \mathcal{H}^{(m)} \right), \]
giving the higher Hamiltonians $\mathcal{H}^{(m)}$ and $\mathcal{H} = \mathcal{H}^{(1)} = A_0^{-1} \tau_{2,1}$ Consequently, we also have their $NL$ eigenvalues,
\[ -\mathcal{H}^{(m)}|p_1, \cdots, p_L\rangle = \sum_{j=1}^{L} (r_j \omega^{p_j})^m |p_1, \cdots, p_L\rangle, \]
with $|p_1, \cdots, p_L\rangle$ denoting the corresponding eigenvector.
Hamiltonian in Generalized Pauli Matrices

\[
\mathcal{H} = - \sum_{j=1}^{L} \sum_{k=j}^{L} \omega^{k-j+(N-1)/2} \frac{d_{2j-2}}{b_{2j-2}} \left( \prod_{\ell=2j-1}^{2k-2} \frac{a_\ell}{b_\ell} \right) \frac{d_{2k-1}}{b_{2k-1}} Z_j \left( \prod_{\ell=j}^{k-1} X_\ell \right) Y_{k-1}^{-1} \\
+ \sum_{j=1}^{L-1} \sum_{k=j+1}^{L} \omega^{k-j-1} \frac{c_{2j-1}}{b_{2j-1}} \left( \prod_{\ell=2j}^{2k-2} \frac{a_\ell}{b_\ell} \right) \frac{d_{2k-1}}{b_{2k-1}} Y_j \left( \prod_{\ell=j}^{k-1} X_\ell \right) Y_{k-1}^{-1} \\
- \sum_{j=1}^{L-1} \sum_{k=j}^{L-1} \omega^{k-j-(N+1)/2} \frac{c_{2j-1}}{b_{2j-1}} \left( \prod_{\ell=2j}^{2k-1} \frac{a_\ell}{b_\ell} \right) \frac{c_{2k}}{b_{2k}} Y_j \left( \prod_{\ell=j}^{k} X_\ell \right) Z_{k+1}^{-1} \\
+ \sum_{j=1}^{L-1} \sum_{k=j}^{L-1} \omega^{k-j} \frac{d_{2j-2}}{b_{2j-2}} \left( \prod_{\ell=2j-1}^{2k-1} \frac{a_\ell}{b_\ell} \right) \frac{c_{2k}}{b_{2k}} Z_j \left( \prod_{\ell=j}^{k} X_\ell \right) Z_{k+1}^{-1}.
\]

For the special case \( N = 2 \), after rotating \( Z_\ell \rightarrow \sigma_\ell^x \), \( X_\ell \rightarrow -\sigma_\ell^z \) and \( Y_\ell \rightarrow \sigma_\ell^y \), we recognize a generalized XY-model, like the spin-chain Hamiltonian that Suzuki introduced to commute with the transfer matrix of the dimer model.
Hamiltonian in Parafermions

We define the basic parafermions as (generalized Jordan–Wigner transform)

\[ \psi_{2j-2} = \left( \prod_{\ell=1}^{j-1} X_\ell \right) Z_j^{-1}, \quad \psi_{2j-1} = \left( \prod_{\ell=1}^{j-1} X_\ell \right) Y_j^{-1}, \quad \psi_0 = \Gamma_0 = Z_1^{-1}, \]

for \( 1 \leq j \leq L \). From the commutation relations of \( X, Y \) and \( Z \), it follows that

\[ \psi_j \psi_k = \omega^{-1} \psi_k \psi_j \quad \text{for } j < k, \quad \psi_j^N = 1. \]

