Frustration and Area law

When the frustration goes odd

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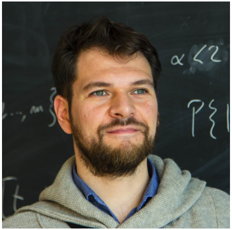

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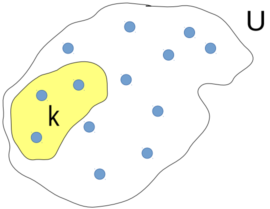
Many Body System and Frustration



Hamiltonian of a many body system

$$H = \sum_k h_k \qquad k \subset U$$

$$h_k \rightarrow \Pi_k \qquad\qquad H \rightarrow \rho_k$$

In the presence of frustration there will be at least one k for which

$$\Pi_k \neq \rho_k$$

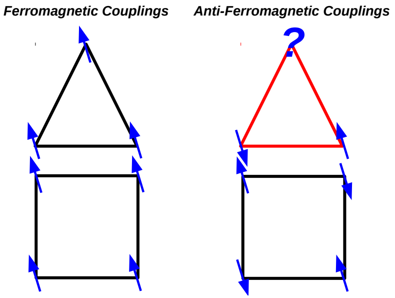
$$F_k = 1 - \mathrm{Tr}(\rho_k \cdot \Pi_k) \geq E_k$$

S. M. Giampaolo *et al.* Phys. Rev. Lett. **107**, 260602 (2011);

U. Marzolino *et al.* Phys. Rev. A **88**, 020301(R) (2013).

Frustration in Classic System

In the classical system, frustration rises from the geometry of the system



Touluse Criterion

If there is a close loop for which $-1^{\mathcal{N}_a} = -1$ the system is frustrated

G. Toulouse, Commun. Phys. **2**, 115 (1977);

J. Vannimenus and G. Toulouse, J. Phys. C **10**, L537 (1977).

Frustration in Quantum System

Monogamy of the entanglement
A spin that share a maximal entangled state with a second spin cannot share entanglement with a third one

Quantum counterpart of classical unfrustrated system are frustrated

It is possible to generalize the Toulouse criterion to quantum world
When Quantum Touluse criterion (QTC) is verified
there is no contribution of the geometry to the frustration

V. Coffman *et al.*, Phys. Rev. A **61**, 052306 (2000);
T. J. Osborne and F. Verstraete, Phys. Rev. Lett. **96**, 220503 (2006);
S. M. Giampaolo *et al.* Phys. Rev. Lett. **107**, 260602 (2011).

Strongly and weakly frustrated system

1. Strongly frustrated – Amount of bond that does not satisfy QTC scales with N
    - Frustration cannot be removed acting on the boundary conditions
    - examples: ANNNI models, Sherrington-Kirkpatrick model
2. Weakly frustrated – Fixed number of bonds that does not satisfy QTC
    - Frustration can be removed acting on boundary condition
    - Difference between odd and even N

Sherrington, D., & Kirkpatrick S. Phys. Rev. Lett. **35** 1792 (1975).

Campostrini, *et al.* Phys. Rev. **E 91**, 042123 (2015).

Weakly frustrated spin chain

$$H = \frac{J}{2}\sum_{l=1}^{N}\left[\left(\frac{1+\gamma}{2}\right)\sigma_l^x\sigma_{l+1}^x + \left(\frac{1-\gamma}{2}\right)\sigma_l^y\sigma_{l+1}^y\right] + \frac{\Delta}{2}\sum_{l=1}^{N}\sigma_l^z\sigma_{l+1}^z - \sum_{l=1}^{N} h\,\sigma_l^z$$

$J = 1$ Anti-ferromagnetic system $\quad$ $J = -1$ Ferromagnetic system

PBC $\sigma_{N+1}^{\alpha} = \sigma_1^{\alpha}$ for $\alpha = x, y, z$

- If $J = -1$ the Ferromagnetic (F) system satisfy the QTC.
- If N is even the Anti-ferromagnetic (AF) system ($J = 1$) satisfy the QTC. It is possible to map the AF system in the F one;
- If N is even the AF system does not satisfy the QTC. It is impossible to map AF system in F system.

