Dynamical Defects in Integrable Field Theories

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...Previously on J.F. Gomes talk...

- Integrable hierarchies $\Longrightarrow \mathcal{G}, Q, E$
- Construction of soliton solutions in terms of representation of Kac-Moody algebras.
- The construction of Gauge-Bäcklund transformation to the entire integrable hierarchy.
- The K-matrix is universal.

Outline

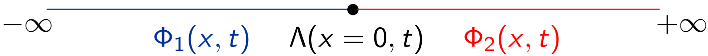
1. Lagrangian formalism
2. Soliton/defect interactions
3. Supersymmetric defects
4. Bäcklund transformations
5. Conclusions and future perspectives.

What is a dynamical defect?

Bowcock, Corrigan, Zambon (2004)

Dynamical (integrable) defects are internal boundary conditions preserving integrability.

- Defects conditions $\equiv$ Bäcklund Transformations



- Local Lagrangian density

$$\mathcal{L} = \theta(-x)\mathcal{L}_1 + \theta(x)\mathcal{L}_2 + \delta(x)\mathcal{L}_D,$$

- Breaking translational symmetry $\Longrightarrow$ integrability?

Lagrangian approach

$-\infty$ $\phi_1(x,t)$ $\lambda_0(x=0,t)$ $\phi_2(x,t)$ $+\infty$

- **Bosonic scalar field**:

$$\mathcal{L}_p = \frac{1}{2}\partial_\mu \phi_p \partial^\mu \phi_p - V_p(\phi_p)$$

- **Defect Lagrangian**:

$$\mathcal{L}_D = \frac{1}{2}\left[\phi_2\partial_t\phi_1 - \phi_1\partial_t\phi_2 - \lambda_0\partial_t(\phi_1-\phi_2) + \partial_t\lambda_0(\phi_1-\phi_2)\right] + \underbrace{\mathcal{D}\left(\phi_1,\phi_2,\lambda_0\right)}_{\text{defect potential}}$$

Lagrangian approach

- Defect conditions at $x = 0$

$$\begin{aligned}
\partial_x \phi_1 - \partial_t \phi_2 + \partial_t \lambda_0 &= -\partial_{\phi_1} \mathcal{D}, \\
\partial_x \phi_2 - \partial_t \phi_1 + \partial_t \lambda_0 &= \partial_{\phi_2} \mathcal{D}, \\
\partial_t (\phi_1 - \phi_2) &= \partial_{\lambda_0} \mathcal{D},
\end{aligned}$$

$$\boxed{\mathcal{D} = \mathcal{D}^{+}(\phi_+ - \lambda_0, \phi_-) + \mathcal{D}^{-}(\phi_-, \lambda_0)} \quad \text{where } \phi_\pm = \phi_1 \pm \phi_2.$$

- Q: Which conditions are satisfied by $\mathcal{D}$ in order to maintain integrability?

Modified Conserved Quantities

Let us consider the canonical energy,

$$E = \int_{-\infty}^{0} dx \left[\frac{1}{2}(\partial_x\phi_1)^2 + \frac{1}{2}(\partial_t\phi_1)^2 + V_1\right] + \int_{0}^{\infty} dx \left[\frac{1}{2}(\partial_x\phi_2)^2 + \frac{1}{2}(\partial_t\phi_2)^2 + V_2\right]$$

By computing its time-derivative, the modified energy given by $\mathcal{E} = E + E_D$ is conserved, with the defect contribution,

$$E_D = \left[\mathcal{D}^{(+)}(\phi_+ - \lambda_0, \phi_-) + \mathcal{D}^{(-)}(\phi_-, \lambda_0)\right]$$

Let us consider the canonical momentum,

$$P = \int_{-\infty}^{0} dx(\partial_t\phi_1)(\partial_x\phi_1) + \int_{0}^{\infty} dx(\partial_t\phi_2)(\partial_x\phi_2).$$

Modified Conserved Quantities

Then we have that, the modified momentum $\mathcal{P} = P + P_D$ is conserved, where the defect contribution is given by,

$$P_D = \left[\mathcal{D}^{(+)}(\phi_+ - \lambda_0, \phi_-) - \mathcal{D}^{(-)}(\phi_-, \lambda_0)\right]$$

If the Poisson Bracket relation is satisfied

$$\boxed{\{\mathcal{D}^+, \mathcal{D}^-\} = \left(\partial_{\phi_-}\mathcal{D}^+\right)\left(\partial_{\lambda_0}\mathcal{D}^-\right) - \left(\partial_{\lambda_0}\mathcal{D}^+\right)\left(\partial_{\phi_-}\mathcal{D}^-\right) = (V_1 - V_2)}$$

$\triangleright$ Higher-orders charges ?? $\longrightarrow$ Defect matrix $\mathcal{K}(\phi_1, \phi_2, \lambda_0)$

[A.R.A, T. Araujo, J.F. Gomes, A.H. Zimerman, JHEP12(2011)056]

Type-I ($\lambda_0 \to 0$) solutions

$$\mathcal{D} = \mathcal{D}^{+}(\phi_{+}) + \mathcal{D}^{-}(\phi_{-}) \quad \Longrightarrow \quad \frac{\partial^3 \mathcal{D}^{\pm}}{\partial \phi_{\pm}^3} = \zeta^2 \frac{\partial \mathcal{D}^{\pm}}{\partial \phi_{\pm}}$$

