# Gauge and Integrable Theories on Loop Spaces

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Workshop: Exactly Solvable Quantum Chains

IIP - Natal 27th June 2018

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- Infinite number of conservation laws
- Inverse scattering method
- Dressing method
- Hirota method
- Classical $r$-matrix
- Quantum $R$-matrix
- etc

# Flatness and path independency

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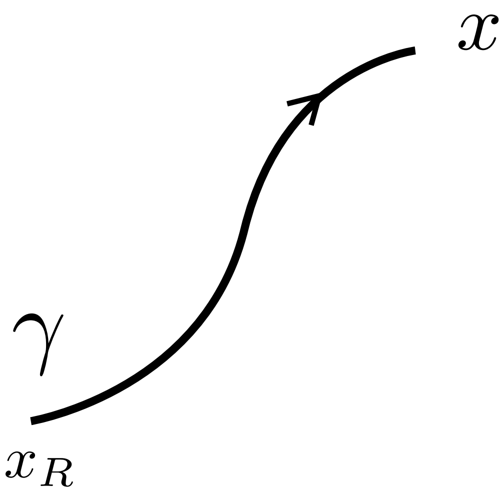

$$\frac{dW}{d\sigma} + A_\mu \frac{dx^\mu}{d\sigma} W = 0$$

$$W(\gamma) = P_1 \, e^{-\int_\gamma A_\mu \frac{dx^\mu}{d\sigma}} \, W_R$$

$$W = 1 - \int_0^\sigma d\sigma_1 A_\mu(\sigma_1) \frac{dx^\mu}{d\sigma_1} + \int_0^\sigma d\sigma_1 A_{\mu_1}(\sigma_1) \frac{dx^{\mu_1}}{d\sigma_1} \int_0^{\sigma_1} d\sigma_2 A_{\mu_2}(\sigma_2) \frac{dx^{\mu_2}}{d\sigma_2} - \ldots$$

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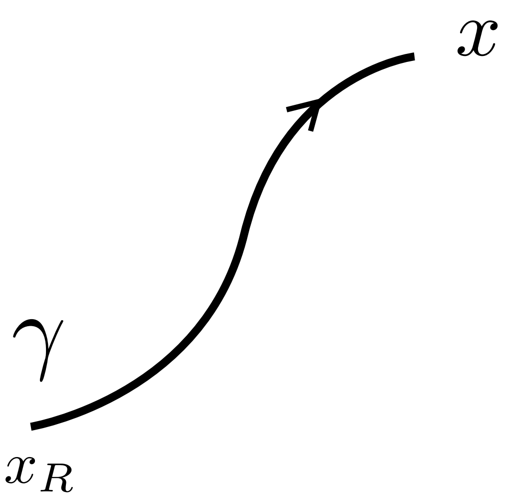

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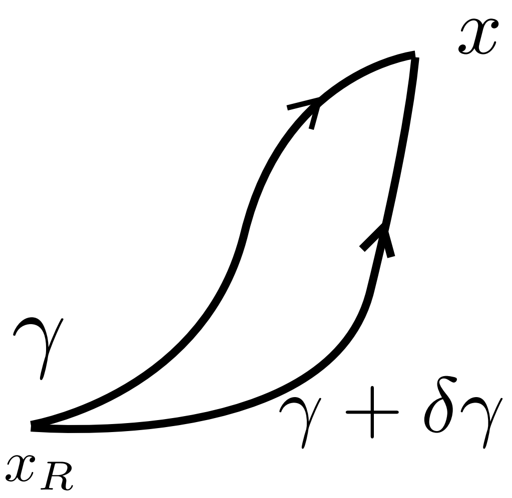

$$W^{-1}(\gamma)\delta W(\gamma) = \int_0^{2\pi} d\sigma\, W^{-1}F_{\mu\nu}W\frac{dx^\mu}{d\sigma}\delta x^\nu$$

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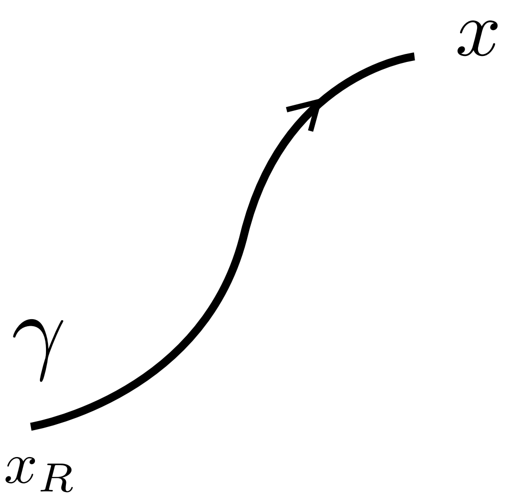

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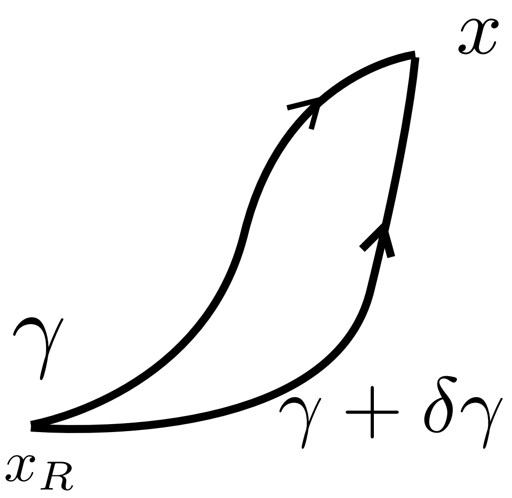

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$F_{\mu\nu} = 0$

$\swarrow$

$W$ is path independent

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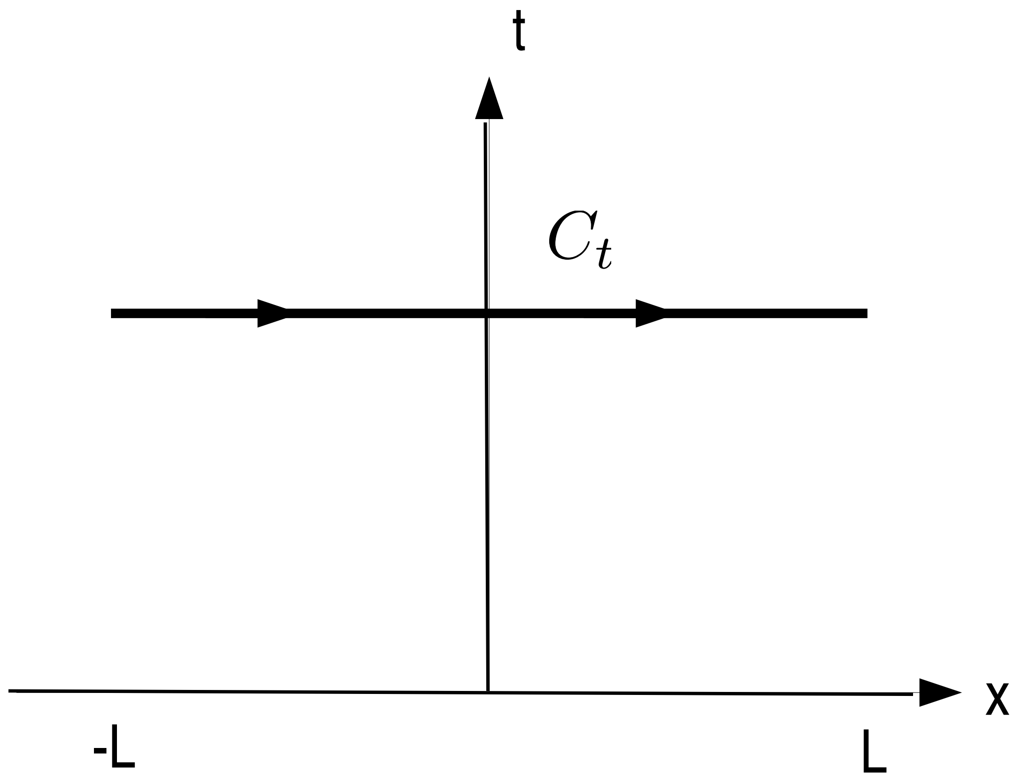

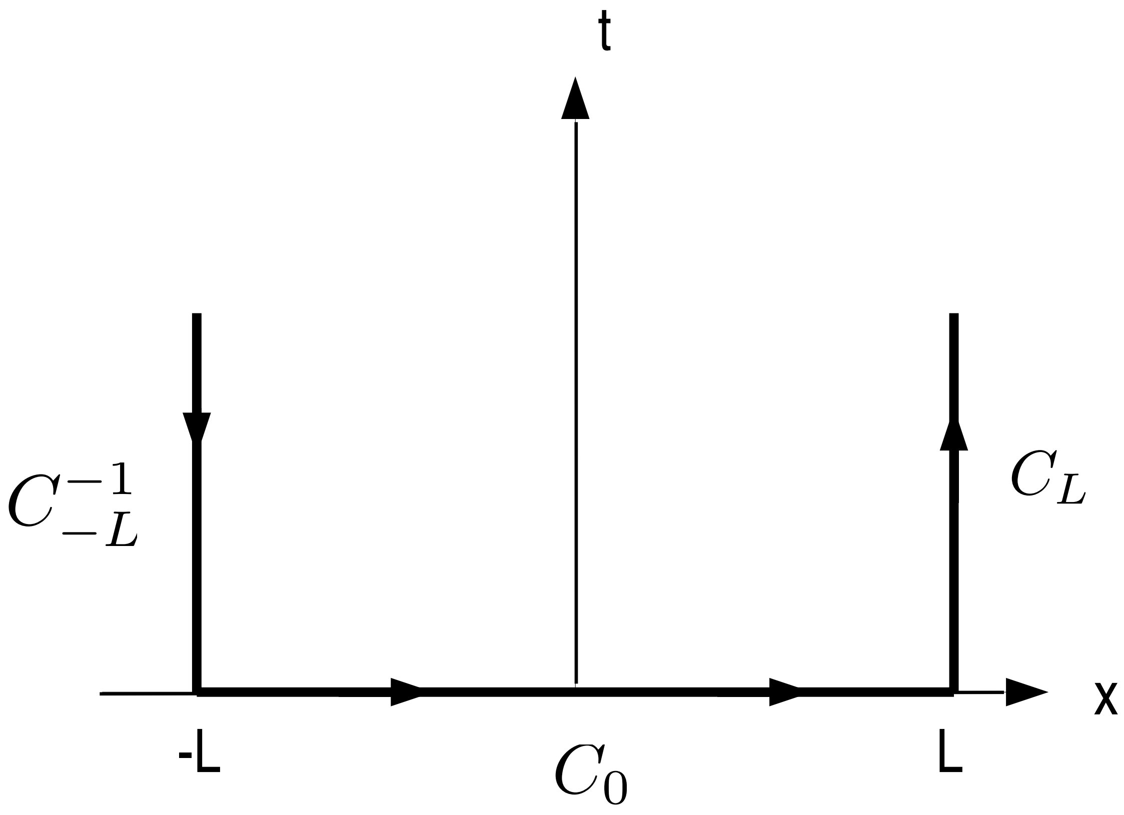

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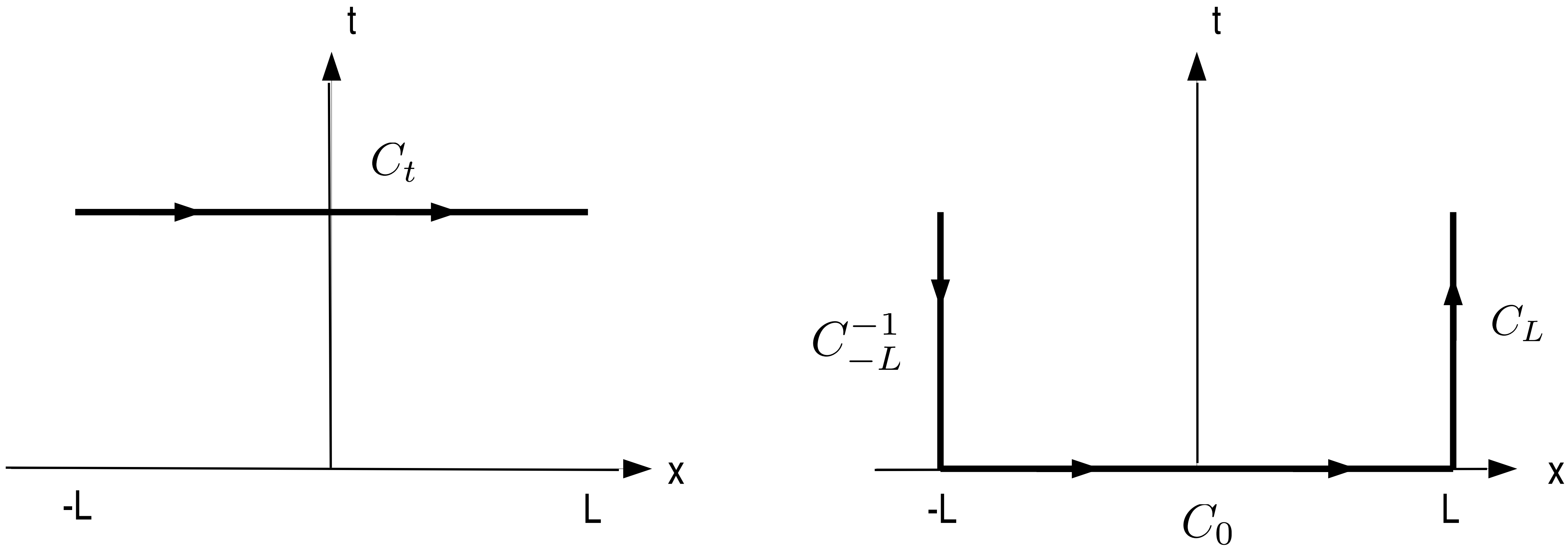