Weakly frustrated spin chain

In agreement with this result one expect for odd N different behaviors for F and AF systems.

However there is no trace in Literature of such difference

We will prove that going at the thermodynamic limit moving on frustrated N, depending on the Hamiltonian parameters, we may arrive in a new phase characterized by:

- A gapless low energy spectrum
- Non degenerate ground state
- Absence of order parameter
- Violation of the area law without the presence of a divergence in the Von Neumann entropy

Analytic Case: Weakly frustrated Ising chain

$$(\gamma = 1\,, \Delta = 0) \qquad H = J\sum_{l=1}^{N} \sigma_l{}^x \sigma_{l+1}{}^x - h\sum_{l=1}^{N} \sigma_j{}^z$$

1. Jordan Wigner Transformations (JWT) map spins in fermions defined in the position space

$$c_l = \prod_{k=1}^{l-1} (\sigma_k^z)\, \sigma_l^- \qquad c_l^\dagger = \prod_{k=1}^{l-1} (\sigma_k^z)\, \sigma_l^+,$$

JWT breaks the invariance under spatial translation

$$J_{1,N} \to J\mathcal{M}$$

The $\mathcal{M} = \otimes_{l=1}^{N} (c_l^\dagger c_l - c_l c_l^\dagger)$ assumes different value in the different sector of parities

P. Jordan, & E. Wigner, Z. Phys. **47**, 631 (1928);

E. Lieb *et al.* Ann. Phys. **16** 407 (1961);

Analytic Case: Weakly frustrated Ising chain

2. Fourier Transform maps fermions in position space in fermions in the momentum space

$$b_k = \frac{1}{\sqrt{N}} \sum_{l=1}^{N} c_l e^{-\imath kl}; \qquad b_k^\dagger = \frac{1}{\sqrt{N}} \sum_{l=1}^{N} c_l^\dagger e^{\imath kl}$$

The momenta used depends on the parity. Chosen to restore the invariance under spatial translation

$$k \in \mathcal{E}_+ \cup \mathcal{E}_- \cup 0 \qquad\qquad k \in \mathcal{O}_+ \cup \mathcal{O}_- \cup \pi$$

$$\mathcal{E}_+ = \{\frac{2\pi}{N}, \ldots, \frac{N-3}{N}\pi, \frac{N-1}{N}\pi\} \qquad \mathcal{O}_+ = \{\frac{\pi}{N}, \ldots, \frac{N-4}{N}\pi, \frac{N-2}{N}\pi\}$$

$$\mathcal{E}_- = \{-\frac{2\pi}{N}, \ldots, -\frac{N-3}{N}\pi, -\frac{N-1}{N}\pi\} \qquad \mathcal{O}_- = \{-\frac{\pi}{N}, \ldots, -\frac{N-4}{N}\pi, -\frac{N-2}{N}\pi\}$$

Fermionic Hamiltonian

The result of the two transformations we obtain

$$H = \frac{\mathbb{1} + \mathbb{P}}{2}\left(H_0 + \sum_{k\in\mathcal{E}_+} H_k\right) + \frac{\mathbb{1} - \mathbb{P}}{2}\left(H_\pi + \sum_{k\in\mathcal{O}_+} H_k\right)$$

$$\mathbb{P} = \bigotimes_{l=1}(c_l^\dagger c_l - c_l c_l^\dagger)$$

$$H_0 = -(J+h)\left(b_0^\dagger b_0 - b_0 b_0^\dagger\right) \qquad H_\pi = -(h-J)\left(b_\pi^\dagger b_\pi - b_\pi b_\pi^\dagger\right)$$

$$H_k = -2\varepsilon_k(b_k^\dagger b_k + b_{-k}^\dagger b_{-k} - 1) - 2\imath\delta_k\left(b_k^\dagger b_{-k}^\dagger - b_{-k} b_k\right)$$

$$\delta_k = J\sin(k); \qquad \varepsilon_k = J\cos(k) + h$$

Local Hamiltonian with $k \neq 0, \pi$

The ground state of H_k is an even state

$$|\phi_k\rangle = \alpha_k |1_k, 1_{-k}\rangle + \beta_k |0_k, 0_{-k}\rangle$$

$$\alpha_k = \imath \frac{\varepsilon_k + \sqrt{\varepsilon_k^2 + \delta_k^2}}{\sqrt{\delta_k^2 + (\varepsilon_k + \sqrt{\varepsilon_k^2 + \delta_k^2})^2}}; \qquad \beta_k = \frac{\delta_k}{\sqrt{\delta_k^2 + (\varepsilon_k + \sqrt{\varepsilon_k^2 + \delta_k^2})^2}}$$