$V_1(\phi_1)$	$V_2(\phi_2)$	$\mathcal{D}(\phi_1, \phi_2)$
0	0	$a_1 e^{(\phi_1 \pm \phi_2)} + a_2 e^{(\phi_1 \pm \phi_2)}$
0	$\frac{\mu^2}{2} e^{2\phi_2}$	$\frac{\mu}{2}\left(\sigma e^{(\phi_1+\phi_2)} + \frac{1}{\sigma} e^{-(\phi_1-\phi_2)}\right)$
$\frac{\mu^2}{2} e^{2\phi_1}$	$\frac{\mu^2}{2} e^{2\phi_2}$	$\frac{\mu}{2}\left(\sigma e^{(\phi_1+\phi_2)} + \frac{1}{\sigma} \cosh(\phi_1 - \phi_2)\right)$
$\frac{m^2}{2} \phi_1^2$	$\frac{m^2}{2} \phi_2^2$	$\frac{m\sigma}{4} (\phi_1 + \phi_2)^2 + \frac{m}{4\sigma} (\phi_1 - \phi_2)^2$
$4m^2 \cosh \phi_1$	$4m^2 \cosh \phi_2$	$2m \left(\sigma \cosh(\frac{\phi_1+\phi_2}{2}) + \frac{1}{\sigma} \cosh(\frac{\phi_1-\phi_2}{2})\right)$

Type-II ($\lambda_0 \neq 0$) solutions: Dynamical defects

1. **Free scalar field**

$$\mathcal{D}^+ = m\left[\frac{(\phi_+ - \lambda_0)^2}{\beta} + \alpha\phi_-^2\right] \qquad \mathcal{D}^- = m\left[\frac{\lambda_0^2}{\alpha} + \beta\phi_-^2\right]$$

2. **sine(h)-Gordon :**

$$\mathcal{D}^+ = -\frac{m}{2\sigma}\left[e^{-i(\phi_+-\lambda_0)}\left(\cosh\phi_- + \gamma\right) + e^{i(\phi_+-\lambda_0)}\right]$$
$$\mathcal{D}^- = -\frac{m\sigma}{2}\left[e^{i\lambda_0}\left(\cosh\phi_- + \gamma\right) + e^{-i\lambda_0}\right]$$

3. **Liouville**

$$\mathcal{D}^+ = -2i\mu\beta^2\, e^{(\phi_+-\lambda_0)}, \qquad \mathcal{D}^- = \frac{i\mu}{\beta^2}\, e^{\lambda_0}\left(\cosh\phi_- + \kappa\right),$$

Type-II ($\lambda_0 \neq 0$) solutions: Dynamical defects

- **Tzitzéica-Bullough-Dodd-Mikhailov-Zhabat** ($a_2^{(2)}$-ATFT)

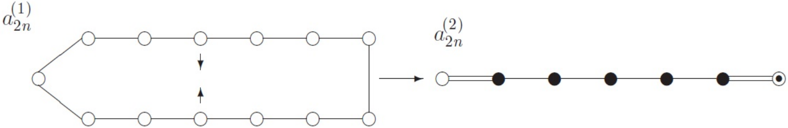
$$\partial_+\partial_-\phi = -e^{\phi} + e^{-2\phi}$$

- Defect potentials:

$$\mathcal{D}^+ = -4i\xi e^{(\phi_+ - \lambda_0)} \cosh\phi_- - \frac{1}{4\xi} e^{-2(\phi_+ - \lambda_0)}$$

$$\mathcal{D}^- = -4\xi e^{-2\lambda_0} \cosh^2\phi_- + \frac{i}{\xi} e^{\lambda_0}$$

- Folding from $a_2^{(1)}$-ATFT(Mikhailov, Olshanetsky, and Perelomov (1981); Olive and Turok (1983); Khastgir and Sasaki (1996); Corrigan (2009); Robertson (2014))



Solitons solutions

- A single sine-Gordon soliton:

$$\phi_1 = 2i \ln \left[\frac{1 - iE}{1 + iE} \right], \quad E = e^{ax+bt+c},$$

where $a = \cosh\theta$, $b = -\sinh\theta$, with θ the rapidity.

- $\theta > 0$, the soliton is moving along the x-axis in a positive direction. An anti-soliton with the same velocity and location is obtained by exchanging $E \to -E$.

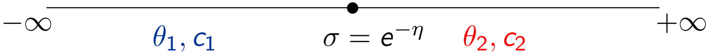
- Supposing a soliton moving in a positive sense along the x-axis, encounters the defect, then a similar, but delayed, soliton emerges

$$e^{i\phi_2/2} = \frac{1 + izE}{1 - izE}, \quad E = e^{ax+bt+c},$$

where z represents the delay,

$$z = \frac{e^{-\theta} + \sigma}{e^{-\theta} - \sigma}$$

Soliton + Type-I Defect



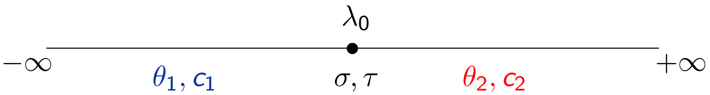
$$\boxed{\theta_1 = \theta_2, \qquad z = \coth\left[\frac{\eta - \theta}{2}\right]}$$

<u>Case $\theta > 0$:</u>

- $\eta > \theta$, implies that $z > 0$ and the soliton is **delayed**, but the effect vanishes in the limit $\theta \to \infty$.
- $\eta = \theta$, then $z = 0$ and the soliton is **infinitely delayed** $\equiv$ "absorbed by the defect"
- $\eta < \theta$, implies that $z < 0$ and **soliton** $\Longrightarrow$ **anti-soliton**.