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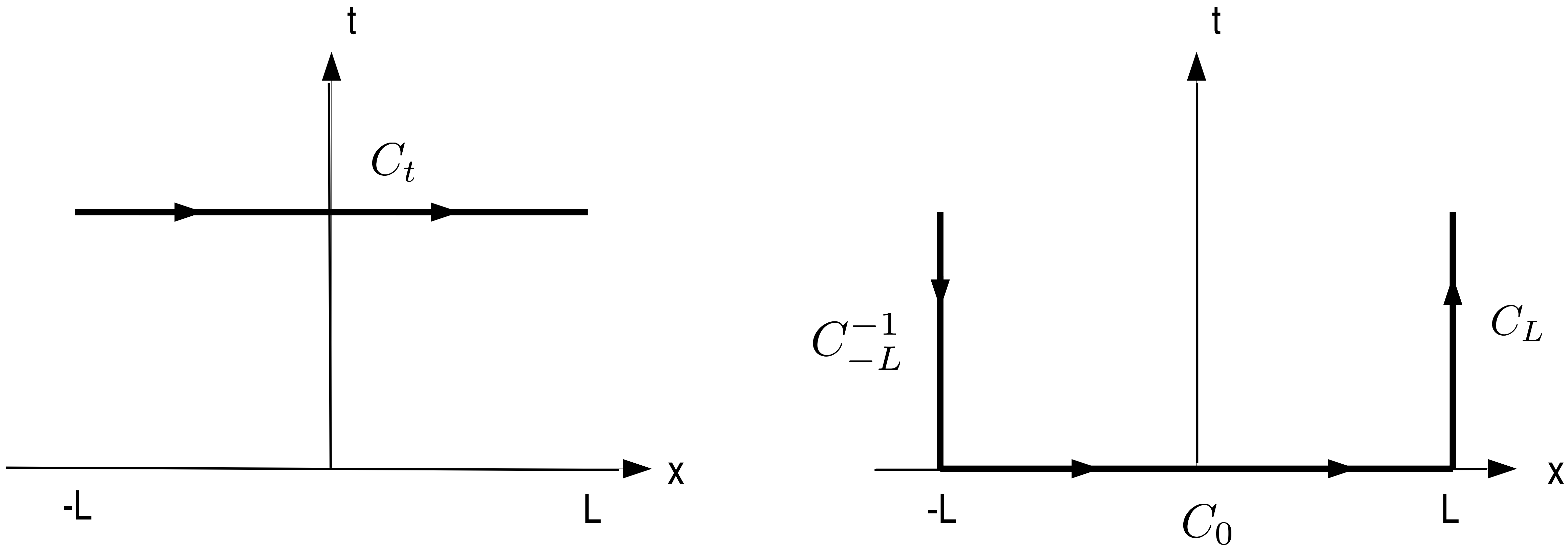

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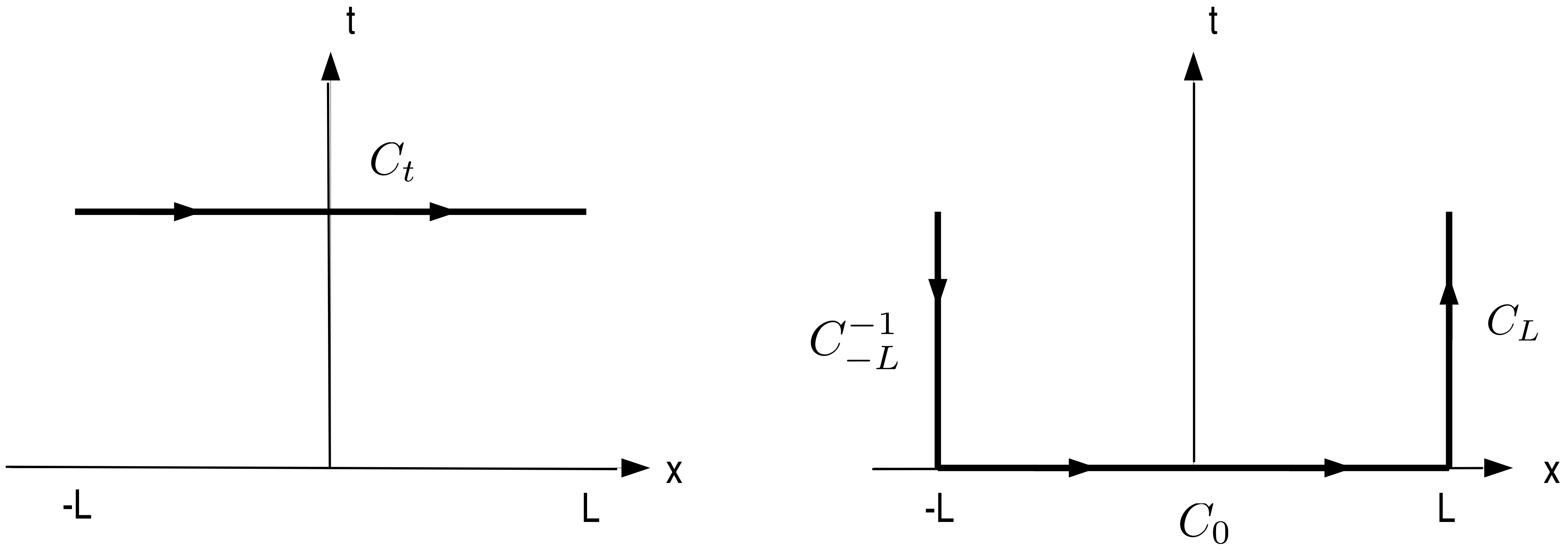

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Eigenvalues of $W(C_t)$ are conserved $\frac{d}{dt}\mathrm{Tr}\,[W(C_t)]^n = 0$

power series in $\lambda$: infinite number of conserved quantities

(no Coleman-Mandula)

# $2 + 1$ dimensions

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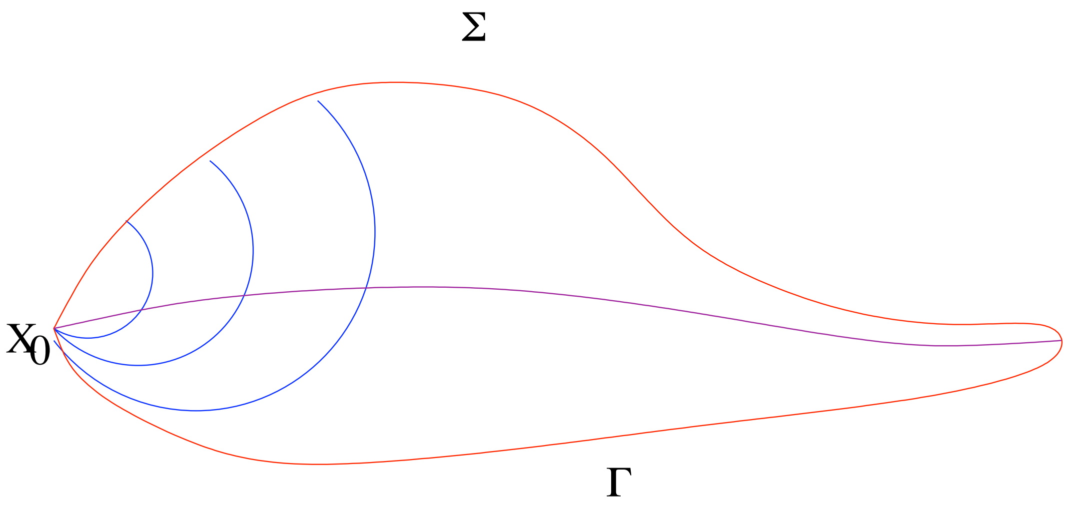
Charges should be integrals over 2d space



space-time surface

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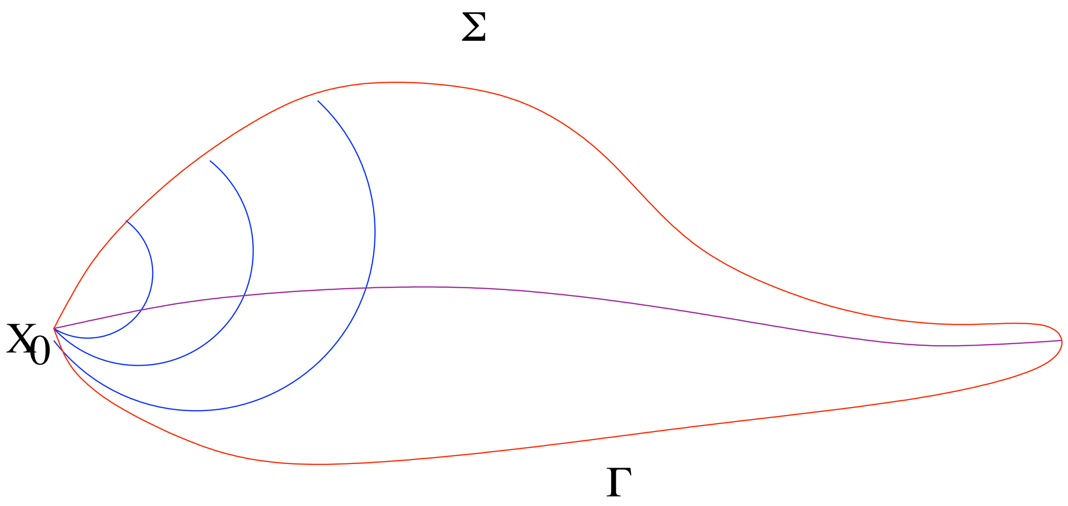

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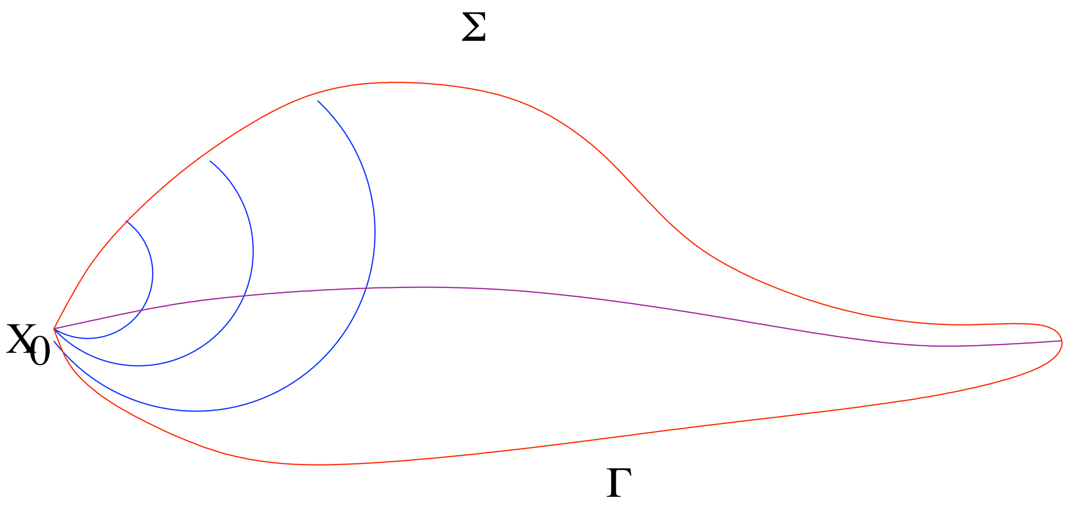

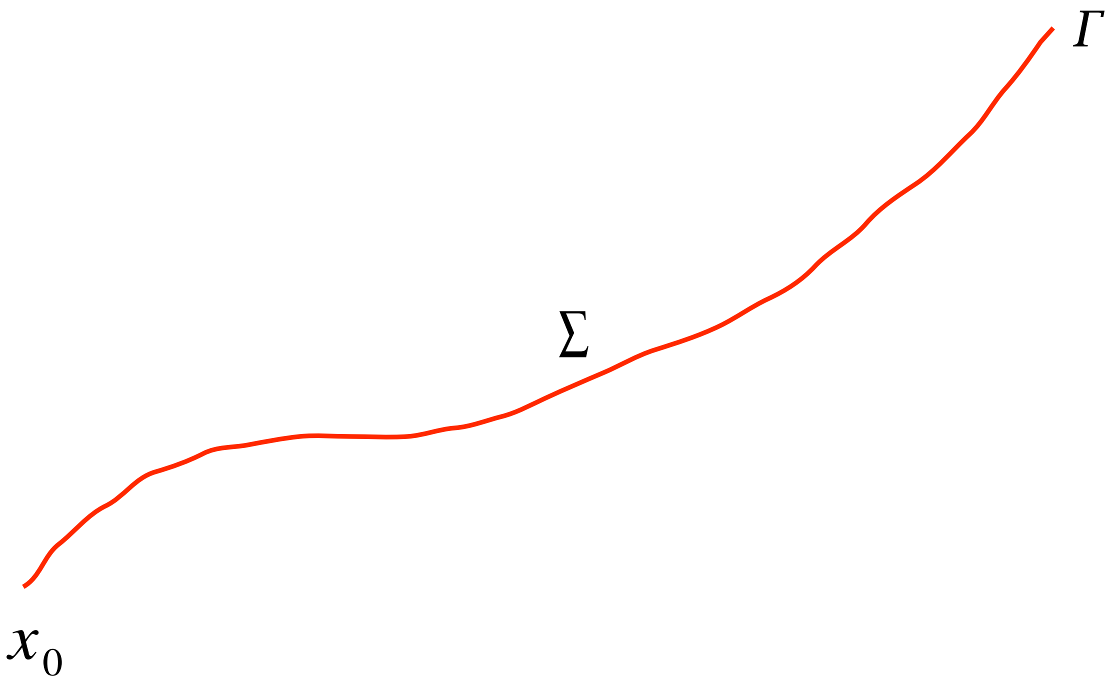

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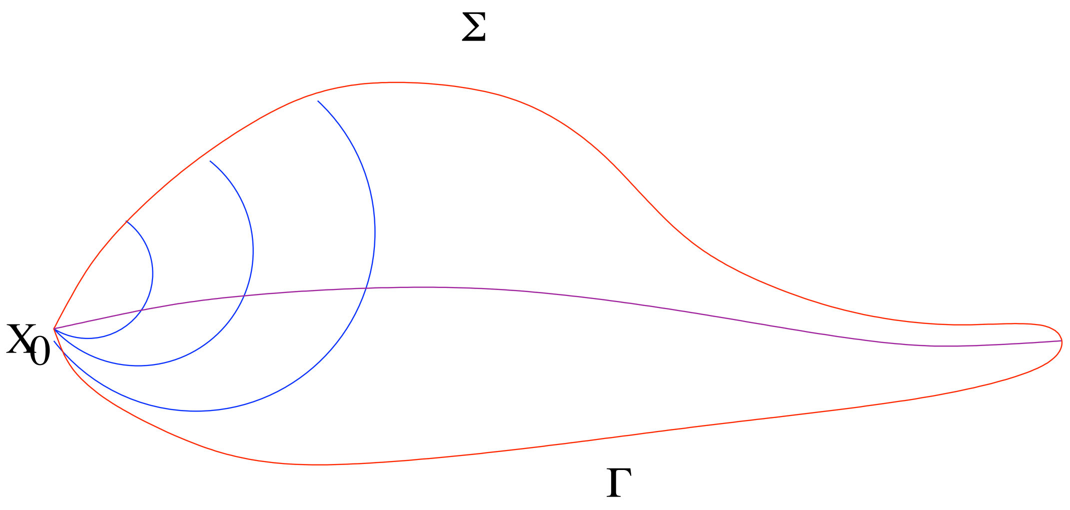