The Hamiltonian may be expressed in terms of these parafermions as†

\[
\mathcal{H} = - \sum_{j=1}^{L} \sum_{m=j}^{L} \omega^{m-j+(N-1)/2} \left( \prod_{\ell=2j-1}^{2m-2} \frac{a_\ell}{b_\ell} \right) \frac{d_{2j-2}d_{2m-1}}{b_{2j-2}b_{2m-1}} \psi_{2j-2}^{-1} \psi_{2m-1} \\
- \sum_{j=1}^{L-1} \sum_{m=j}^{L-1} \omega^{m-j} \left[ \omega^{-(N+1)/2} \left( \prod_{\ell=2j}^{2m-1} \frac{a_\ell}{b_\ell} \right) \frac{c_{2j-1}c_{2m}}{b_{2j-1}b_{2m}} \psi_{2j-1}^{-1} \psi_{2m} \\
- \left( \prod_{\ell=2j-1}^{2m-1} \frac{a_\ell}{b_\ell} \right) \frac{d_{2j-2}c_{2m}}{b_{2j-2}b_{2m}} \psi_{2j-2}^{-1} \psi_{2m} - \left( \prod_{\ell=2j}^{2m} \frac{a_\ell}{b_\ell} \right) \frac{c_{2j-1}d_{2m+1}}{b_{2j-1}b_{2m+1}} \psi_{2j-1}^{-1} \psi_{2m+1} \right].
\]

† The special Baxter case studied by Fendley follows setting all \( a_\ell \)'s zero.
The Fendley–Baxter Suggestion

Define recursively

\[ \Gamma_0 = Z_1^{-1}, \quad \Gamma_{j+1} = (\omega^{-1} - 1)^{-1}(\mathcal{H}\Gamma_j - \Gamma_j\mathcal{H}), \quad (j \geq 0), \]

Using \( \Gamma_0 = \psi_0 \), it is straightforward to show that

\[ \Gamma_1 = \frac{d_0}{b_0} \left[ \sum_{m=1}^{L} \omega^{m+(N-1)/2} \left( \prod_{\ell=1}^{2m-2} \frac{a_\ell}{b_\ell} \right) \frac{d_{2m-1}}{b_{2m-1}} \psi_{2m-1} - \sum_{m=1}^{L-1} \omega^m \left( \prod_{\ell=1}^{2m-1} \frac{a_\ell}{b_\ell} \right) \frac{c_{2m}}{b_{2m}} \psi_{2m} \right], \]

which is rather complicated. Nevertheless, we can easily show

\[ \Gamma_0\Gamma_1 = \omega^{-1}\Gamma_1\Gamma_0. \]

Based on numerical evidence, Baxter found that the infinite sequence of the \( \Gamma_j \) truncates, as he conjectured that the \( \Gamma \) matrices satisfy the equation

\[ s_0 \Gamma_{NL+j} + s_1 \Gamma_{N(L-1)+j} + \cdots + s_L \Gamma_j = 0, \quad \text{for } j = 0, \]

with the same coefficients \( s_\ell \) as defined earlier in

\[ \tau_2(t)\tau_2(\omega t) \cdots \tau_2(\omega^{N-1}t) = (s_0 t^{NL} + s_1 t^{N(L-1)} + s_2 t^{N(L-2)} + \cdots + s_L) 1. \]

If the conjecture holds for \( j = 0 \), then by recurrence also for all \( j > 0 \). It has been proved using using the partially Fourier transformed vertex model weights \( S^{(pf)} \). (See section 4 of H. Au-Yang and J.H.H. Perk, J. Phys. A 47 (2014) 315002.)
The partial Fourier gauge transform (pf) cancels out in the row-to-row transfer matrix, giving the diamond:
As before we set \( q = (a_q, b_q, c_q, d_q) \), \( q' = (b_q, \omega^2 a_q, d_q, c_q) \), so that \( S^{(pf)} \) becomes \( \mathcal{L}_{\tau_2} \), a \( \tau_2 \) R-matrix with \( \sigma_{1,2} = 0, 1 \) and \( n_{1,2} = 0, 1, \ldots, N - 1 \).