<u>Case $\theta < 0$:</u> Similar behaviour by exchanging soliton for antisoliton.

Soliton + Type-II Defect



$$\theta_1 = \theta_2, \qquad z_1 = \tanh\left[\frac{\eta-\theta+\tau}{2}\right]\tanh\left[\frac{\eta-\theta-\tau}{2}\right], \qquad z_2 = 1/z_1$$

$$e^{\lambda_0} = \frac{1}{\cosh\tau}\,\frac{(1+E_0)(1+zE_0)}{(1+\rho_+E_0)(1+\rho_-E_0)}, \qquad \rho_\pm = \tanh\left(\frac{\theta-\eta\pm\tau}{2}\right),$$

- $\theta = \eta - \tau$ or $\theta = \eta + \tau$, then $z_1 = 0$ and the soliton is **absorbed**.
- $\eta - \tau < \theta < \eta + \tau$, implies that $z_1 < 0$ and **soliton** $\Longrightarrow$ **anti-soliton**.
- $\tau \to 0$, the defect is behaving like another soliton of rapidity η.

Defects in $a_r^{(1)}$ ATFT

The $a_r^{(1)}$ ATFT is described by the bulk density Lagrangian,

$$\mathcal{L} = \frac{1}{2}\left(\partial_\mu \phi_a\right)\left(\partial^\mu \phi_a\right) - \frac{m^2}{\beta^2}\sum_{i=0}^{r} n_i \left(e^{\beta(\alpha_i \cdot \phi)} - 1\right),$$

where $a = 1, 2, ..., r$, and the simple roots α_i and the extra root $\alpha_0 = -\sum_{i=1}^{r} n_i \alpha_i$.

The type-II defect Lagrangian,

$$\begin{aligned}\mathcal{L} \quad &= \quad \frac{1}{2}\Big[\phi \cdot \mathbf{A}\phi_t + \phi \cdot \mathbf{B}\psi_t - \phi_t \cdot \mathbf{B}\psi - \psi \cdot \mathbf{C}\psi_t + \Lambda \cdot (\phi - \psi)_t - (\phi - \psi) \cdot \Lambda_t \\ &- \underbrace{\Lambda \cdot \mathbf{M}\Lambda_t}_{\text{defect couplings}} \Big] - \mathcal{D}(\phi, \psi, \Lambda),\end{aligned}$$

Type-II defects in $a_r^{(1)}$ ATFT

$\mathbf{M} = 0$: E. Corrigan and C. Zambon, Nucl. Phys. B848 (2011) 545.

$$\mathcal{D}(\lambda, p, q) = \frac{m}{\beta^2} \sum_{j=0}^{r} \left(\sigma e^{i\beta\alpha_j\cdot(p+\lambda)/2} D_j(q) + \frac{1}{\sigma} e^{i\beta\alpha_j\cdot(p-\lambda)/2} D_{j+1}(q) \right),$$

where

$$D_j(q) = \gamma\, e^{i\beta\alpha_j\cdot Gq/2} + \frac{1}{\gamma}\, e^{-i\beta\alpha_j\cdot Gq/2},$$

The constant matrix G is constructed as follows,

$$G = 2\sum_{a=1}^{n} (w_a - w_{a+1})\, w_a^T, \quad \alpha_i \cdot w_j = \delta_{ij} \quad i, j = 1, \ldots, r$$

where w_i are the fundamental weights of the Lie algebra $a_r^{(1)}$, and σ, γ are the two defect parameters.

Generalized type-II defects in $a_r^{(1)}$ ATFT

$\mathbf{M} \neq 0$: From the modified momentum conservation we get

$$\dot{\mathcal{P}} = \Lambda_t \cdot \nabla_\Lambda \mathcal{D} - [\mathbf{A}\phi_t + \mathbf{B}\psi_t - \Lambda_t] \cdot \nabla_\phi \mathcal{D} - \left[\mathbf{B}^{\mathbf{T}}\phi_t + \mathbf{C}\psi_t - \Lambda_t\right] \cdot \nabla_\psi \mathcal{D}$$
$$+ \frac{1}{2}\left[(\nabla_\phi \mathcal{D})^2 - (\nabla_\psi \mathcal{D})^2 - 2(\mathcal{V} - \mathcal{W})\right]. \quad (1)$$

The modified momentum $\mathcal{P} + [\mathcal{D}^+ - \mathcal{D}^-]$ is conserved if the PB relation is satisfied:

$$2\,(\mathcal{V} - \mathcal{W}) = \left[\widehat{\nabla}_q \mathcal{D}^{(+)} \cdot \widehat{\nabla}_\Lambda \mathcal{D}^{(-)} - \widehat{\nabla}_\Lambda \mathcal{D}^{(+)} \cdot \widehat{\nabla}_q \mathcal{D}^{(-)}\right]$$
$$+ \left[\widehat{\nabla}_q \mathcal{D}^{(+)} \cdot \mathbf{M}\widehat{\nabla}_q \mathcal{D}^{(-)} - \widehat{\nabla}_\Lambda \mathcal{D}^{(+)} \cdot \mathbf{A}\widehat{\nabla}_\Lambda \mathcal{D}^{(-)}\right]$$

where,

$$\hat{q} = q - \frac{\mathbf{M}\Lambda}{2}, \quad \widehat{\Lambda} = \frac{\Lambda}{2} - \mathbf{A}q,$$