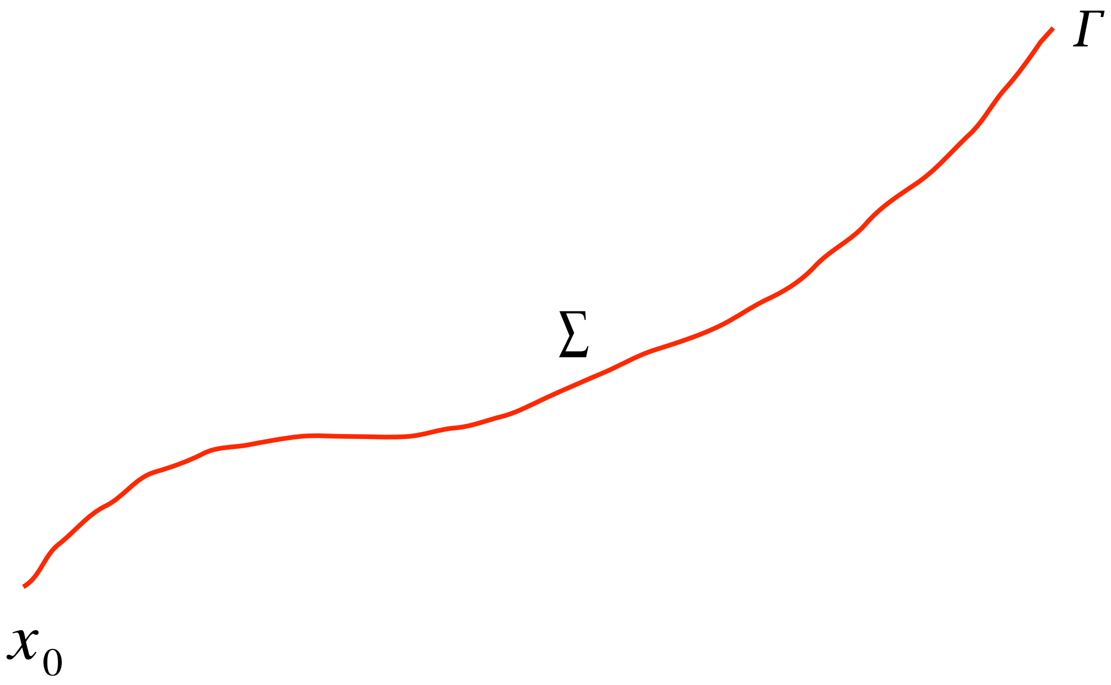

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**Loop Space:** $\Omega^{(1)} = \{ f : S^1 \to M \mid \text{north pole } \to x_0 \}$

Introduce a flat connection $\mathcal{A}$ in loop space

$$\mathcal{F} = \delta \mathcal{A} + \mathcal{A} \wedge \mathcal{A} = 0$$

Construct the charges using path independency!

# The one-form connection on loop space

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$$\mathcal{A}[x(\sigma)] = \int_0^{2\pi} d\sigma \; W(\sigma)^{-1} B_{\mu\nu}(x(\sigma)) W(\sigma) \frac{dx^\mu}{d\sigma} \delta x^\nu(\sigma)$$

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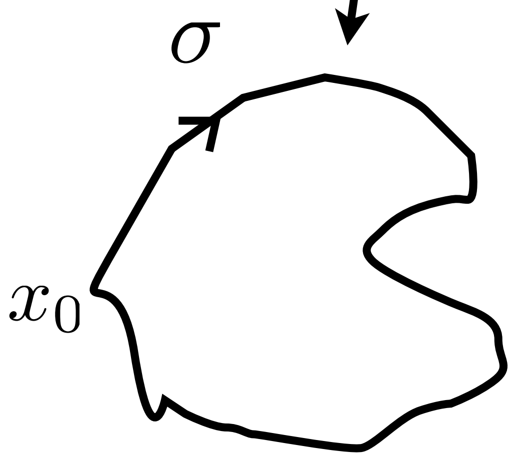

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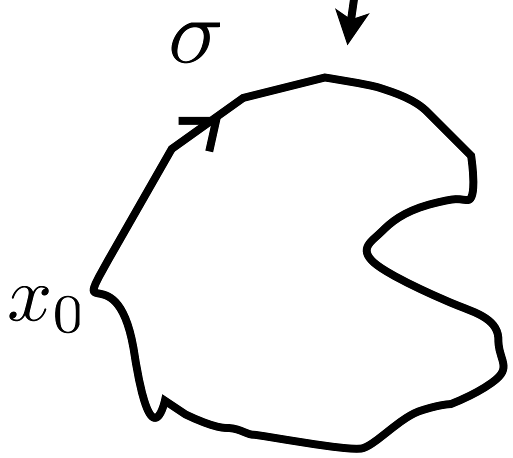

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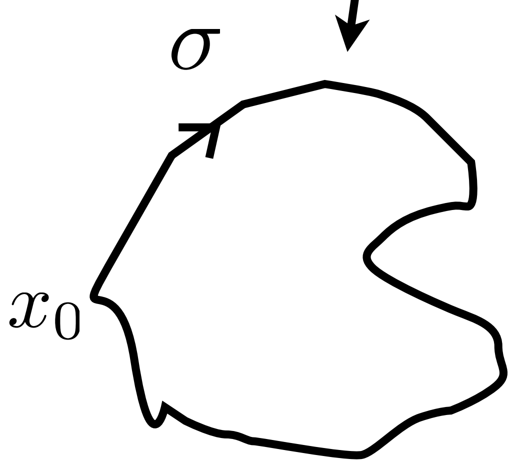

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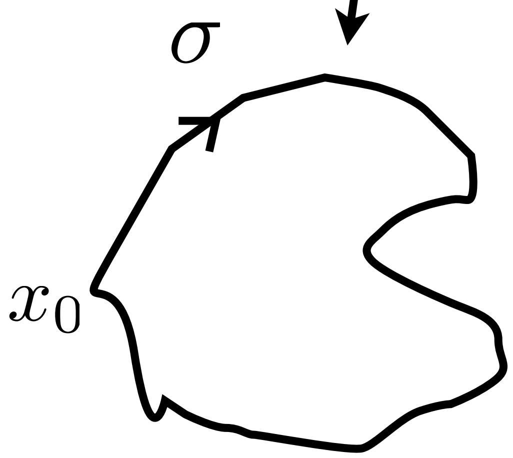

variations perpendicular to the loop

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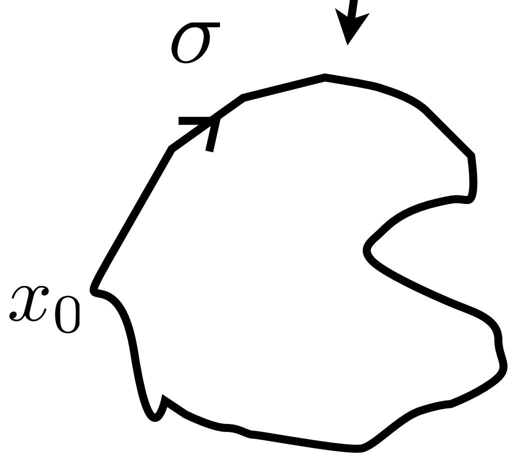

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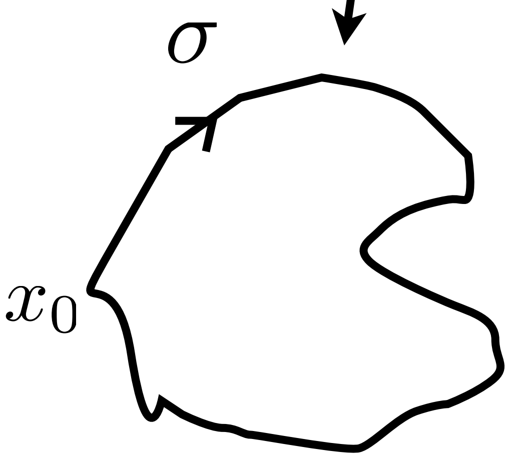

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$$B_{\mu\nu} \to g\, B_{\mu\nu}\, g^{-1} \qquad B^W_{\mu\nu} \to g(x_0)\, B^W_{\mu\nu}\, g^{-1}(x_0) \qquad (B^W_{\mu\nu} \equiv W^{-1}\, B_{\mu\nu}\, W)$$

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$$\mathcal{F} = \delta\mathcal{A} + \mathcal{A} \wedge \mathcal{A}$$

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Problems to have $\mathcal{F} = 0$:

- Non-locality
- Dependency upon reparameterization
- Hard to reconcile with local field theories

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$CP^1$-model
Skyrme model
Skyrme-Faddeev model
Self-dual YM, etc

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Orlando Alvarez, LAF and J. Sánchez Guillén
hep-th/9710147, *Nucl. Phys.* **B529** (1998) 689-736
*IJMPA*, **24** (2009) 1825 - 1888; arXiv:0901.1654 [hep-th].

## Local conditions:

$$[T_a\, ,\, T_b] = i\, f_{abc}\, T_c$$

$$[T_a\, ,\, P_i] = P_j\, R_{ji}(T_a)$$

$$[P_i\, ,\, P_j] = 0$$

$$A_\mu = A^a_\mu\, T_a \qquad F_{\mu\nu} = 0$$

$$B_{\mu\nu} = B^i_{\mu\nu}\, P_i \qquad D \wedge B = 0$$

$CP^1$-model
Skyrme model
Skyrme-Faddeev model
Self-dual YM, etc

# Revisit integrable field theories in $1+1$ dimensions

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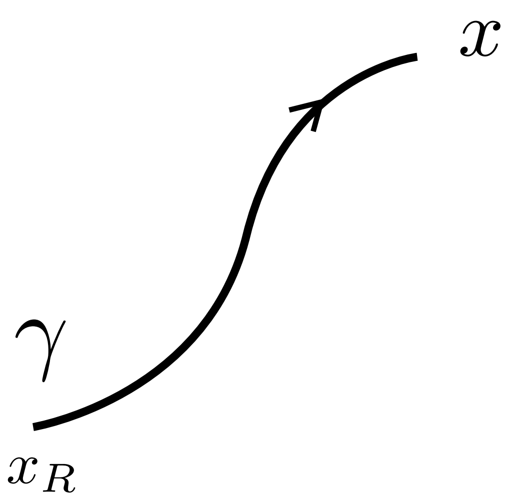

Let $g(x)$ and $A_\mu$ be functionals of the fields of the theory and impose on any curve $\gamma$ the integral equation

$$g\,(x)\;g^{-1}\,(x_R) = P_1\,e^{-\int_\gamma d\sigma\,A_\mu\,\frac{d\,x^\mu}{d\sigma}}$$

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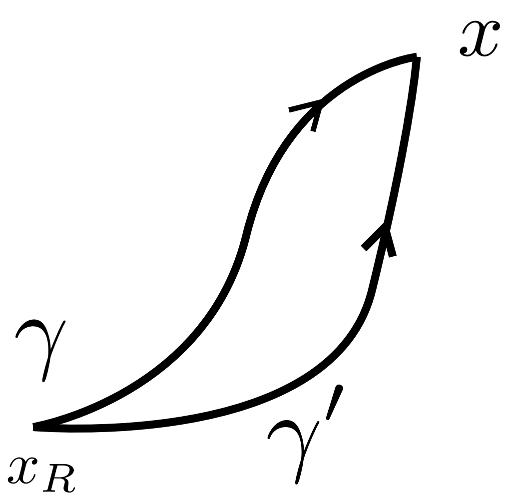

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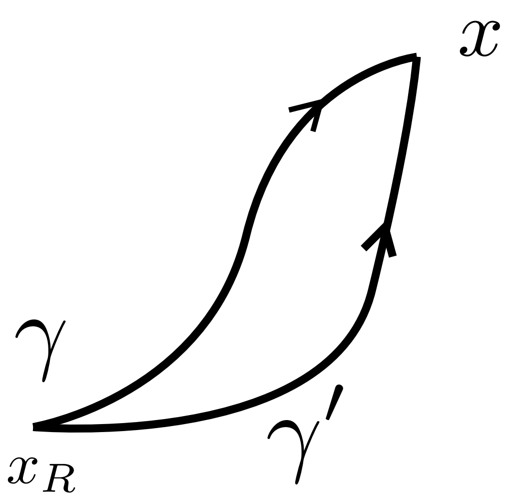

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path independent

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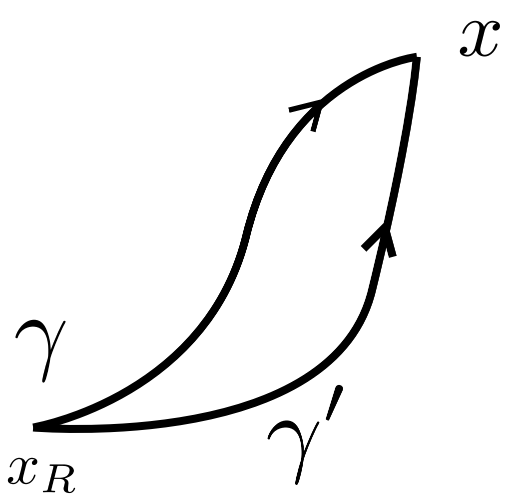

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path independent $\rightarrow$ conservation laws

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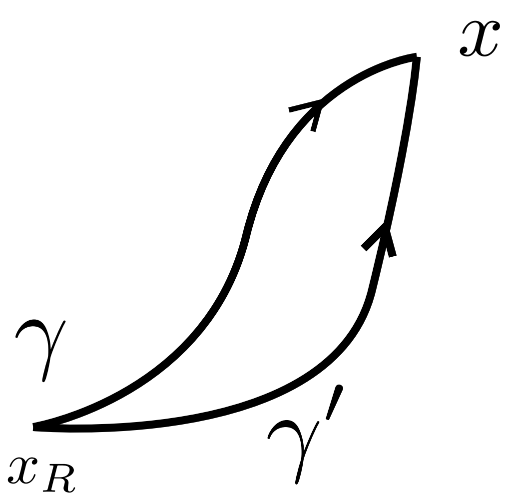