As standard in quantum inverse scattering we construct the monodromy matrix using \( L + 1 \) copies for \( j = 0, \ldots, L \), summing over the states on internal edges:

\[
\mathcal{M}^{0,L}(t), \quad \text{where} \quad \mathcal{M}^{m,n}(t) \equiv \prod_{j=m}^{n} \mathcal{L}_j(t) = \begin{pmatrix} A_{m,n}(t) & B_{m,n}(t) \\ C_{m,n}(t) & D_{m,n}(t) \end{pmatrix}.
\]

After setting \( a_{-1} = d_{-1} = c_{-1} = c_{2L} = a_{2L} = d_{2L} = 0 \) again, \( \mathcal{L}_0 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \), and so that \( \tau_2(t) = A^{1,L}(t) \) for the open boundary case on sites 1, \( \ldots, L \).
The Monodromy Matrices by Recurrence

\[ \mathcal{M}^{m,n}(t) = \mathcal{M}^{m,k}(t)\mathcal{M}^{k+1,n}(t) = \begin{pmatrix} A^{m,n}(t) & B^{m,n}(t) \\ C^{m,n}(t) & D^{m,n}(t) \end{pmatrix}, \]

\[ \mathcal{M}^{j,j}(t) = \mathcal{L}_j(t) = \begin{pmatrix} A^{j,j}(t) & B^{j,j}(t) \\ C^{j,j}(t) & D^{j,j}(t) \end{pmatrix}, \]

\[
\begin{aligned}
\mathcal{L}_j(0, 0) &= A^{j,j}(t) = b_{2j-2}b_{2j-1} - \omega td_{2j-2}d_{2j-1}X_j, \\
\mathcal{L}_j(0, 1) &= B^{j,j}(t) = (-\omega t)Z_j(b_{2j-2}c_{2j-1} - d_{2j-2}a_{2j-1}X_j), \\
\mathcal{L}_j(1, 0) &= C^{j,j}(t) = Z_j^{-1}(c_{2j-2}b_{2j-1} - \omega a_{2j-2}d_{2j-1}X_j), \\
\mathcal{L}_j(1, 1) &= D^{j,j}(t) = \omega a_{2j-2}a_{2j-1}X_j - \omega tc_{2j-2}c_{2j-1},
\end{aligned}
\]

\[ \tau_2(t) = A^{1,L}(t), \quad A^{m,n}(t) = A^{m,k}(t)A^{k+1,n}(t) + B^{m,k}(t)C^{k+1,n}(t). \]

The technical proofs of Baxter’s conjecture just mentioned and the next one use this recurrence and the Yang–Baxter equation for the monodromy matrices \( \mathcal{R}^{6\nu}(t, t') \mathcal{M}^{m,n}(t) \mathcal{M}^{m,n}(t) = \mathcal{M}^{m,n}(t') \mathcal{M}^{m,n}(t) \mathcal{R}^{6\nu}(t, t') \). For the details we refer to our paper,* as it would take too much time to explain it here.

Rewriting Baxter’s First Conjecture

We have just outlined what we needed to show that the recurrence

\[ \Gamma_0 = Z_1^{-1}, \quad \Gamma_{j+1} = (\omega^{-1} - 1)^{-1}(\mathcal{H}\Gamma_j - \Gamma_j\mathcal{H}), \quad (j \geq 0), \]

closes through

\[ s_0\Gamma_{NL+j} + s_1\Gamma_{N(L-1)+j} + \cdots + s_L\Gamma_j = 0, \quad \text{for } j = 0. \]

This then obviously holds for all \( j \) also. We can now rewrite

\[ \Gamma_j\mathcal{H} - \mathcal{H}\Gamma_j = (1 - \omega^{-1})\Gamma_{j+1} = (1 - \omega^{-1}) \sum_{k=0}^{NL-1} h_{jk}\Gamma_k = (1 - \omega^{-1})(\mathbf{H} \cdot \Gamma)_j, \]

where \((j = 0, \cdots, NL - 1)\) and

\[ h_{ij} = \delta_{i,j-1}, \quad (0 \leq i < NL - 1), \]

\[ h_{NL-1,mN} = -s_{L-m}/s_0, \quad (0 \leq m < L), \quad h_{NL-1,j} = 0, \quad (j \neq 0 \mod N). \]
Baxter’s Second Conjecture

Baxter next conjectured:

\[ t \nu_j = \mu_{j-1}, \]

where

\[ \mu_j \equiv \Gamma_j \tau_2(t) - \tau_2(t) \Gamma_j, \quad \nu_j \equiv \omega \Gamma_j \tau_2(t) - \tau_2(t) \Gamma_j. \]

We have proved this with the same tools in the paper just cited. Again the details are too technical to present.