Two massive free fields:

$$\mathcal{D}^+ = m\left[\frac{1}{\beta_1}\left(p_1 - \frac{(1+\mu a)\Lambda_1}{2}\right)^2 + \frac{1}{\beta_2}\left(p_2 - \frac{(1+\mu a)\Lambda_2}{2}\right)^2 + \alpha_1\left(q_1 - \frac{\mu\Lambda_2}{2}\right)^2 + \alpha_2\left(q_2 + \frac{\mu\Lambda_1}{2}\right)^2\right],$$

$$\mathcal{D}^- = m\left[\beta_1\, q_1^2 + \beta_2\, q_2^2 + \frac{1}{\alpha_1}\left(\frac{\Lambda_1}{2} - aq_2\right)^2 + \frac{1}{\alpha_2}\left(\frac{\Lambda_2}{2} + aq_1\right)^2\right].$$

Two Liouville fields:

$$\mathcal{D}^+ = 2\, e^{\left[p_1-(1+\mu k_2)\left(\frac{\Lambda_1}{2}+k_1 q_2\right)\right]} + 2\, e^{\left[p_2-(1-\mu k_1)\left(\frac{\Lambda_2}{2}+k_2 q_1\right)\right]},$$

$$\mathcal{D}^- = e^{(1+\mu k_2)\left(\frac{\Lambda_1}{2}+k_1 q_2\right)}\left(e^{q_1} + e^{-q_1} + \gamma_1\right) + e^{(1-\mu k_1)\left(\frac{\Lambda_2}{2}+k_2 q_1\right)}\left(e^{q_2} + e^{-q_2} + \gamma_2\right).$$

The $a_2^{(1)}$ ATFT

$$\mathcal{D}^{+} = \frac{m}{2\sigma_1} e^{2\left(p_1 - \frac{(\lambda_1 - a q_2)}{2}\right)} \beta_1(q) + \frac{m}{2\sigma_2} e^{-\left(p_1 - \frac{(\lambda_1 - a q_2)}{2}\right) - \sqrt{3}\left(p_2 - \frac{(\lambda_2 + a q_1)}{2}\right)} \beta_2(q) + \frac{m}{2\sigma_3} e^{-\left(p_1 - \frac{(\lambda_1 - a q_2)}{2}\right) + \sqrt{3}\left(p_2 - \frac{(\lambda_2 + a q_1)}{2}\right)} \beta_3(q),$$

$$\mathcal{D}^{-} = \frac{m\sigma_1}{2} e^{(\lambda_1 - a q_2)} \beta_3(q) + \frac{m\sigma_2}{2} e^{-\frac{1}{2}\left((\lambda_1 - a q_2) + \sqrt{3}(\lambda_2 + a q_1)\right)} \beta_1(q) + \frac{m\sigma_3}{2} e^{-\frac{1}{2}\left((\lambda_1 - a q_2) - \sqrt{3}(\lambda_2 + a q_1)\right)} \beta_2(q),$$

with

$$\beta_1(q) = \sigma_0 e^{q_1 + \frac{q_2}{\sqrt{3}}} + \frac{1}{\sigma_0} e^{-\left(q_1 + \frac{q_2}{\sqrt{3}}\right)},$$

$$\beta_2(q) = \frac{1}{\sigma_0} e^{\frac{2q_2}{\sqrt{3}}} + \sigma_0 e^{-\frac{2q_2}{\sqrt{3}}},$$

$$\beta_3(q) = \frac{1}{\sigma_0} e^{q_1 - \frac{q_2}{\sqrt{3}}} + \sigma_0 e^{-\left(q_1 - \frac{q_2}{\sqrt{3}}\right)},$$

Supersymmetric extensions ?

- All of the type-I defects have a supersymmetric extension (Ymai,Gomes,Zimerman)
- Q: **Is it possible to find any supersymmetric extension of the type-II defects ???**
- Lagrangian formalism $\longrightarrow$ Superfield formalism.
- Defects $\longrightarrow$ SuperBäcklund transformations.
- Lax formulation $\longrightarrow$ Super-Lax connections valued on Lie superalgebras: $osp(1|2), sl(1|2), \ldots$

Type-II supersymmetric defect

Let us consider the supersymmetric field theory with type-II defects,

$$\begin{aligned}\mathcal{L} &= \theta(-x)\mathcal{L}_1 + \theta(x)\mathcal{L}_2 + \delta(x)\mathcal{L}_D, \\ \mathcal{L}_p &= \frac{1}{2}\left(\partial_x\phi_p\right)^2 - \frac{1}{2}\left(\partial_t\phi_p\right)^2 + i\bar{\psi}_p\left(\partial_x - \partial_t\right)\bar{\psi}_p + i\psi_p\left(\partial_t + \partial_x\right)\psi_p \\ &\quad + V_p + W_p,\end{aligned}$$

The defect Lagrangian, $\mathcal{L}_D = \mathcal{L}_b + \mathcal{L}_f$, with

$$\begin{aligned}\mathcal{L}_b &= \frac{1}{2}\left(\phi_2\partial_t\phi_1 - \phi_1\partial_t\phi_2 - \lambda_0\partial_t(\phi_1 - \phi_2) + (\phi_1 - \phi_2)\partial_t\lambda_0\right) \\ &\quad + \mathcal{D}_0^{(+)}(\phi_+ - \lambda_0, \phi_-) + \mathcal{D}_0^{(-)}(\phi_-, \lambda_0), \\ \mathcal{L}_f &= i\bar{\psi}_1\bar{\psi}_2 - i\psi_1\psi_2 - if_1\partial_t f_1 - i\tilde{f}_1\partial_t\tilde{f}_1 + \mathcal{D}_1^{(+)}(\phi_+ - \lambda_0, \phi_-, \bar{\psi}_+, f_1, \tilde{f}_1) \\ &\quad + \mathcal{D}_1^{(-)}(\phi_-, \lambda_0, \psi_+, f_1, \tilde{f}_1).,\end{aligned}$$

where $\lambda_0, f_1, \tilde{f}_1$ are auxiliary fields living at the defect point $x = 0$.