It is separated by a gap

$$\Delta E(k) = 2\sqrt{\varepsilon_k^2 + \delta_k^2} = -2\sqrt{(J\cos(k) + h)^2 + J^2 \sin^2(k)}.$$

from a two fold degenerate odd subspace

$$|1_k, 0_{-k}\rangle \text{ and } |0_k, 1_{-k}\rangle$$

Local Hamiltonian with $k = 0, \pi$

The parity of the ground state of these two local Hamiltonian depends on h and J

$k = 0$

- $h < -J \rightarrow |1_0\rangle$ odd state
- $h > -J \rightarrow |0_0\rangle$ even state

$$\Delta E(0) = 2|j + h|$$

$k = \pi$

- $h > J \rightarrow |1_\pi\rangle$ odd state
- $h < J \rightarrow |0_\pi\rangle$ even state

$$\Delta E(\pi) = 2|j - h|$$

The energetic gap $\Delta E(k)$ is a continuous function in $k = 0$ and $k = \pi$

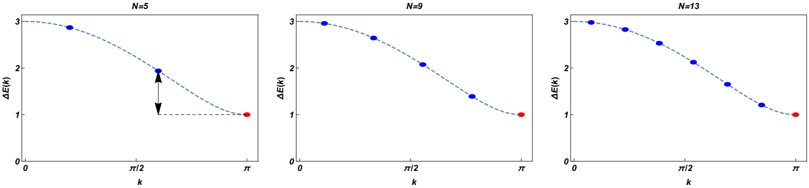
AFM chain ($J = 1$) with $h > 0$ Odd sector

$$h > 1 \qquad |\psi^{(>)}_{a,o}\rangle = |1_\pi\rangle \otimes \left(\bigotimes_{k \in \mathcal{O}_+} |\phi_k\rangle \right)$$

Gap in the Thermodynamic Limit $= 2(h-1)$

$$0 < h < 1 \qquad |\psi^{(<)}_{a,o}\rangle = |0_\pi\rangle \otimes \left(\bigotimes_{k \in \mathcal{O}_+} |\phi_k\rangle \right)$$

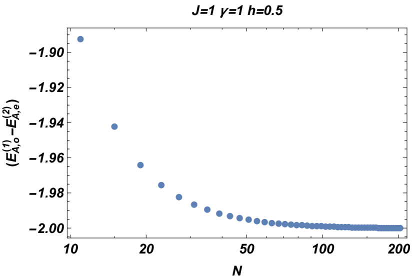
This state has the wrong parity. Cannot be accepted



Similar results hold also for the even sector

The system becomes gapless in thermodynamic limit for $|h| < 1$

AFM Gap and degeneracy



The gap between even and odd sector does not close. The ground state is non degenerate in Thermodynamic limit

The ground state stay unique even in the thermodynamic limit

Both these two results are in contrast with the results obtained for Ferromagnetic system

E. Lieb *et al.* Ann. Phys. **16** 407 (1961);
E. Barouch *et al.* Phys. Rev. A **2** 1075 (1970);

Fermionic Correlation Functions (Fermionic C.F.)