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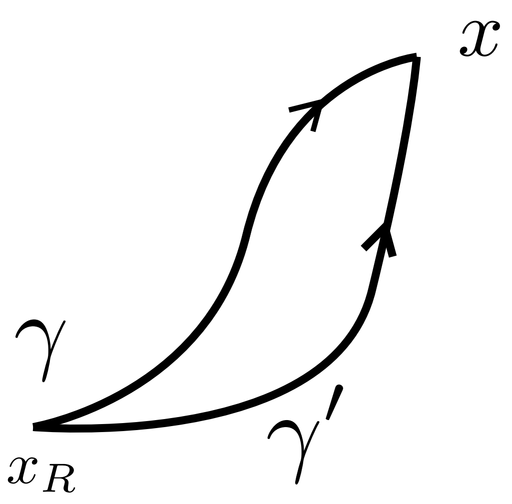

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So, $A_\mu$ is flat and we have Lax-Zakharov-Shabat equation

$$\partial_\mu A_\nu - \partial_\nu A_\mu + [A_\mu\,,\, A_\nu] = 0$$

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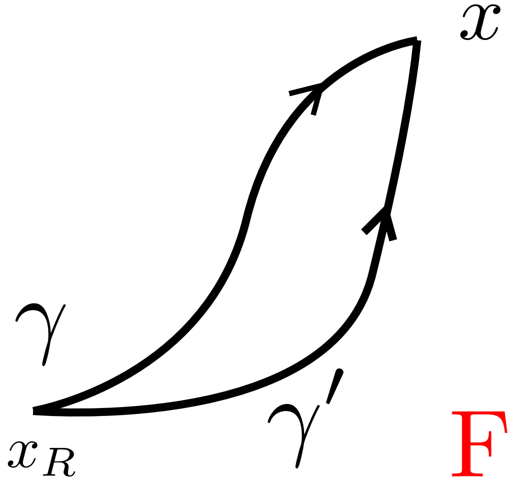

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Flux

charge

path independent

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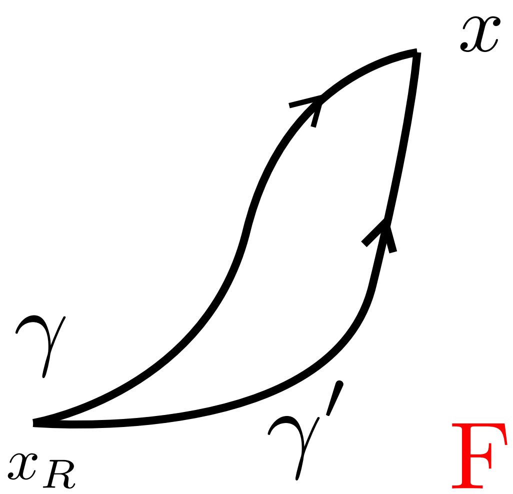

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Look for integral equations!! $P_{d-1}e^{\int_{\partial\Omega}\mathcal{A}} = P_d e^{\int_\Omega \mathcal{F}}$

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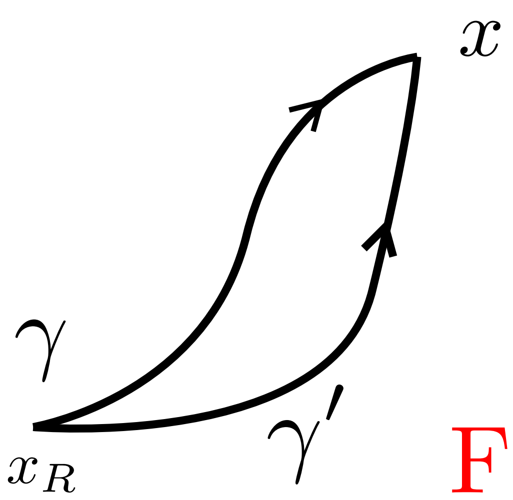

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Basic property of gauge theories: Flux=Charge

# Non-Abelian Stokes Theorem

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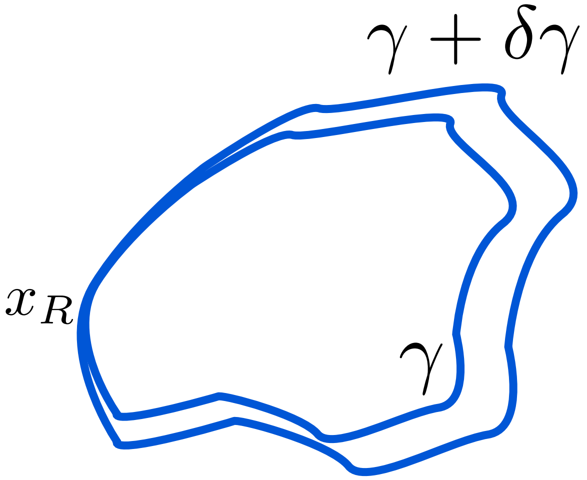

$$W^{-1}(\gamma)\delta W(\gamma) = \int_0^{2\pi} d\sigma\, W^{-1} F_{\mu\nu} W \frac{dx^\mu}{d\sigma}\delta x^\nu$$

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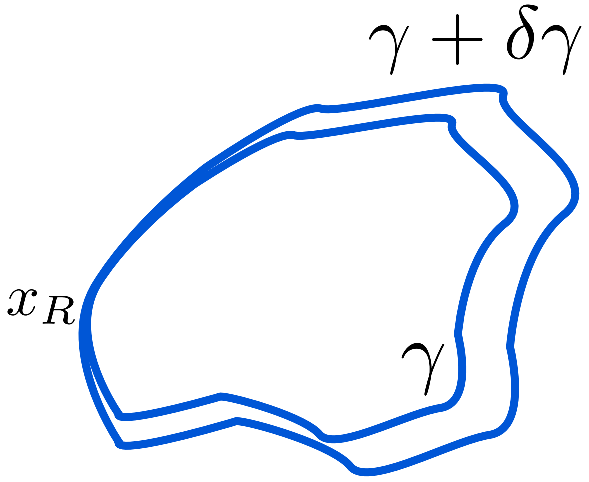

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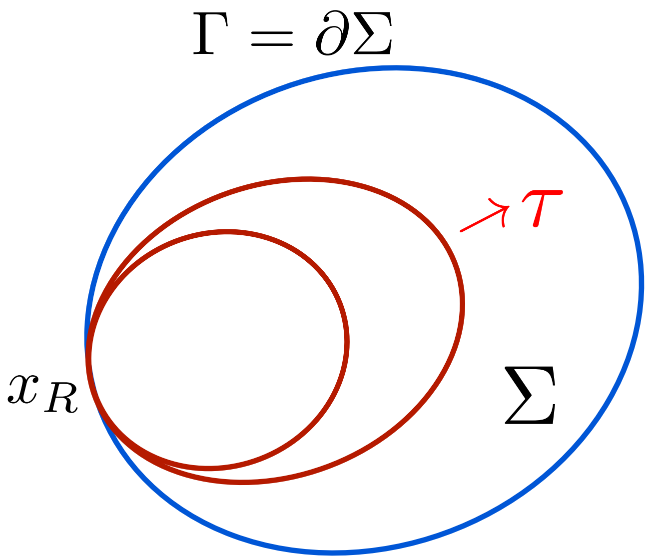

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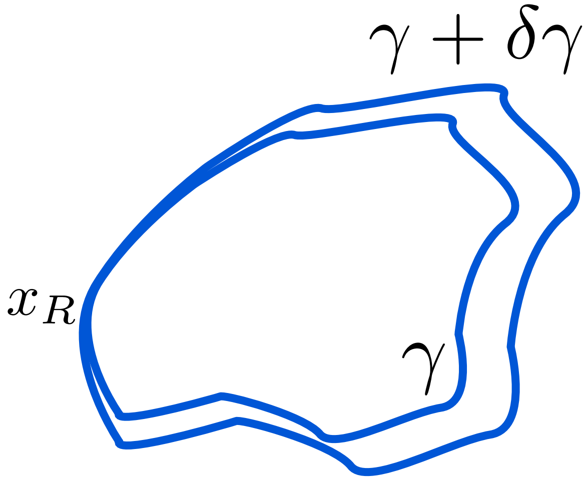

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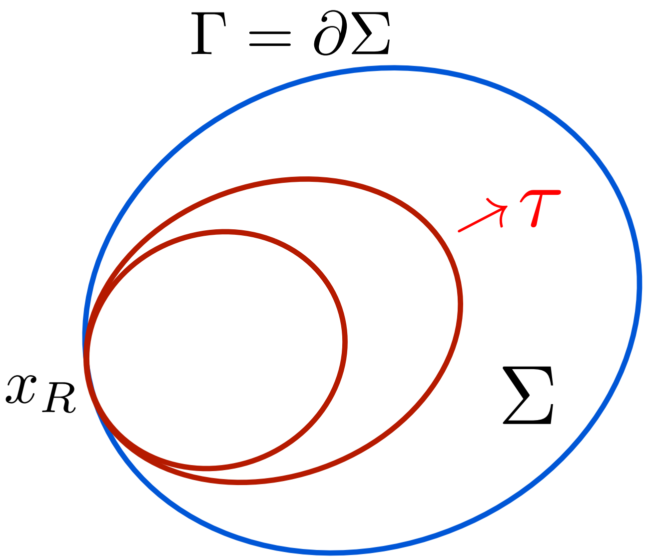

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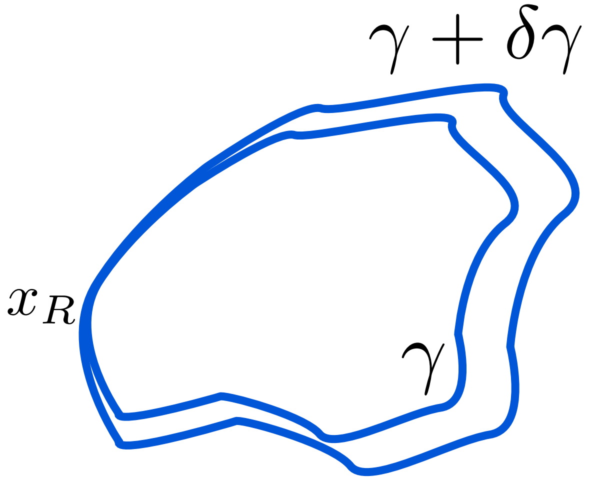

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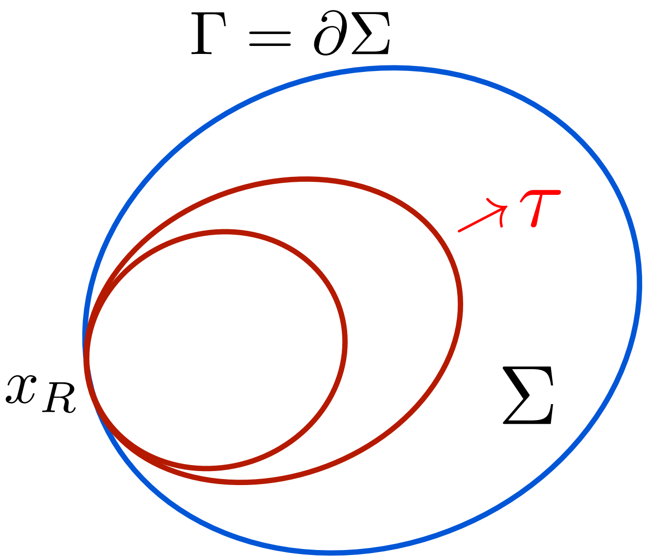

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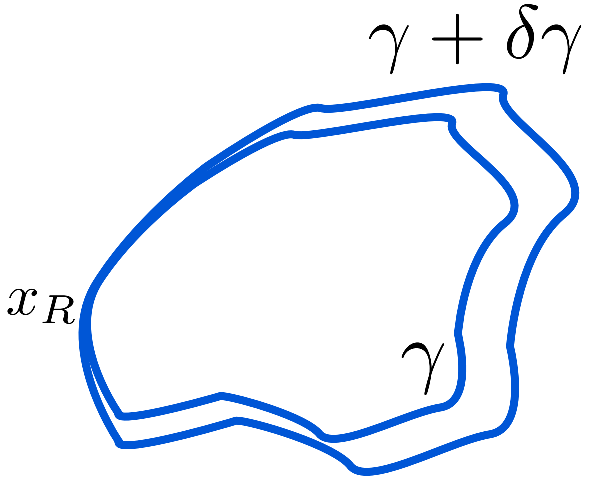

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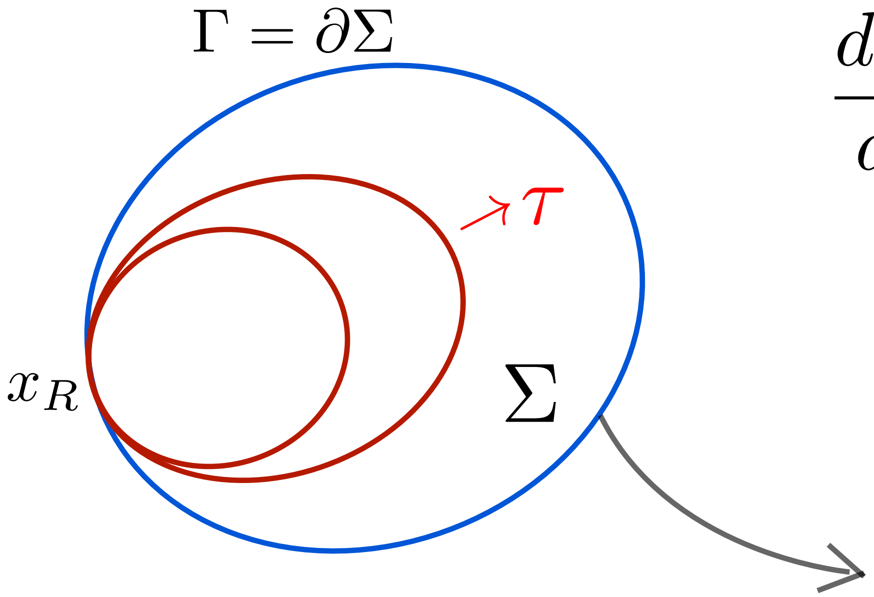