Modified Conserved Quantities

Let us consider the canonical energy,

$$E = \int_{-\infty}^{0} dx \left[\frac{1}{2}(\partial_x\phi_1)^2 + \frac{1}{2}(\partial_t\phi_1)^2 - i\bar{\psi}_1\partial_x\bar{\psi}_1 + i\psi_1\partial_x\psi_1 + V_1 + W_1\right] + \\ + \int_{0}^{\infty} dx \left[\frac{1}{2}(\partial_x\phi_2)^2 + \frac{1}{2}(\partial_t\phi_2)^2 - i\bar{\psi}_2\partial_x\bar{\psi}_2 + i\psi_2\partial_x\psi_2 + V_2 + W_2\right],$$

By computing its time-derivative, the modified energy given by $\mathcal{E} = E + E_D$ is conserved, with the defect contribution,

$$E_D = \left[\mathcal{D}_0^{(+)}(\phi_+ - \lambda_0, \phi_-) + \mathcal{D}_0^{(-)}(\phi_-, \lambda_0)\right] + \frac{i}{2}\left[\bar{\psi}_+\bar{\psi}_- - \psi_+\psi_-\right] \\ + \left[\mathcal{D}_1^{(+)}(\phi_+ - \lambda_0, \phi_-, \bar{\psi}_+, f_1, \tilde{f}_1) + \mathcal{D}_1^{(-)}(\phi_-, \lambda_0, \psi_+, f_1, \tilde{f}_1)\right].$$

Let us consider the canonical momentum,

$$P = \int_{-\infty}^{0} dx \Big[(\partial_t\phi_1)(\partial_x\phi_1) - i\bar{\psi}_1\partial_x\bar{\psi}_1 - i\psi_1\partial_x\psi_1\Big] \\ + \int_{0}^{\infty} dx \Big[(\partial_t\phi_2)(\partial_x\phi_2) - i\bar{\psi}_2\partial_x\bar{\psi}_2 - i\psi_2\partial_x\psi_2\Big].$$

Modified Conserved Quantities

The modified momentum $\mathcal{P} = P + P_D$ is conserved, where

$$P_D = \left[\mathcal{D}_0^{(+)}(\phi_+ - \lambda_0, \phi_-) - \mathcal{D}_0^{(-)}(\phi_-, \lambda_0)\right] + \frac{i}{2}\left[\bar{\psi}_+\bar{\psi}_- + \psi_+\psi_-\right]$$
$$+ \left[\mathcal{D}_1^{(+)}(\phi_+ - \lambda_0, \phi_-, \bar{\psi}_+, f_1, \tilde{f}_1) - \mathcal{D}_1^{(-)}(\phi_-, \lambda_0, \psi_+, f_1, \tilde{f}_1)\right].$$

If the Poisson Bracket relations are satisfied

$$\left(\partial_{\phi_-}\mathcal{D}_0^{(+)}\partial_{\lambda_0}\mathcal{D}_0^{(-)} - \partial_{\lambda_0}\mathcal{D}_0^{(+)}\partial_{\phi_-}\mathcal{D}_0^{(-)}\right) = V_1 - V_2,$$

$$\left(\partial_{\phi_-}\mathcal{D}_0^{(+)}\partial_{\lambda_0}\mathcal{D}_1^{(-)} - \partial_{\lambda_0}\mathcal{D}_0^{(+)}\partial_{\phi_-}\mathcal{D}_1^{(-)}\right) - \left(\partial_{\phi_-}\mathcal{D}_0^{(-)}\partial_{\lambda_0}\mathcal{D}_1^{(+)} - \partial_{\lambda_0}\mathcal{D}_0^{(-)}\partial_{\phi_-}\mathcal{D}_1^{(+)}\right)$$
$$-i\left(\partial_{f_1}\mathcal{D}_1^{(+)}\partial_{f_1}\mathcal{D}_1^{(-)} + \partial_{\tilde{f}_1}\mathcal{D}_1^{(+)}\partial_{\tilde{f}_1}\mathcal{D}_1^{(-)}\right) = W_1 - W_2.$$

Solution I

- ARA, J. Phys. Conf. Ser. 474 (2013) 012001 [arXiv:1312.3463]
- ARA, J Gomes, N Spano, Zimerman, JHEP 02 (2015) 175, JHEP06(2015)125

Super-Liouville equations:

$$\begin{aligned} \partial_x^2\phi_p - \partial_t^2\phi_p &= \mu^2 e^{2\phi_p} + i\mu e^{\phi_p}\bar{\psi}_p\psi_p \\ (\partial_x + \partial_t)\psi_p &= i\mu e^{\phi_p}\bar{\psi}_p \\ (\partial_x - \partial_t)\bar{\psi}_p &= -i\mu e^{\phi_p}\psi_p, \qquad p = 1, 2, \end{aligned}$$

$$\begin{aligned} V_p &= \frac{1}{2}\mu^2 e^{2\phi_p}, & W_p &= 2i\mu e^{\phi_p}\bar{\psi}_p\psi_p, \\ \mathcal{D}_0^+ &= -i\mu\beta^2 e^{(\phi_+ - \lambda_0)}, & \mathcal{D}_0^- &= i\mu\omega 2\beta^2\, e^{\lambda_0}\left(\cosh\phi_- + \kappa\right), \\ \mathcal{D}_1^+ &= \sqrt{\mu}\beta\, e^{\frac{(\phi_+ - \lambda_0)}{2}}\bar{\psi}_+ f_1, & \mathcal{D}_1^- &= \sqrt{\mu}\omega\beta\, e^{\frac{\lambda_0}{2}}\sinh\left(\frac{\phi_-}{2}\right)\psi_+ f_1 \end{aligned}$$