Using \( \Gamma_{j+1} = (H \cdot \Gamma)_j \), we find

\[ \mu_j = \Gamma_j \tau_2(t) - \tau_2(t) \Gamma_j = t \nu_{j+1} = \omega t (H \cdot \Gamma)_j \tau_2(t) - t \tau_2(t) (H \cdot \Gamma)_j, \]

or

\[ \Gamma - \tau_2(t) \Gamma \tau_2(t)^{-1} = \omega t H \cdot \Gamma - t H \cdot \tau_2(t) \Gamma \tau_2(t)^{-1}, \]

or

\[ \tau_2(t) \Gamma \tau_2(t)^{-1} = \frac{1 - \omega t H}{1 - t H} \cdot \Gamma, \]

first written down by Baxter. With this we can prove Baxter’s final conjecture.
Diagonalization of Matrix $\mathbf{H}$ by a Vandermonde

$$
\mathbf{H} = \begin{pmatrix}
0 & 1 & 0 & 0 & 0 & \cdots & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & \cdots & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & \cdots & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & \cdots & 0 & 0 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & 0 & 0 & 0 & \cdots & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & \cdots & 0 & 1 \\
* & * & * & * & * & * & \cdots & 0 & 0
\end{pmatrix}
= \mathbf{P} \cdot \mathbf{H}_d \cdot \mathbf{P}^{-1},
$$

with in the last row $h_{NL-1,mN} = -s_{L-m}/s_0$, \((0 \leq m < L)\), and 0 otherwise. The eigenvalues are given by $\sum s_k \lambda^{N(L-k)} = 0$, i.e. $\lambda_{Nj+i+1} = r_j \omega^i$ seen before, and $\mathbf{P}$ is the Vandermonde matrix with columns $(\lambda_m)^k$, $(k = 0, \cdots, NL-1)$. To deal with the inverse, we used Prony’s 1795 result

$$f_m(z) = \prod_{n=1, n \neq m}^{NL} \frac{z - \lambda_n}{\lambda_m - \lambda_n} = \sum_{k=0}^{NL-1} (P^{-1})_m z^k, \text{ satisfying } f_m(\lambda_n) = \delta_{mn}.$$
Cyclic Raising Operators and Projection Operators

Baxter defined the candidate free parafermion operators

\[ \hat{\Gamma}_i \equiv \sum_{j=0}^{NL-1} P_{ij}^{-1} \Gamma_j, \quad \mathcal{H}\hat{\Gamma}_j - \hat{\Gamma}_j \mathcal{H} = (\omega^{-1} - 1)\lambda_j \hat{\Gamma}_j. \]

Generalizing Fendley, we also introduce the projection operators

\[ \mathcal{P}_{\omega^p,k} = - \sum_{\ell=0}^{L-1} \sum_{q=0}^{N-1} P_{Nk+p,\ell N+q}^{-1} \mathcal{H}^{(\ell N+q)}. \]

Multiplying both sides with the Vandermonde, we find

\[ \mathcal{H}^{(m)} = - \sum_{k=1}^{L} \sum_{p=0}^{N-1} (r_k \omega^p)^m \mathcal{P}_{\omega^p,k}, \]

which all commute, so that

\[ [\mathcal{P}_{\omega^p,k}, \mathcal{P}_{\omega^q,\ell}] = 0. \]
Remember
\[ \mathcal{H}^{(m)} | n_1, n_2, \cdots, n_L \rangle = - \sum_{k=1}^{L} (r_k \omega^{n_k})^m | n_1, n_2, \cdots, n_L \rangle, \]

so that we must have
\[ \mathcal{P}_{\omega^p,k} | n_1, n_2, \cdots, n_L \rangle = \delta_{p,n_k} | n_1, n_2, \cdots, n_L \rangle, \]

from which the projection operator properties follow:
\[ \mathcal{P}^2_{\omega^p,k} = \mathcal{P}_{\omega^p,k}, \quad \mathcal{P}_{\omega^p,k} \mathcal{P}_{\omega^q,k} = \delta_{p,q} \mathcal{P}_{\omega^p,k}, \quad \sum_{p=0}^{N-1} \mathcal{P}_{\omega^p,k} = 1. \]