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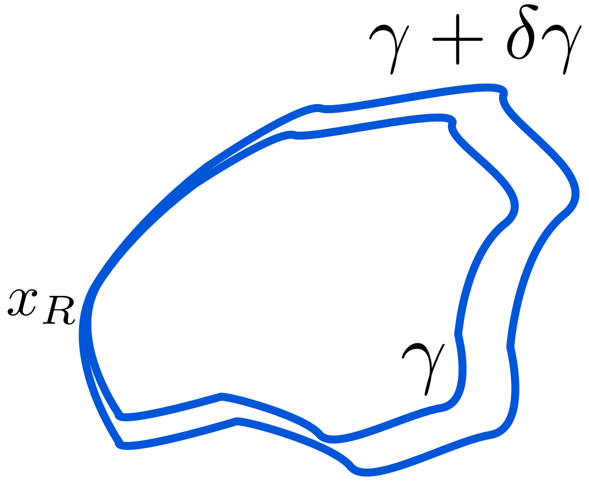

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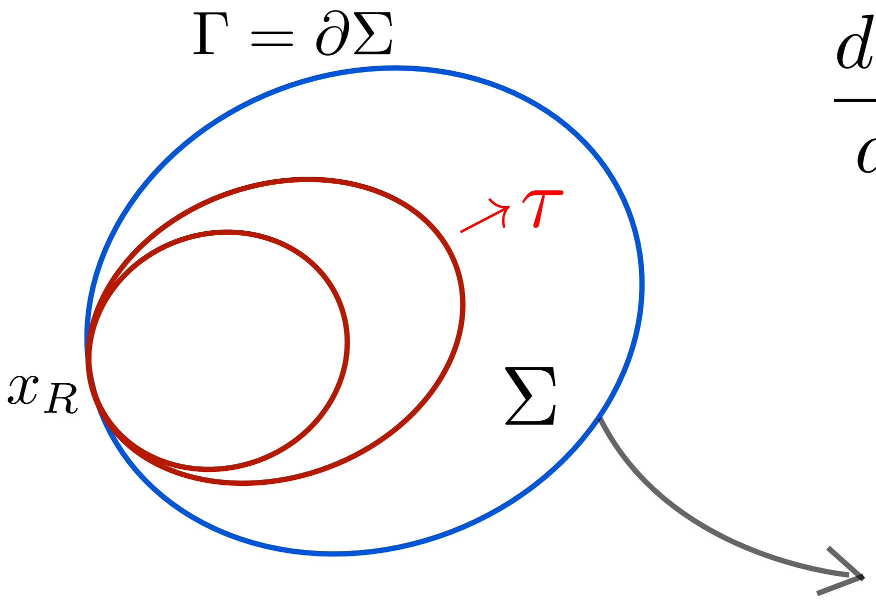

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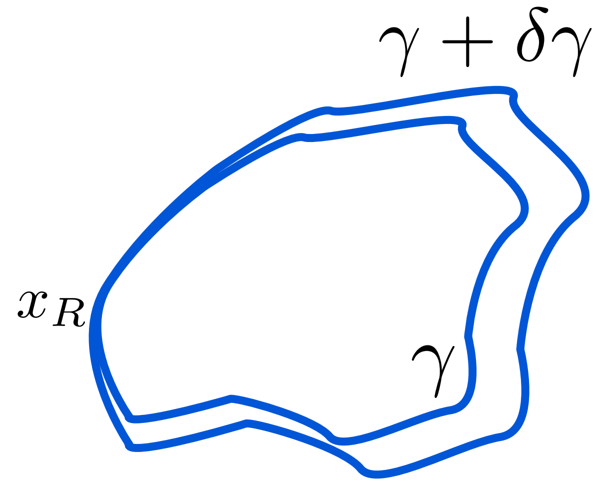

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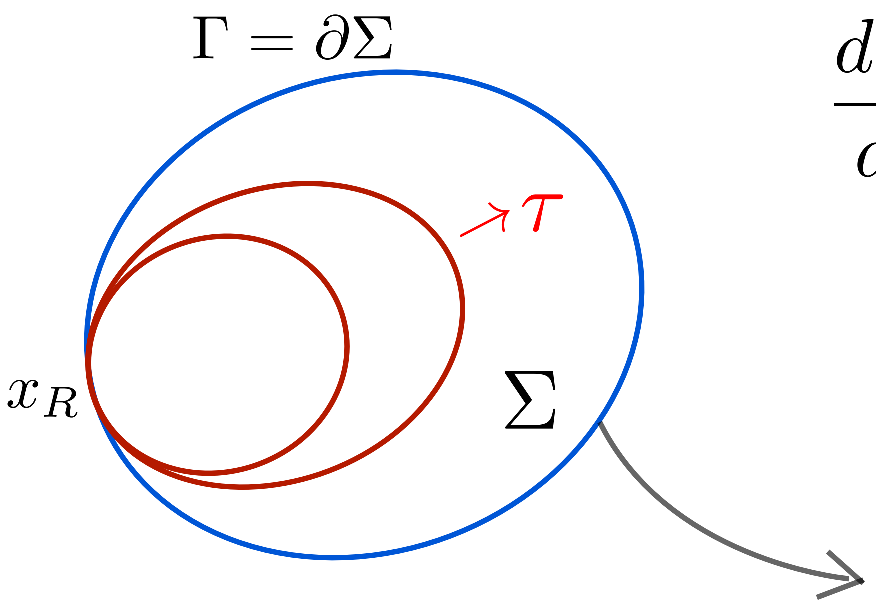

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$$\int_{\partial\Sigma} A = \int_\Sigma d \wedge A$$

# An example: Chern-Simons theory

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$A_\mu \in$ Lie algebra $\mathcal{G}$

Eq. of motion → $$F_{\mu\nu} = \frac{1}{\kappa}\varepsilon_{\mu\nu\rho}J^\rho \equiv \frac{1}{\kappa}\tilde{J}_{\mu\nu}$$

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For an infinitesimal $\Sigma$ one gets the differential equation $F_{\mu\nu} = \tilde{J}_{\mu\nu}$

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For a closed surface $\Sigma_c$ the integral equation implies

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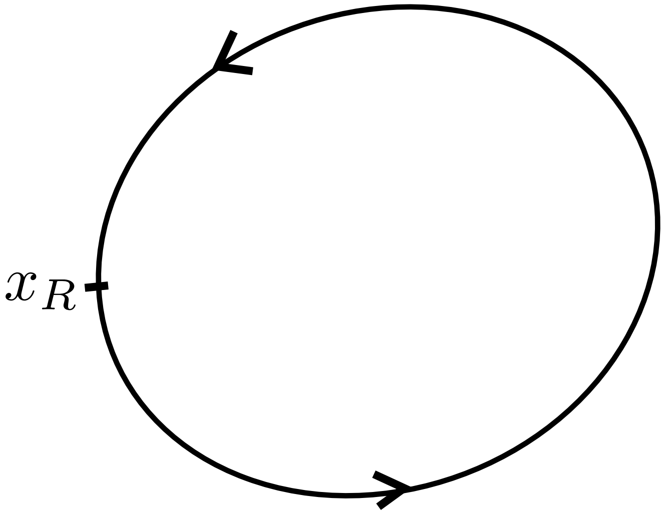
On the loop space $\Sigma_c \equiv$ closed path



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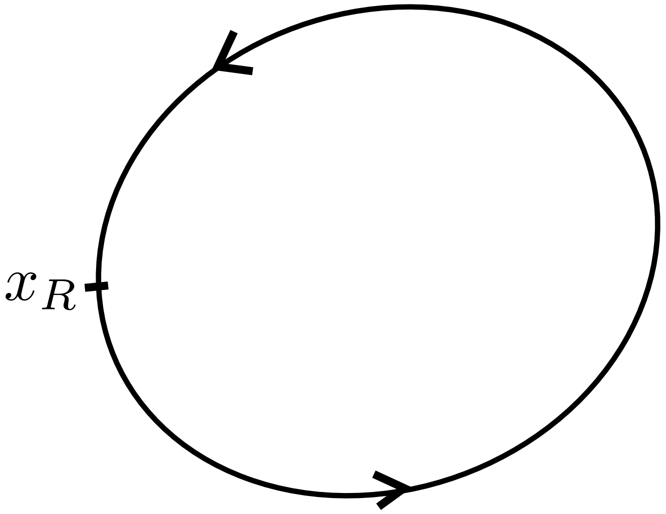

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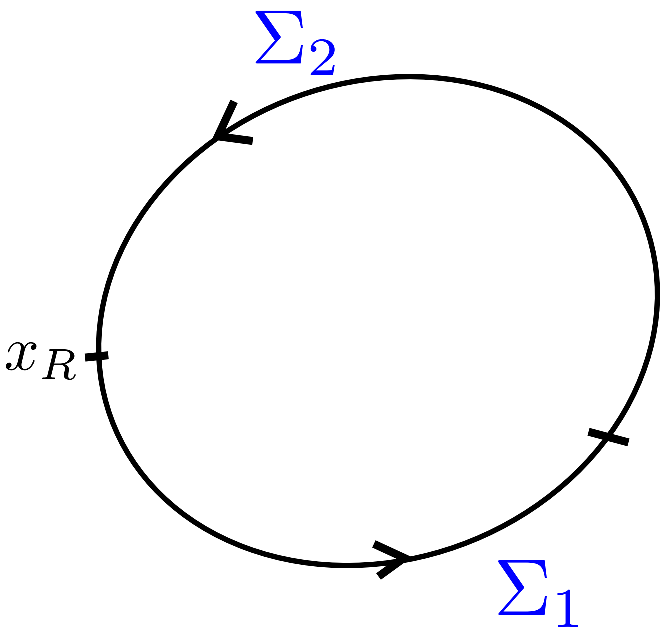

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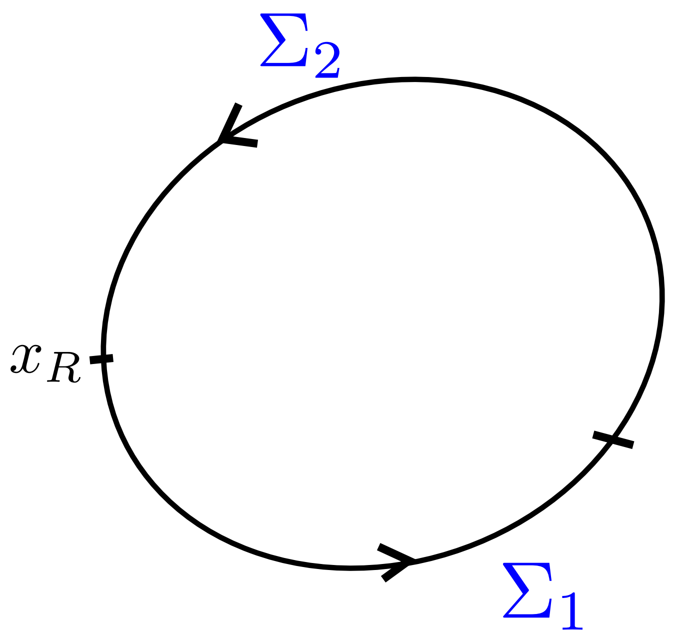

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$$\searrow$$

$$P_2\, e^{\int_{\Sigma_1} \mathcal{A}} = P_2\, e^{\int_{\Sigma_2^{-1}} \mathcal{A}}$$

Path (surface) independency

# Construction of conserved charges

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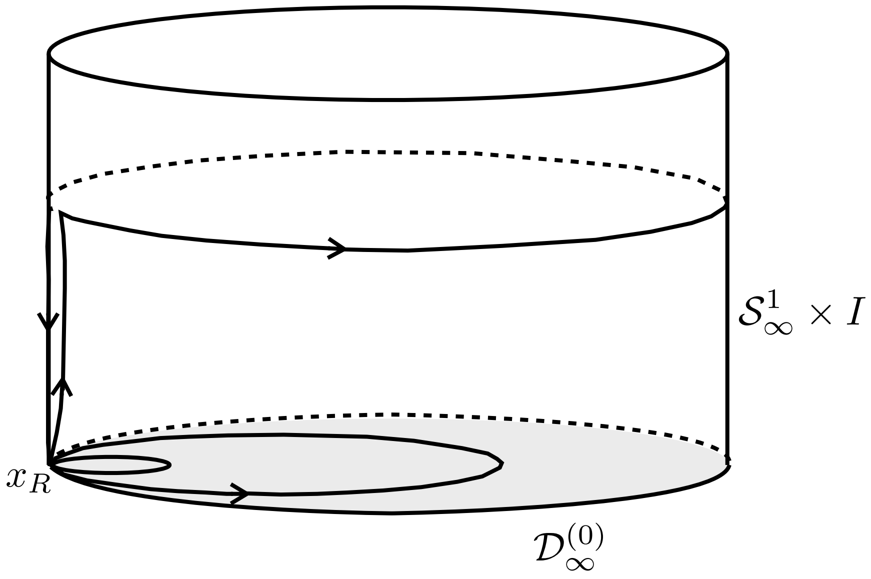
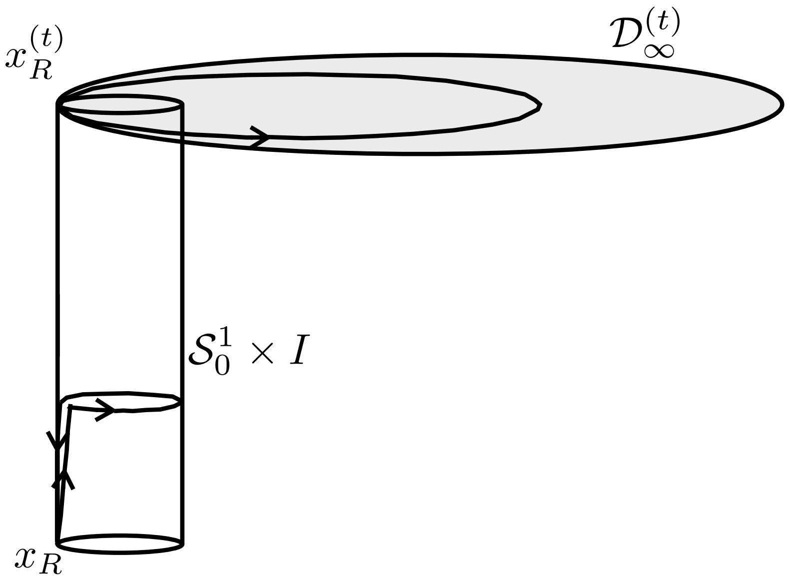