Solution II

Super sinh-Gordon equations:

$$\begin{aligned}
\partial_x^2\phi_p - \partial_t^2\phi_p &= 2m^2\sinh(2\phi_p) - 4im\,\bar{\psi}_p\psi_p\sinh\phi_p,\\
(\partial_x - \partial_t)\bar{\psi}_p &= -2m\,\psi_p\cosh\phi_p,\\
(\partial_x + \partial_t)\psi_p &= -2m\,\bar{\psi}_p\cosh\phi_p, \qquad p = 1, 2.
\end{aligned}$$

$$\begin{aligned}
\mathcal{D}_0^{(+)} &= m\sigma\left[e^{(\phi_+-\lambda_0)} + e^{-(\phi_+-\lambda_0)}\left(\sinh^2\left(\frac{\phi_-}{2}\right) + \cosh^2\tau\right)\right],\\
\mathcal{D}_0^{(-)} &= \frac{m}{\sigma}\left[e^{-\lambda_0} + e^{\lambda_0}\left(\sinh^2\left(\frac{\phi_-}{2}\right) + \cosh^2\tau\right)\right],\\
\mathcal{D}_1^{(+)} &= -i\sqrt{m\sigma}\left[\left(e^{\frac{(\phi_+-\lambda_0)}{2}} + e^{-\frac{(\phi_+-\lambda_0)}{2}}\cosh\tau\right)\bar{\psi}_+ f_1 + e^{-\frac{(\phi_+-\lambda_0)}{2}}\sinh\left(\frac{\phi_-}{2}\right)\bar{\psi}_+\tilde{f}_1\right]\\
&\quad + im\sigma\left(1 + e^{-(\phi_+-\lambda_0)}\cosh\tau\right)\cosh\left(\frac{\phi_-}{2}\right)f_1\tilde{f}_1,\\
\mathcal{D}_1^{(-)} &= -i\sqrt{\frac{m}{\sigma}}\left[\left(e^{-\frac{\lambda_0}{2}} + e^{\frac{\lambda_0}{2}}\cosh\tau\right)\psi_+\tilde{f}_1 - e^{\frac{\lambda_0}{2}}\sinh\left(\frac{\phi_-}{2}\right)\psi_+ f_1\right]\\
&\quad + \frac{im}{\sigma}\left(1 + e^{\lambda_0}\cosh\tau\right)\cosh\left(\frac{\phi_-}{2}\right)f_1\tilde{f}_1.
\end{aligned}$$

More solutions

- $N = 2$ super sinh-Gordon model with type-II defect (Terrific long expressions !)

- The fusing procedure has been used.

- ARA, J Gomes, N Spano, Zimerman, JHEP 02 (2015) 175, JHEP06(2015)125

- ARA, Gomes, Spano, Zimerman, J. Phys.: Conf. Ser. 670 (2016) 012049

Defects in the super-Liouville theory

- Let Φ be a bosonic superfield,

$$\Phi = \phi - i\theta_1 \bar{\psi} + i\theta_2 \psi - i\theta_1\theta_2 F,$$

and

$$D_+ = -i\partial_{\theta_1} + \theta_1\partial_+, \quad D_- = i\partial_{\theta_2} + \theta_2\partial_-, \quad D_\pm^2 = \mp i\partial_\pm$$
$$\{D_+, D_-\} = 0.$$

- The supersymmetric Liouville equation is written as

$$D_+D_-\Phi = -i\mu e^{\Phi},$$

- We extend the holomorphic field Λ_0 to a chiral superfield

$$\Lambda = \lambda_0 + \theta\lambda_1$$

The super Bäcklund transformation

We propose the type-II super Bäcklund transformation to take the following form,

$$
\begin{aligned}
D_-(\Phi_+ - \Lambda) &= \frac{i\sqrt{\mu}}{\beta}\,\Sigma\, e^{\frac{\Lambda}{2}}\cosh\left(\frac{\Phi_-}{2}\right),\\
D_+\Phi_- &= i\sqrt{\mu}\beta\,\Sigma\,\exp\left(\frac{\Phi_+ - \Lambda}{2}\right),\\
D_-\Phi_- &= \frac{i\sqrt{\mu}}{\beta}\,\Sigma\, e^{\frac{\Lambda}{2}}\sinh\left(\frac{\Phi_-}{2}\right),\\
D_+\Lambda &= 0\\
D_-\Sigma &= -\frac{2\sqrt{\mu}}{\beta} e^{\frac{\Lambda}{2}}\sinh\left(\frac{\Phi_-}{2}\right),\\
D_+\Sigma &= 2\sqrt{\mu}\beta\exp\left(\frac{\Phi_+ - \Lambda}{2}\right).
\end{aligned}
$$

where $\Sigma = f_1 + \theta b_1 + \bar{\theta} b_2 + \bar{\theta}\theta f_2$. $\Lambda = 0$ is the case of the Bäcklund transformation proposed by M. Chaichain and P.P. Kulish.