Fermionic Operators

$$A_l = c_l^\dagger + c_l = \bigotimes_{k=1}^{l-1} (\sigma_k^z) \otimes \sigma_l^x \qquad B_l = \imath(c_l - c_l^\dagger) = \bigotimes_{k=1}^{l-1} (\sigma_k^z) \otimes \sigma_l^y$$

All spin correlator can be written in terms of operator A_l and B_l

Because no superposition of the ground state $\langle A_l \rangle = \langle B_l \rangle = 0$

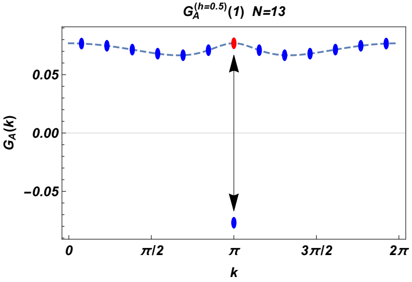
$$\langle A_l A_{l+r} \rangle = \langle B_l B_{l+r} \rangle = \delta_{r,0} \qquad \langle A_l B_{l+r} \rangle = \imath G_A(r)$$

$\langle \cdot \rangle$ expectation value on the ground state
G. C. Wick, Phys. Rev. **80**, 268 (1950);

Fermionic Correlation functions ($h > 0$)

$$G_A^{(h>0)}(r) = \frac{1}{N}\left[\cos(\pi r) + \sum_{k\in\mathcal{O}} \frac{(J\cos(k)+h)\cos(kr)}{\sqrt{(J\cos(k)+h)^2 + J^2\sin^2(k)}} + \sum_{k\in\mathcal{O}} \frac{J\sin(k)\sin(kr)}{\sqrt{(J\cos(k)+h)^2 + J^2\sin^2(k)}}\right]$$

$$\mathcal{O} = \mathcal{O}_+ \cup \mathcal{O}_-$$



$$G_A(r) = (-1)^r G_F(r) + \frac{2\cos(\pi r)}{N}$$

If we take the thermodynamic limit of $G_A(r)$ we obtain the usual fermionic C.F. $G_F(r)$

Spin Correlation functions

Is the right thing to do? NO!

we are interested in spin correlation function.
Hence the right way is:

1. determine the expression of the correlation functions for a finite size system
2. make the thermodynamic limit

Two different families of spin correlation functions

Local spin C.F.
expressed in terms of a number of fermionic c.f. that does not scale with N
same behavior of ferromagnetic models

Non-Local spin C.F.
expressed in terms of a number of fermionic c.f. that scales with N
????

Local spin correlation functions

Example: C.F. along z between two spins at distance R

$$\langle \sigma_l^z \sigma_{l+R}^z \rangle = \langle \sigma_l^z \rangle_F^2 - \frac{c_1^z(h)}{R^2} \left(\frac{h^2}{J^2} \right)^R + \frac{4 \langle \sigma_l^z \rangle_F^2}{N} \left[1 + c_2^z(h) (-1)^R \left| \frac{h}{J} \right| \right]$$

$c_1^z(h)$ and $c_2^z(h)$ do not depend on R or N

Exponential decay to saturation (correlation length $\xi = -\frac{1}{\ln h^2/J^2}$)

Perfect agreement with the usual solutions

E. Barouch & B. M. McCoy, Phys. Rev. A **3** 786 (1971).

Non-Local spin correlation functions

Example: C.F. along x between two spins at distance R

$$\langle \sigma_l^x \sigma_{l+R}^x \rangle = (-1)^R \left(1 - \frac{h^2}{J^2}\right)^{1/4} \left[1 + \frac{c^x(h)}{R^2} \left(\frac{h^2}{J^2}\right)^R \right] \left(1 - \frac{2R}{N}\right)$$

$c^x(h)$ does not depend on R or N

Polynomial decay of the Non-Local Spin C.F.

$$m_x = \lim_{N \to \infty} \sqrt{\langle \sigma_l^x \sigma_{l+\frac{N-1}{2}}^x \rangle} = 0$$

Absence of ordered phase!!

J.-J. Dong, & P. Li, Mod. Phys. Lett. **B 31**, 1750061 (2017)

Von Neumann Entropy

We consider a bipartition of the chain:

- a subsystem α made by R contiguous spins;
- The rest of the system β made by $N - R$ spins.