Surface $\Sigma_1 = \mathcal{D}^{(0)}_\infty \cup \left(S^1_\infty \times I\right)$



Surface $\Sigma_2 = \left(S^1_0 \times I\right) \cup \mathcal{D}^{(t)}_\infty$

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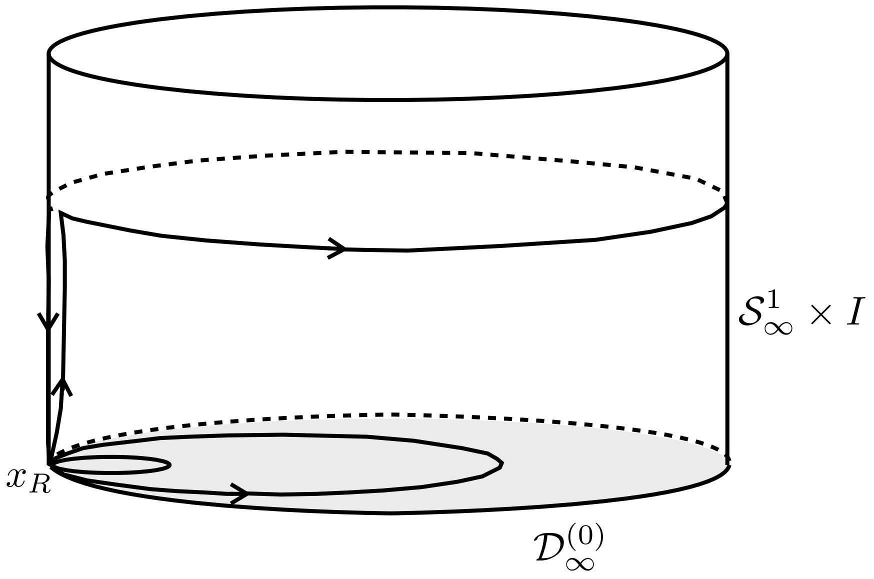

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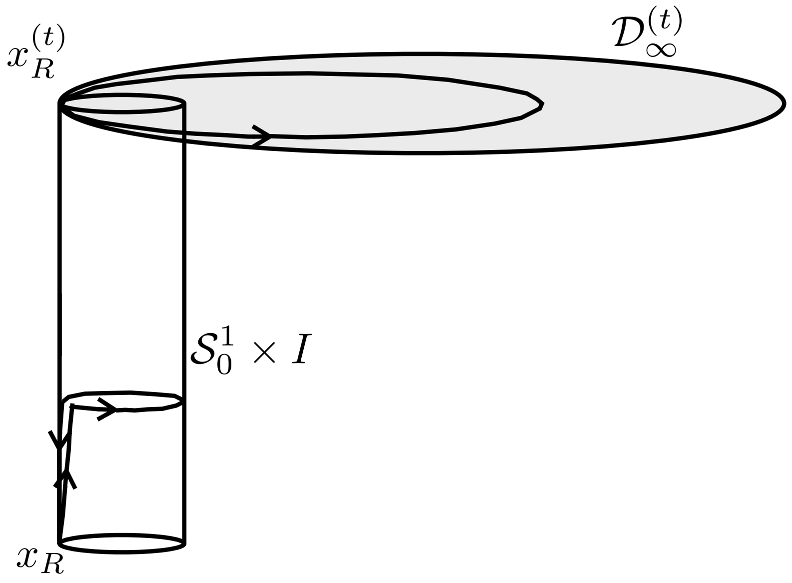
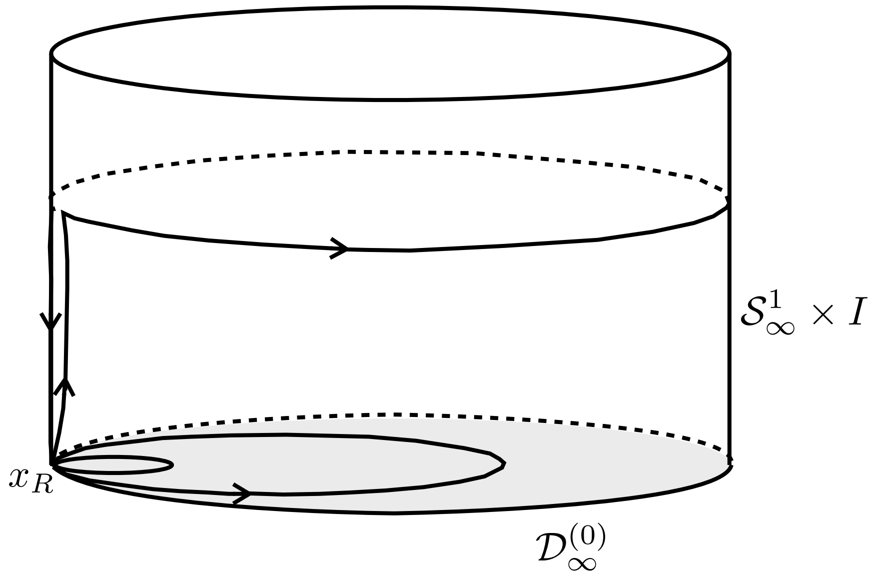


Surface $\Sigma_2 = \left(S^1_0 \times I\right) \cup \mathcal{D}^{(t)}_\infty$

Path independency: $P_2\, e^{\int_{\mathcal{D}^{(0)}_\infty} \mathcal{A}}\, P_2\, e^{\int_{S^1_\infty \times I} \mathcal{A}} = P_2\, e^{\int_{S^1_0 \times I} \mathcal{A}}\, P_2\, e^{\int_{\mathcal{D}^{(t)}_\infty} \mathcal{A}}$

# Construction of conserved charges



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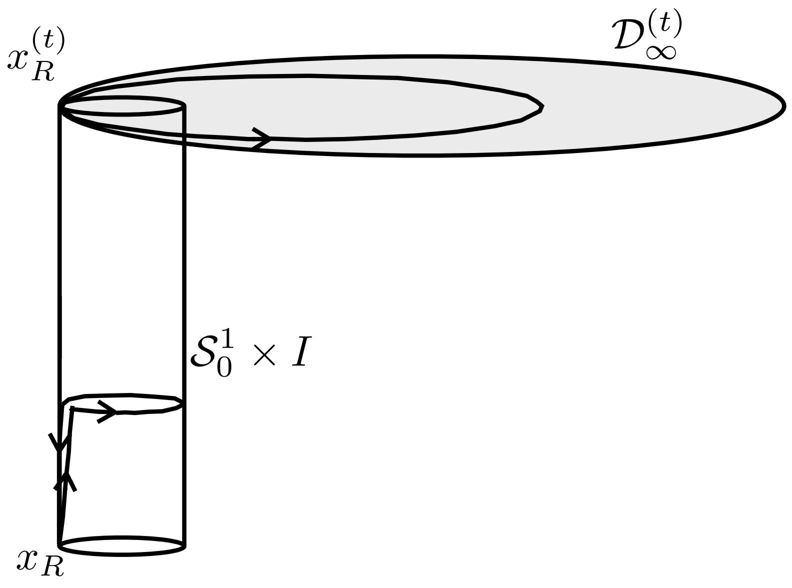

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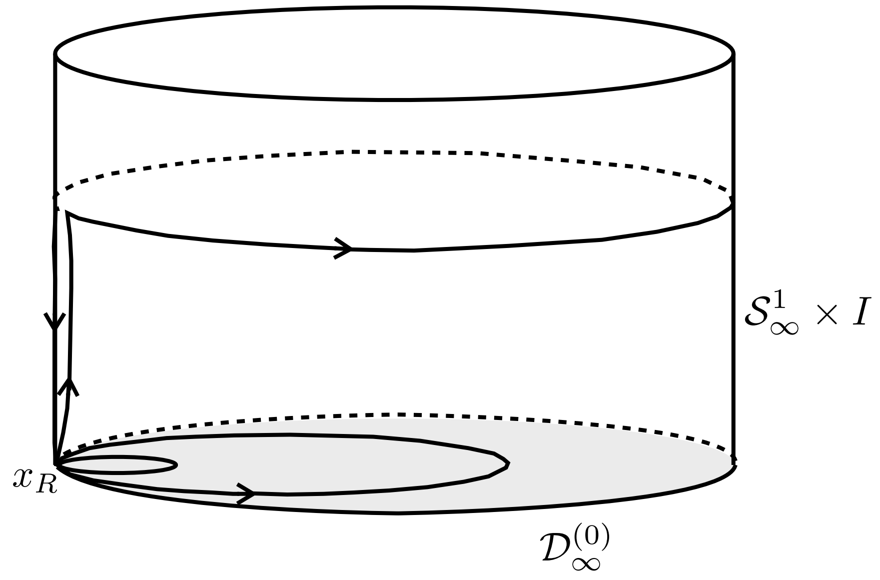

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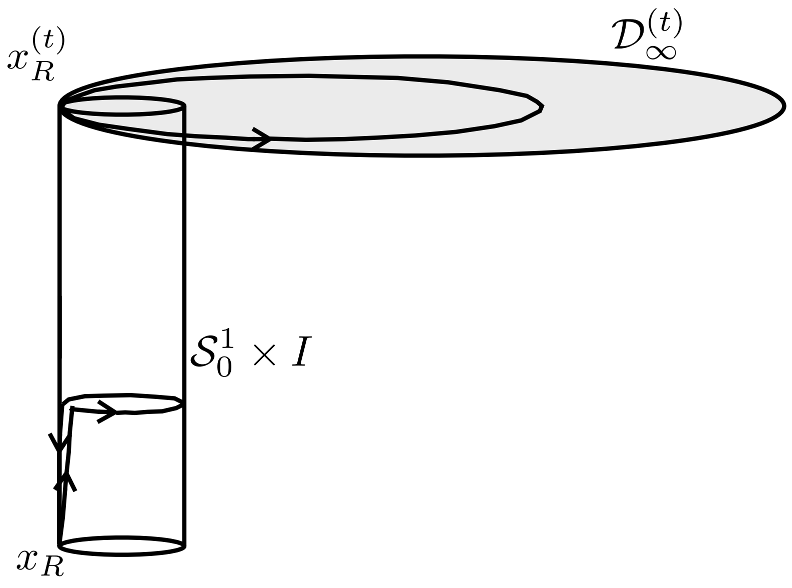

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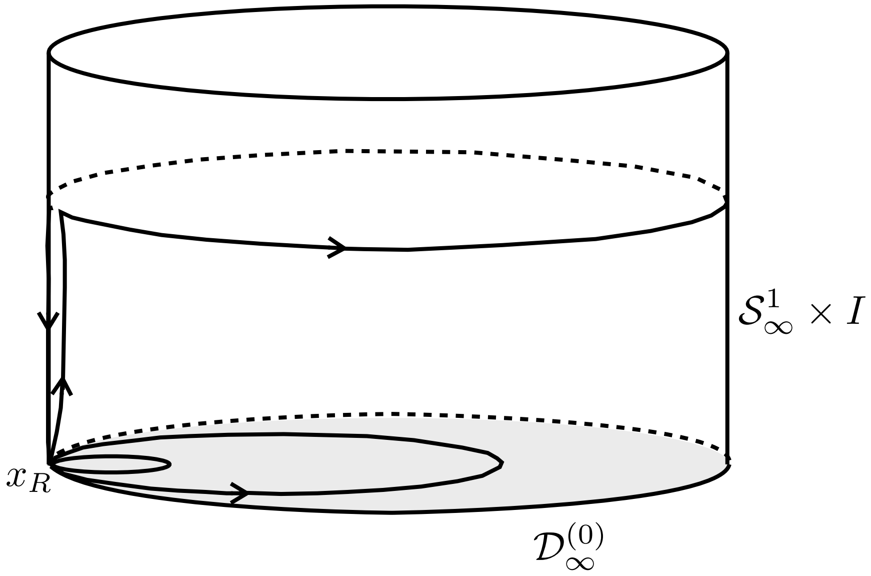
$$P_2\, e^{\int_{\mathcal{D}^{(t)}_\infty} \mathcal{A}}\, |_{x_R^{(t)}} = W\left(x_R^{(t)}, x_R\right)\, P_2\, e^{\int_{\mathcal{D}^{(t)}_\infty} \mathcal{A}}\, |_{x_R}\; W^{-1}\left(x_R^{(t)}, x_R\right)$$

# Construction of conserved charges



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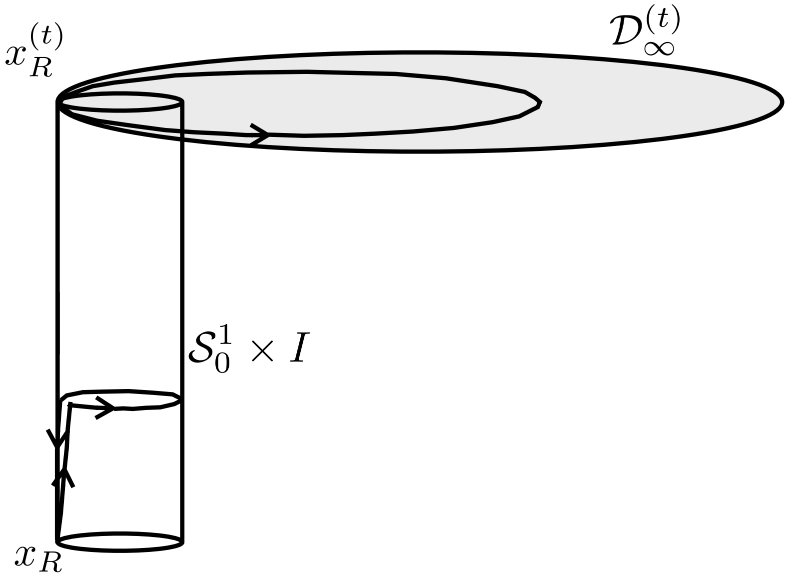

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Conserved charges → eigenvalues of the operator:

$$\boxed{V_{x_R^{(t)}}\left(\mathcal{D}^{(t)}_\infty\right) = P_2\, e^{\frac{ie}{\kappa}\int_{\mathcal{D}^{(t)}_\infty} d\tau\, d\sigma\, W^{-1} \widetilde{J}_{\mu\nu} W \frac{dx^\mu}{d\sigma} \frac{dx^\nu}{d\tau}} = P_1 e^{-ie \oint_{S^1_\infty} d\sigma\, A_\mu \frac{dx^\mu}{d\sigma}}}$$