On-shell supersymmetry

These defect equations are invariant under the supersymmetric transformations,

$$\begin{aligned}
\delta\phi_p &= \varepsilon\,\psi_p + \bar{\varepsilon}\,\bar{\psi}_p,\\
\delta\psi_p &= -\varepsilon\,\partial\phi_p - i\mu\,\bar{\varepsilon}\,e^{\phi_p},\\
\delta\bar{\psi}_p &= -\bar{\varepsilon}\,\bar{\partial}\phi_p + i\mu\,\varepsilon\,e^{\phi_p},\\
\delta\lambda_0 &= \varepsilon\,\lambda_1,\\
\delta\lambda_1 &= -\varepsilon\partial\lambda_0,\\
\delta f_1 &= \frac{2i\varepsilon\sqrt{\mu}}{\beta}\,e^{\frac{\lambda_0}{2}}\sinh\left(\frac{\phi_1-\phi_2}{2}\right) - 2i\sqrt{\mu}\beta\,\bar{\varepsilon}\,e^{\frac{(\phi_1+\phi_2-\lambda_0)}{2}}.
\end{aligned}$$

Modified Conserved Supercharges

Considering the two supercharges

$$Q_\varepsilon = -\int_{-\infty}^{\infty} dx \left(\psi \partial \phi + i\mu e^{\phi} \bar{\psi}\right), \qquad \bar{Q}_{\bar{\varepsilon}} = \int_{-\infty}^{\infty} dx \left(\bar{\psi} \bar{\partial} \phi - i\mu e^{\phi} \psi\right)$$

we obtain

$$\mathcal{Q} = Q_\varepsilon - \frac{2\sqrt{\mu}}{\beta}\, e^{\frac{\lambda_0}{2}} \sinh\left(\frac{\phi_-}{2}\right) f_1,$$

$$\bar{\mathcal{Q}} = \bar{Q}_{\bar{\varepsilon}} - 2\sqrt{\mu}\beta\, e^{\frac{(\phi_+ - \lambda_0)}{2}} f_1.$$

This turns to be one of the first examples of a type-II defect enclosed within a supersymmetric model, and presumably super-extensions for most of the other ATFT can be found.

The sBT for sshG - type I

- $\mathcal{N} = 1$ **bosonic superfield**: $\Phi = \phi - i\theta_1\,\bar{\psi} + i\theta_2\,\psi - i\theta_1\theta_2 F$
- **The sshG equation**:

$$D_+ D_- \Phi = im \sinh \Phi$$

- M. Chaichian and P.P. Kulish, Phys. Lett. B 78 (1978) 413

$$\begin{aligned} D_-\Phi_+ &= \omega\,\Sigma \cosh\left(\frac{\Phi_-}{2}\right), & D_-\Sigma &= -\frac{2im}{\omega}\sinh\left(\frac{\Phi_-}{2}\right),\\ D_+\Phi_- &= \frac{1}{\omega}\Sigma \cosh\left(\frac{\Phi_+}{2}\right), & D_+\Sigma &= 2im\omega\,\sinh\left(\frac{\Phi_+}{2}\right), \end{aligned}$$

Type-II sBT for sshG - type II

$$
\begin{aligned}
D_-(\Phi_+ - \Lambda) &= -\sqrt{\frac{m}{\sigma}}\; e^{\frac{\Lambda}{2}}\, \Sigma \cosh\left(\frac{\Phi_-}{2}\right), \\
D_+\Lambda &= \sqrt{m\sigma}\; e^{-\frac{(\Phi_+ - \Lambda)}{2}}\, \tilde{\Sigma} \cosh\left(\frac{\Phi_-}{2}\right), \\
D_+\Phi_- &= -\sqrt{m\sigma}\left[\left(e^{\frac{(\Phi_+ - \Lambda)}{2}} + \cosh\tau\, e^{-\frac{(\Phi_+ - \Lambda)}{2}}\right)\Sigma + e^{-\frac{(\Phi_+ - \Lambda)}{2}} \sinh\left(\frac{\Phi_-}{2}\right)\tilde{\Sigma}\right] \\
D_-\Phi_- &= \sqrt{\frac{m}{\sigma}}\left[\left(e^{-\frac{\Lambda}{2}} + \cosh\tau\, e^{\frac{\Lambda}{2}}\right)\tilde{\Sigma} - e^{\frac{\Lambda}{2}} \sinh\left(\frac{\Phi_-}{2}\right)\Sigma\right],
\end{aligned}
$$

with

$$
D_+\Sigma = -i\sqrt{m\sigma}\left[e^{\frac{(\Phi_+ - \Lambda)}{2}} - \cosh\tau\, e^{-\frac{(\Phi_+ - \Lambda)}{2}}\right], \qquad D_-\Sigma = i\sqrt{\frac{m}{\sigma}}\; e^{\frac{\Lambda}{2}} \sinh\left(\frac{\Phi_-}{2}\right),
$$

$$
D_-\tilde{\Sigma} = i\sqrt{\frac{m}{\sigma}}\left[e^{-\frac{\Lambda}{2}} - \cosh\tau\, e^{\frac{\Lambda}{2}}\right], \qquad D_+\tilde{\Sigma} = i\sqrt{m\sigma}\; e^{-\frac{(\Phi_+ - \Lambda)}{2}} \sinh\left(\frac{\Phi_-}{2}\right),
$$

where $\Lambda, \Sigma, \tilde{\Sigma}$ are three auxiliary superfields, and σ, τ two parameters.