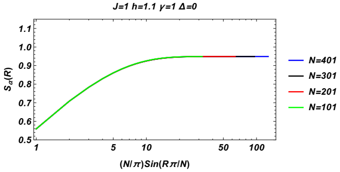
$$\rho_\alpha(R) = \mathrm{Tr}_\beta |GS\rangle\langle GS|$$

Von Neumann Entropy is defined as

$$S_\alpha(R) = -\mathrm{Tr}\left[\rho_\alpha(R) \ln \rho_\alpha(R)\right]$$

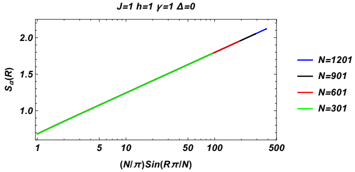
Area Law: Main Results

Gapped Models



$$S_\alpha(R) = a + b\exp(-cR)$$

CFT Models



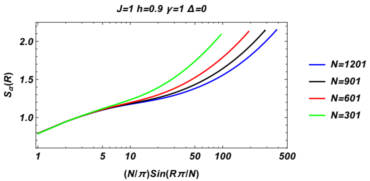
$$S_\alpha(R) = a + b\log(R)$$

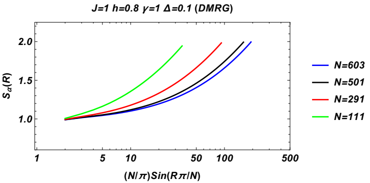
Two different Universal behavior

G. Vidal *et al.* Phys. Rev. Lett. **90**, 227902 (2003)
V. E. Korepin Phys. Rev. Lett. **92**, 096402 (2004)
P. Calabrese & J. Cardy, JSTAT **0406**, P002 (2004)

Entopy in weakly frustrated models

Weakly frustrated models





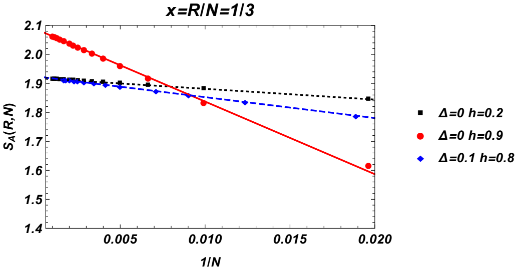
$$S_\alpha(R) = a_0(N) + b_0(N) R^{c_0(N)}$$

- No Plateau
- Non Universal behavior
- For small R close to transition appears a logarithmic universal behavior

Violation of the area law!!!

Entopy in weakly frustrated models

Weakly frustrated models



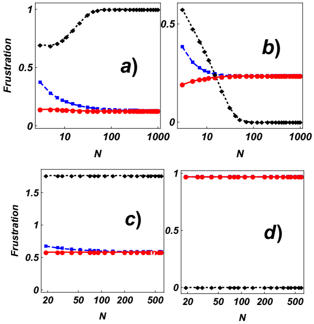
$$S_\alpha(R)\big|_{x=cost} = a_1 - \frac{b_1}{N}$$

- No divergence of the entropy
- convergence slower for systems close the quantum critical point

Relevance of the geometric frustration

Evaluation of the effect of the geometric frustration

$$N\sum_{i=1}^{N}\left(F_{i,i+1}^{(A)}-F_{i,i+1}^{(F)}\right)$$



$\Delta = 0$ a) $h = 0.9$ b) $h = 0.1$

$\Delta = 0.1$ c) $h = 0.8$ d) $h = 2.0$

S. M. Giampaolo *et al.* Phys. Rev. Lett. **107**, 260602 (2011);

U. Marzolino *et al.* Phys. Rev. A **88**, 020301(R) (2013).

Conclusion

Quantum spin chains with a weak frustration:

- develop a new quantum phase of matter, which present a mixture of correlation functions: some decaying exponentially and some decaying algebraically.
- The power-law correlations are very slowly decaying, since the relevant parameter is $x = \frac{R}{N}$.
- Show violation of the area law with an algebraic growth with subsystem size, which does not lead to a divergence of the EE with large systems
- As for non critial system the total amount of entanglement is finite, but, similarly to critical systems, the entanglement is distributed through the whole chain, with the possibility of distilling Bell-pairs with arbitrary distance (possible QI applications).
- Such behavior seams not to be affected by the integrability of the model
- Further application: Generalization of the results (2D, spin greater than 1/2); Different kind of interactions (cluster, Dzyaloshinskii-Moriya) other measures of the entanglement, applications

Workshop in Zagreb

https://maqp.irb.hr