# Integral Equations for Yang-Mills in $(2+1)$-Dimensions

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$$P_1 e^{-ie \oint_{\partial\Sigma} d\sigma \, \left(A_\mu + \beta \, \widetilde{F}_\mu\right) \frac{dx^\mu}{d\sigma}} = P_2 \, e^{ie \int_\Sigma d\tau \, d\sigma \, W^{-1} \left(F_{\mu\nu} - \beta \, \widetilde{J}_{\mu\nu} + ie \, \beta^2 \left[\widetilde{F}_\mu \, , \widetilde{F}_\nu\right]\right) W \frac{dx^\mu}{d\sigma} \frac{dx^\nu}{d\tau}}$$

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Conserved charges are obtained the same way as for Chern-Simons

# Generalizing Faraday: Non-Abelian integrals

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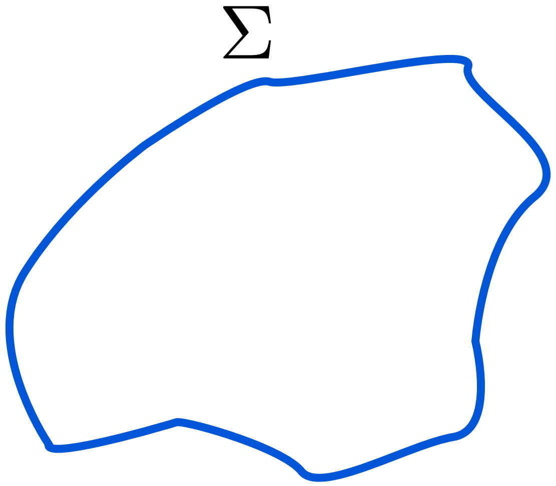

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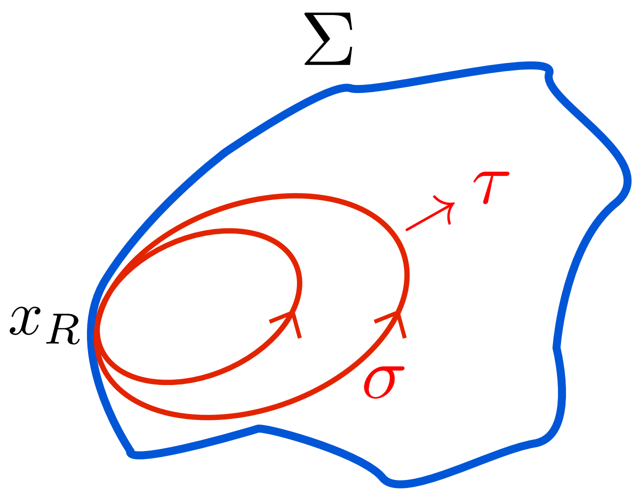

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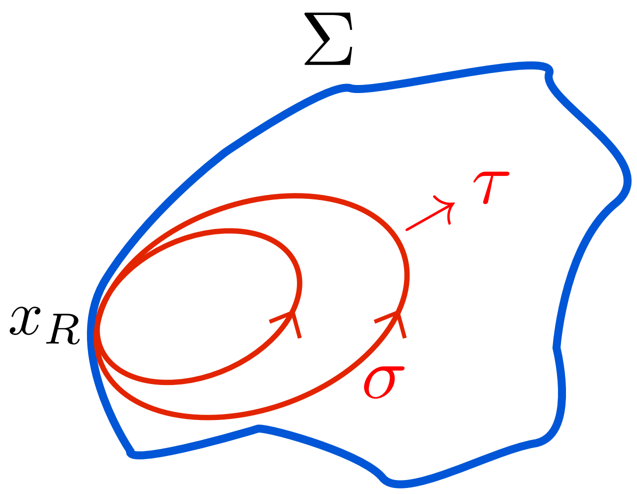

$$\frac{d\,V}{d\,\tau} - V\,T\,(A, B, \tau) = 0$$

$$T\,(B, A, \tau) \equiv \int_0^{2\,\pi} d\sigma\;W^{-1}\,B_{\mu\nu}\,W\,\frac{d\,x^{\mu}}{d\,\sigma}\,\frac{d\,x^{\nu}}{d\,\tau}$$

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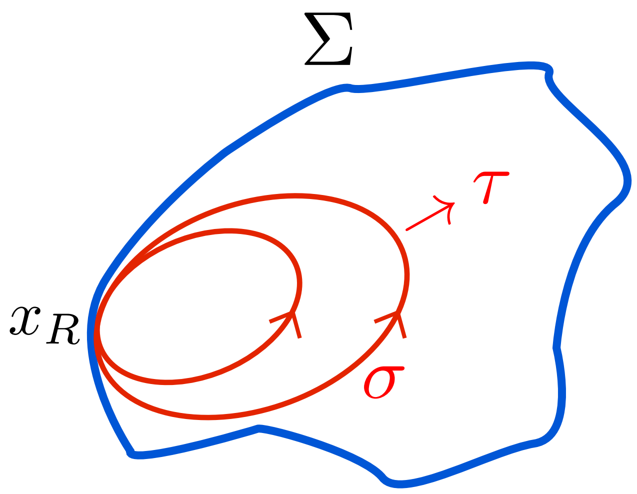

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It is a surface ordered integral

$$V(\Sigma) = V_R\; P_2\, e^{\int_{\Sigma} d\sigma d\tau\; W^{-1}\; B_{\mu\nu}\; W\; \frac{d\,x^{\mu}}{d\,\sigma}\; \frac{d\,x^{\nu}}{d\,\tau}}$$

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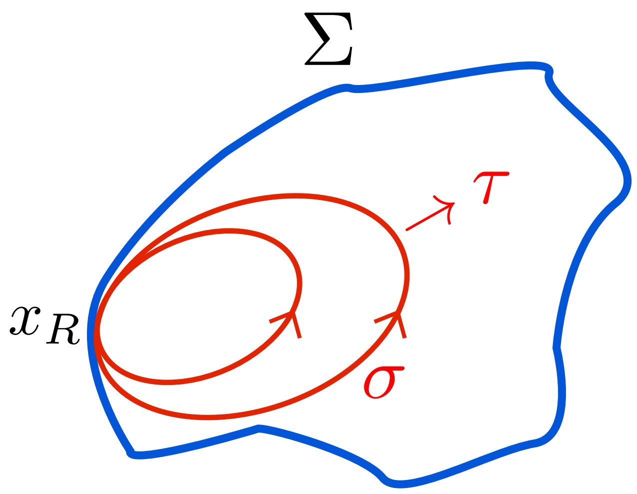

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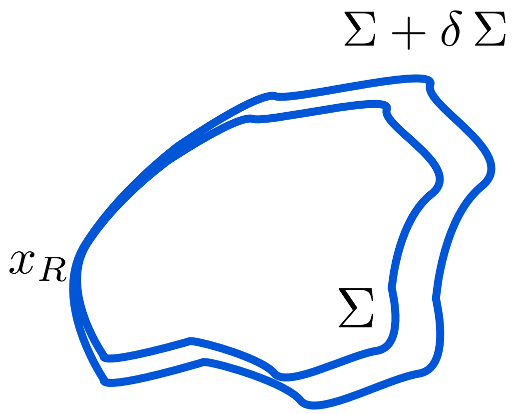

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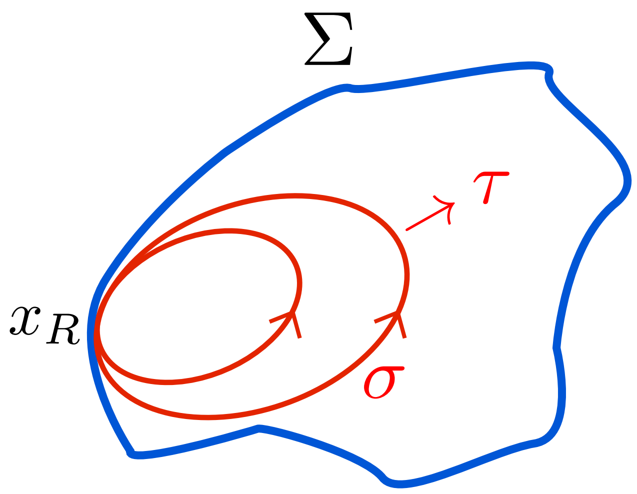

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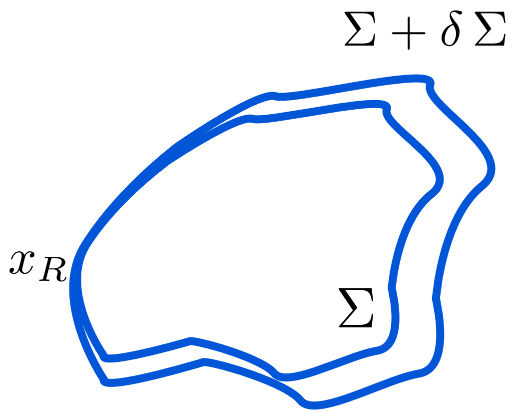

$$\delta V\, V^{-1} \equiv \int_0^{2\pi} d\tau \int_0^{2\pi} d\sigma\, V\,(\tau)\; \{$$

$$W^{-1}\; [D_\lambda B_{\mu\nu} + D_\mu B_{\nu\lambda} + D_\nu B_{\lambda\mu}]\; W \frac{d\,x^\mu}{d\,\sigma}\, \frac{d\,x^\nu}{d\,\tau}\, \delta x^\lambda$$

$$- \int_0^\sigma d\sigma' \left[\, B^W_{\kappa\rho}\,(\sigma') - ieF^W_{\kappa\rho}\,(\sigma')\,,\, B^W_{\mu\nu}\,(\sigma)\, \right] \frac{dx^\kappa}{d\sigma'}\, \frac{dx^\mu}{d\sigma}$$

$$\times \left( \frac{d\,x^\rho\,(\sigma')}{d\,\tau} \delta x^\nu\,(\sigma) - \delta x^\rho\,(\sigma')\, \frac{d\,x^\nu\,(\sigma)}{d\,\tau} \right) \Big\}\, V^{-1}\,(\tau)$$

# The generalized non-abelian Stokes Theorem

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$$\boxed{V_R\, P_2 e^{\int_{\partial\Omega} d\tau d\sigma W^{-1} B_{\mu\nu} W \frac{dx^{\mu}}{d\sigma} \frac{d\,x^{\nu}}{d\,\tau}} = P_3 e^{\int_{\Omega} d\zeta\, \mathcal{K}}\, V_R}$$

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$$\begin{aligned}
\mathcal{K} \equiv & \int_0^{2\,\pi} d\tau \int_0^{2\,\pi} d\sigma\, V\,(\tau)\, \Big\{ \\
& W^{-1}\, \left[D_{\lambda} B_{\mu\nu} + D_{\mu} B_{\nu\lambda} + D_{\nu} B_{\lambda\mu}\right] W \frac{d\,x^{\mu}}{d\,\sigma}\, \frac{d\,x^{\nu}}{d\,\tau}\, \frac{d\,x^{\lambda}}{d\,\zeta} \\
& - \int_0^{\sigma} d\sigma'\, \left[\, B^{W}_{\kappa\rho}\,(\sigma') - ieF^{W}_{\kappa\rho}\,(\sigma')\,,\, B^{W}_{\mu\nu}\,(\sigma)\,\right] \frac{dx^{\kappa}}{d\sigma'}\, \frac{dx^{\mu}}{d\sigma} \\
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O. Alvarez, L. A. Ferreira and J. Sanchez Guillen,
Nucl. Phys. B **529**, 689 (1998) [arXiv:hep-th/9710147].
Int. J. Mod. Phys. A **24**, 1825 (2009) [arXiv:0901.1654 [hep-th]]

# The Integral Equations for Yang-Mills

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$$P_2 e^{ie \int_{\partial\Omega} d\tau d\sigma \left[\alpha F^{W}_{\mu\nu} + \beta \widetilde{F}^{W}_{\mu\nu}\right] \frac{dx^{\mu}}{d\sigma} \frac{dx^{\nu}}{d\tau}} = P_3 e^{\int_{\Omega} d\zeta d\tau V \mathcal{J} V^{-1}}$$

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\mathcal{J} \equiv \int_0^{2\pi} d\sigma \Bigg\{ & ie\beta \widetilde{J}^W_{\mu\nu\lambda} \frac{dx^\mu}{d\sigma} \frac{dx^\nu}{d\tau} \frac{dx^\lambda}{d\zeta} + e^2 \int_0^\sigma d\sigma' \\
& \times \left[ \left( (\alpha - 1)\, F^W_{\kappa\rho} + \beta \widetilde{F}^W_{\kappa\rho} \right) (\sigma') \, , \left( \alpha F^W_{\mu\nu} + \beta \widetilde{F}^W_{\mu\nu} \right) (\sigma) \right] \\
& \times \frac{d\, x^\kappa}{d\, \sigma'} \frac{d\, x^\mu}{d\, \sigma} \left( \frac{d\, x^\rho\, (\sigma')}{d\, \tau} \frac{d\, x^\nu\, (\sigma)}{d\, \zeta} - \frac{d\, x^\rho\, (\sigma')}{d\, \zeta} \frac{d\, x^\nu\, (\sigma)}{d\, \tau} \right) \Bigg\}
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\mathcal{J} \equiv \int_0^{2\pi} d\sigma \Bigg\{ & ie\beta \widetilde{J}^W_{\mu\nu\lambda}\frac{dx^\mu}{d\sigma}\frac{dx^\nu}{d\tau}\frac{dx^\lambda}{d\zeta} + e^2 \int_0^\sigma d\sigma' \\
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$$B_{\mu\nu} \rightarrow \alpha F_{\mu\nu} + \beta \widetilde{F}_{\mu\nu}$$

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\end{aligned}$$

$$B_{\mu\nu} \to \alpha F_{\mu\nu} + \beta \widetilde{F}_{\mu\nu} \qquad D^\mu F_{\mu\nu} = J_\nu \qquad D^\mu \widetilde{F}_{\mu\nu} = 0$$

$$J^\mu = \frac{1}{3!} \varepsilon^{\mu\nu\rho\lambda}\, \widetilde{J}_{\nu\rho\lambda}$$

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$B_{\mu\nu} \rightarrow \alpha F_{\mu\nu} + \beta \widetilde{F}_{\mu\nu}$ $\qquad D^\mu F_{\mu\nu} = J_\nu$ $\qquad D^\mu \widetilde{F}_{\mu\nu} = 0$

Direct consequence of Stokes theorem and Yang-Mills eqs.