Defects in the smkdV hierarchy

- The $N = 1$ smKdV equation is described by a fermionic superfield $\Psi(x,\theta) = \sqrt{i}\bar{\psi}(x) + \theta u(x)$,

$$D_{t_3}\Psi = D^6\Psi - 3(D\Psi)D^2(\Psi D\Psi),$$

where, $D = \partial_\theta + \theta\partial_x$, and then

$$4\partial_{t_3}u = \partial_x^3 u - 6u^2\partial_x u + 3i\bar{\psi}\partial_x\left(u\partial_x\bar{\psi}\right),$$
$$4\partial_{t_3}\bar{\psi} = \partial_x^3\bar{\psi} - 3u\partial_x\left(u\bar{\psi}\right).$$

- In terms of a bosonic superfield $\Phi(x,\theta) = \phi(x) - \sqrt{i}\theta\bar{\psi}(x)$, such that $\Psi = -D\Phi$,

$$D_{t_3}\Phi = D^6\Phi - 2(D^2\Phi)^3 + 3(D\Phi)(D^2\Phi)(D^3\Phi).$$

sBT for the smkdV hierarchy

$$
\begin{aligned}
D\Phi_- &= \frac{4i}{\omega}\cosh\left(\frac{\Phi_+}{2}\right)\Sigma, \\
D\Sigma &= -\frac{2i}{\omega}\sinh\left(\frac{\Phi_+}{2}\right), \\
D_{t_3}\Phi_- &= -\frac{32}{\omega^6}\sinh^3\Phi_+ + \frac{i}{\omega}\cosh\left(\frac{\Phi_+}{2}\right)\left[D^4\Phi_+ D\Phi_+ - D^2\Phi_+ D^3\Phi_+\right]\Sigma \\
&+\frac{i}{\omega}\sinh\left(\frac{\Phi_+}{2}\right)\left[2D^5\Phi_+ - (D^2\Phi_+)^2(D\Phi_+)\right]\Sigma \\
&+\frac{2}{\omega^2}\sinh\Phi_+\left[(D\Phi_+)(D^3\Phi_+) - (D^2\Phi_+)^2\right] + \frac{4}{\omega^2}\cosh\Phi_+(D^4\Phi_+) \\
&-\frac{96i}{\omega^5}\left[\sinh\left(\frac{\Phi_+}{2}\right) + 4\sinh^3\left(\frac{\Phi_+}{2}\right) + 3\sinh^5\left(\frac{\Phi_+}{2}\right)\right](D\Phi_+)\Sigma \\
D_{t_3}\Sigma &= \frac{i}{2\omega}\cosh\left(\frac{\Phi_+}{2}\right)\left[(D\Phi_+)(D^2\Phi_+)^2 - 2(D^5\Phi_+)\right] \\
&+\frac{i}{2\omega}\sinh\left(\frac{\Phi_+}{2}\right)\left[(D^2\Phi_+)(D^3\Phi_+) - (D\Phi_+)(D^4\Phi_+)\right] \\
&-\frac{12}{\omega^4}\sinh\Phi_+\cosh^2\left(\frac{\Phi_+}{2}\right)(D^2\Phi_+)\Sigma + \frac{12i}{\omega^5}\sinh^2\Phi_+\cosh\left(\frac{\Phi_+}{2}\right)(D\Phi_+).
\end{aligned}
$$

Conclusions and further interesting aspects

- The conservation of modified momentum turns out to be a sufficient condition for integrability of the model with type-II defects.
- There exist supersymmetric extensions for some integrable field theories supporting type I and II defects
- The remaining point is if for any integrable field theory supporting such a defects should exist a supersymmetric extension : $B(0, 2n)$ ...,
- We have proposed type-II super Bäcklund transformations to describe the defect conditions corresponding to type-II defects.

Some questions to be addressed

- Folding procedure to generate (super) type-II defects for non-simply laced cases. (C. Robertson).
- Type-II defect in ATFT based on Dynkin diagrams with "forks": $D_4^{(1)}$ (Recent approach R. Bristow and P. Bowcock)
- Other algebraic aspects: The classical r-matrix description for (super) type II defects following the on-shell (Habibullin,Kundu, 2007) and off-shell (Avan, Doikou, 2011), and Multisymplectic approach for supersymetric cases (Caudrelier, Kundu).
- Beyond integrability: Deformations (D. Bazeia), Quasi-integrable (L.A. Ferreira) defects, Radiating and moving defects (Corrigan)

Thanks a lot for your patience !

Chronology

- Bowcock, Corrigan, Zambon ('04) : Local Lagrangian density.
- Free scalar field, Liouville, sine(h)-Gordon, $A_r^{(1)}$ affine Toda.
- Corrigan, Zambon ('06): Non-relativistic field theories NLS, (m)KdV.
- Gomes, Ymai, Zimerman: ('06) $N = 1$ and ('08) $N = 2$ sshG model
- Caudrelier, Kundu ('08): On-shell Lax approach, defect matrix $\mathcal{K}$, conserved $\mathcal{I}^{(n)}$.
- A.A., Gomes, Ymai, Zimerman ('09, '11): Lagrangian approach for Thirring model
- Important models missing e.g Tzitzeica-Bullogh-Dodd-Zhakarov-Shabat or $A_2^{(2)}$ ATFT.
- Generalization of the framework $\longrightarrow$ Classification in Type-I and -II.
- Corrigan, Zambon ('09): Type-II defect Lagrangian with additional degree of freedom