Also,
\[ \tau_2(t) = A_0 \prod_{k=1}^{L} \prod_{p=0}^{N-1} (1 - r_k \omega^{1+p} \mathcal{P}_{\omega^p,k}) = A_0 \prod_{k=1}^{L} \left( 1 - \omega t \sum_{p=0}^{N-1} r_k \omega^p \mathcal{P}_{\omega^p,k} \right), \]

as this produces the correct eigenvalues seen before.
Proof of Commutation Relation of Cyclic Raising Operators

From

\[ \mathcal{H} \hat{\Gamma}_j - \hat{\Gamma}_j \mathcal{H} = (\omega^{-1} - 1) \lambda_j \hat{\Gamma}_j, \quad \mathcal{H} = \mathcal{H}^{(1)} = - \sum_{k=1}^{L} \sum_{p=0}^{N-1} (r_k \omega^p) \mathcal{P}_{\omega^p, k}, \]

we find

\[ \sum_{k=1}^{L} \sum_{p=0}^{N-1} (r_k \omega^p) [\mathcal{P}_{\omega^p, k} \hat{\Gamma}_{N\ell+q} - \hat{\Gamma}_{N\ell+q} \mathcal{P}_{\omega^p, k}] = r_\ell (\omega^{q-1} - \omega^q) \hat{\Gamma}_{N\ell+q}. \]

This implies the relation,

\[ [\mathcal{P}_{\omega^p, k} \hat{\Gamma}_{N\ell+q} - \hat{\Gamma}_{N\ell+q} \mathcal{P}_{\omega^p, k}] = \delta_{k, \ell} (\delta_{p, q-1} - \delta_{p, q}) \hat{\Gamma}_{N\ell+q}. \]

We used

\[ \tau_2(t) \hat{\Gamma} \tau_2(t)^{-1} = \frac{1 - \omega t \mathcal{H}_d}{1 - t \mathcal{H}_d} \cdot \hat{\Gamma}, \quad (1 - r_\ell \omega^q t) \tau_2(t) \hat{\Gamma}_{N\ell+q} = (1 - r_\ell \omega^{q+1} t) \hat{\Gamma}_{N\ell+q} \tau_2(t), \]

implying that \( \hat{\Gamma}_{N\ell+q} \) only acts on the \( n_\ell \) in \( |n_1, \cdots, n_L\rangle \).
From this we can also conclude

\[ \hat{\Gamma}_{Nk+p} \hat{\Gamma}_{Nk+p'} = 0, \quad \text{if } p' \neq p - 1 \mod N. \]

Finally, we could prove the third conjecture of Baxter,

\[ (r_k \omega^p - r_{k'} \omega^{p'+1}) \hat{\Gamma}_{Nk+p} \hat{\Gamma}_{Nk'+p'} + (r_{k'} \omega^{p'} - r_k \omega^{p+1}) \hat{\Gamma}_{Nk'+p'} \hat{\Gamma}_{Nk+p} = 0, \]

which gives the commutation relation between these operators. We can now create all the basis states by acting on \(|0,0,\cdots,0\rangle\).

The eigenstates of the \(\tau_2\) model are useful—and have been used—as a starting point to study the chiral Potts model, the first model found with rapidities (spectral parameters) on a curve of high genus.

Thank you!
Some References

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sl(2) chiral Potts model from six-vertex, odd $N$ only:


sl(2) chiral Potts model from six-vertex, all $N$:

Korepanov’s tau-2 model from six-vertex:

4. Onsager algebra.


Original 2D Ising model sources:

Discovery of superintegrable chiral Potts chain by Dolan–Grady criterium:

General “cluster XY model” of Suzuki:
5. Free parafermions.