$J^\mu = \frac{1}{3!}\varepsilon^{\mu\nu\rho\lambda}\,\widetilde{J}_{\nu\rho\lambda}$

# The Integral Equations for Yang-Mills

$$\boxed{P_2 e^{ie\int_{\partial\Omega} d\tau d\sigma\left[\alpha F^W_{\mu\nu}+\beta \widetilde{F}^W_{\mu\nu}\right]\frac{dx^\mu}{d\sigma}\frac{dx^\nu}{d\tau}} = P_3 e^{\int_\Omega d\zeta d\tau V \mathcal{J} V^{-1}}}$$

$$\begin{aligned}\mathcal{J} \equiv \int_0^{2\pi} d\sigma \Bigg\{ & ie\beta \widetilde{J}^W_{\mu\nu\lambda}\frac{dx^\mu}{d\sigma}\frac{dx^\nu}{d\tau}\frac{dx^\lambda}{d\zeta} + e^2\int_0^\sigma d\sigma' \\ & \times \left[\left((\alpha-1)\,F^W_{\kappa\rho}+\beta\widetilde{F}^W_{\kappa\rho}\right)(\sigma')\,,\,\left(\alpha F^W_{\mu\nu}+\beta\widetilde{F}^W_{\mu\nu}\right)(\sigma)\right] \\ & \times \frac{d\,x^\kappa}{d\,\sigma'}\frac{d\,x^\mu}{d\,\sigma}\left(\frac{d\,x^\rho\,(\sigma')}{d\,\tau}\frac{d\,x^\nu\,(\sigma)}{d\,\zeta}-\frac{d\,x^\rho\,(\sigma')}{d\,\zeta}\frac{d\,x^\nu\,(\sigma)}{d\,\tau}\right)\Bigg\}\end{aligned}$$

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L.A. Ferreira and G. Luchini

1) [arXiv:1205.2088 [hep-th]], Phys. Rev. D 86, 085039 (2012)

2) [arXiv:1109.2606 hep-th]], Nuclear Physics B 858PM (2012) 336-365

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$$P_2 e^{ie \int_{\partial\Omega} d\tau d\sigma \left[\alpha F^{W}_{\mu\nu} + \beta \widetilde{F}^{W}_{\mu\nu}\right] \frac{dx^{\mu}}{d\sigma} \frac{dx^{\nu}}{d\tau}} = P_3 e^{\int_{\Omega} d\zeta d\tau V \mathcal{J} V^{-1}}$$

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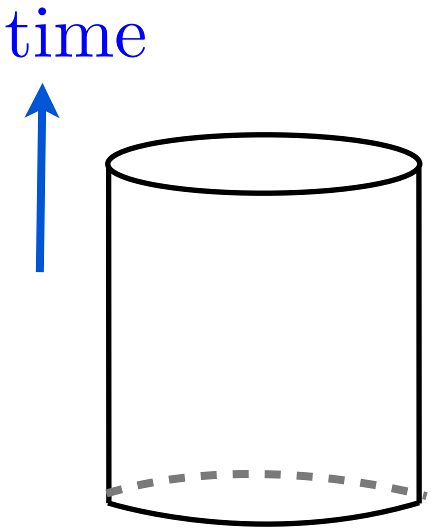

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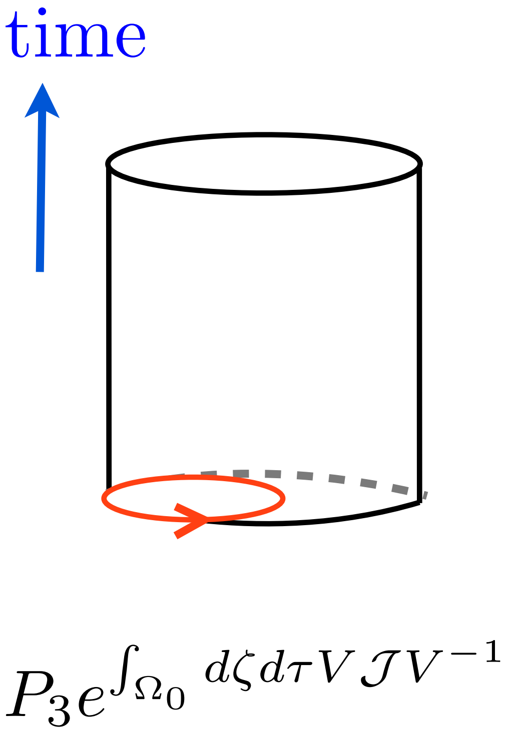

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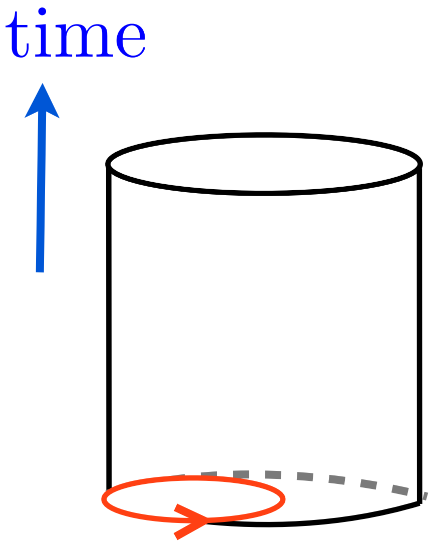

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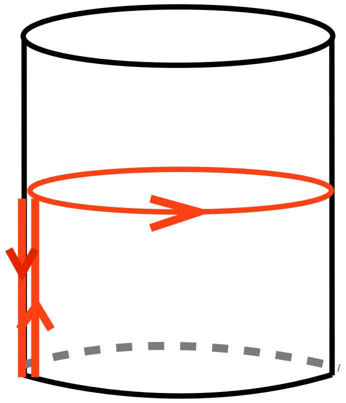

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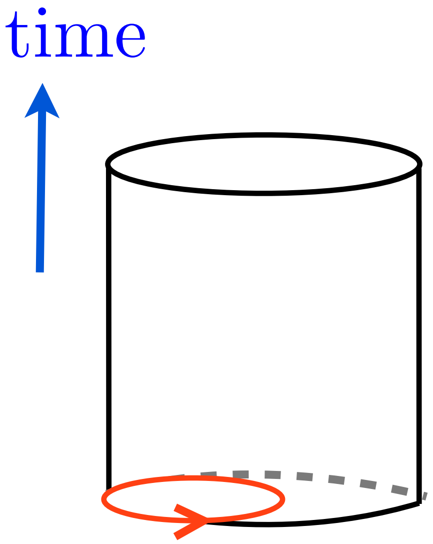

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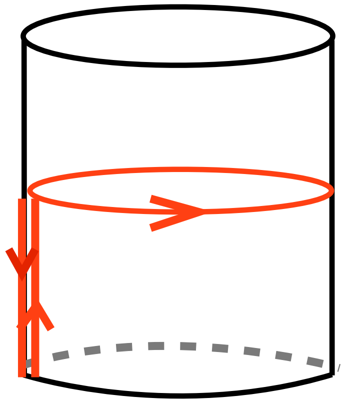

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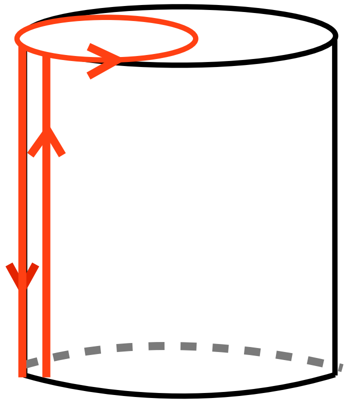

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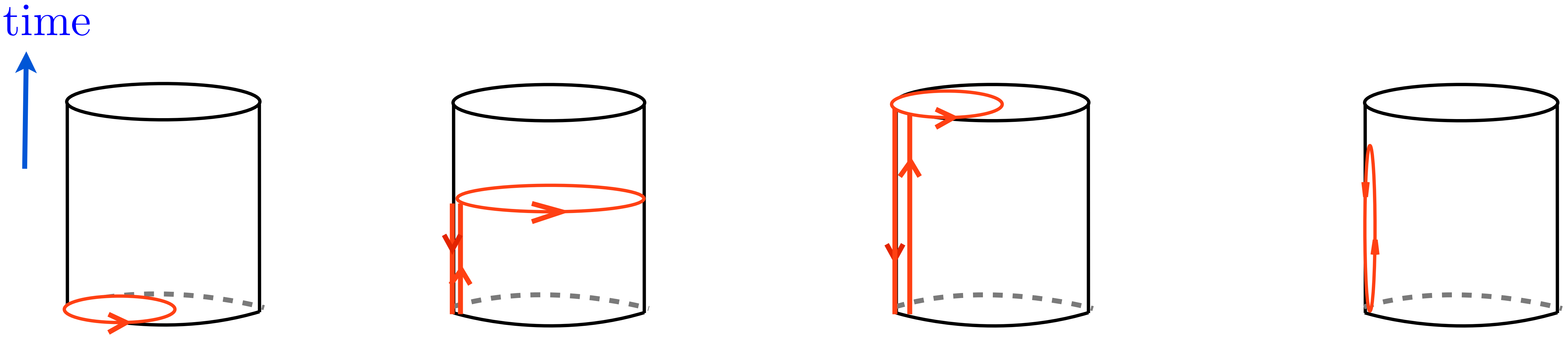

$P_3 e^{\int_{\Omega_0} d\zeta d\tau V\mathcal{J}V^{-1}}$ $P_3 e^{\int_{S^2_\infty\times I} d\zeta d\tau V\mathcal{J}V^{-1}}$ $P_3 e^{\int_{\Omega_t^{-1}} d\zeta d\tau V\mathcal{J}V^{-1}}$ $P_3 e^{\int_{S^2_0\times I} d\zeta d\tau V\mathcal{J}V^{-1}}$

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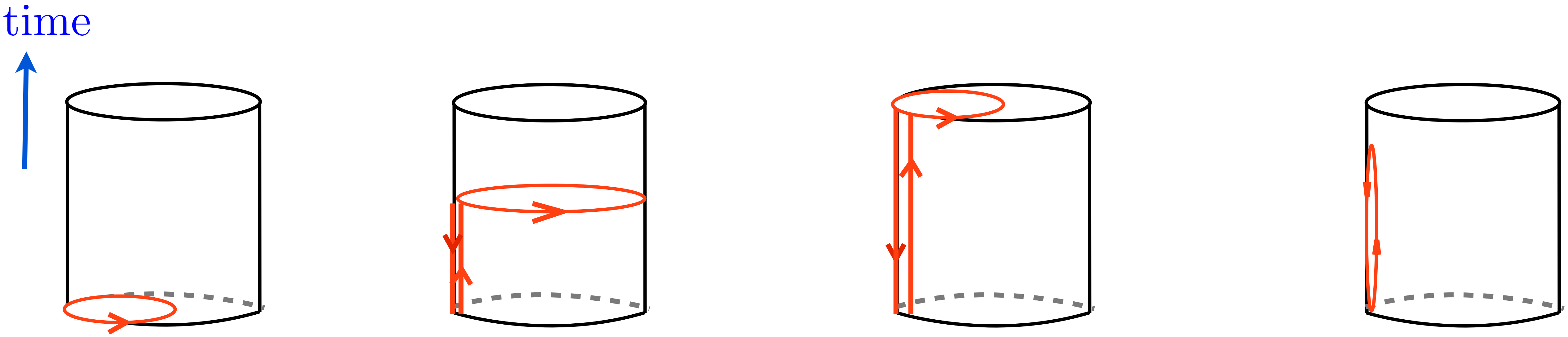

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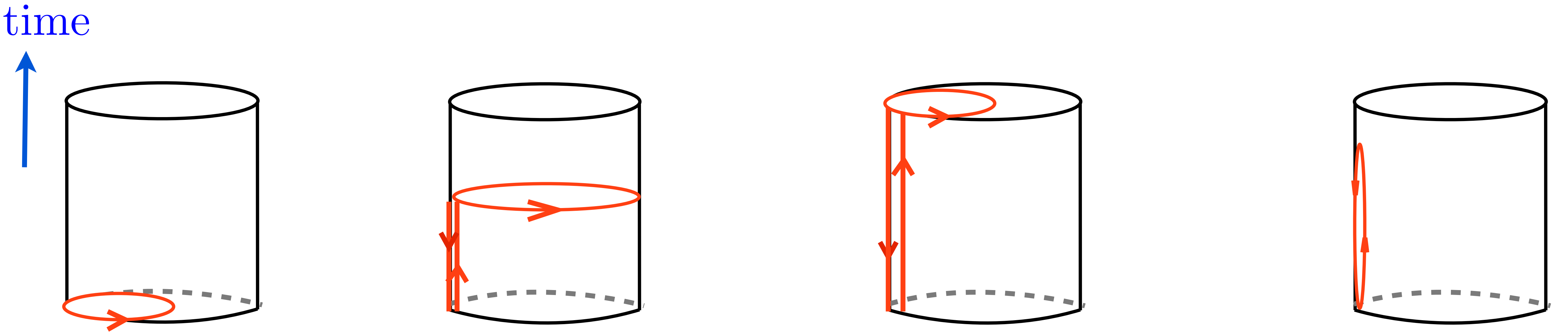

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**Iso-spectral evolution:** $\boxed{V(\Omega_t) = U(t) \cdot V(\Omega_0) \cdot U^{-1}(t)}$

Eigenvalues of $V(\Omega_t)$ are constant in time

# Conserved charges are eigenvalues of the operator

$$V(\Omega_t) = P_2 \; e^{ie \int_{\mathcal{S}^{2,(t)}_{\infty}} d\tau \; d\sigma \; \left(\alpha F^{W}_{\mu\nu} + \beta \widetilde{F}^{W}_{\mu\nu}\right) \frac{dx^{\mu}}{d\sigma} \frac{dx^{\nu}}{d\tau}} = P_3 \; e^{\int_{\Omega_t} d\zeta \; d\tau \; V \mathcal{J} V^{-1}}$$

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5) Relevant for the global aspects of Yang-Mills theory

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# Wu-Yang and 't Hooft-Polyakov monopoles

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At spatial infinity:

$$A_i = -\frac{1}{e}\,\varepsilon_{ijk}\,\frac{\hat{r}_j}{r}\,T_k \qquad F_{ij} = \frac{1}{e}\,\varepsilon_{ijk}\,\frac{\hat{r}_k}{r^2}\,\hat{r}\cdot T \qquad [T_i\,,\,T_j] = i\,\varepsilon_{ijk}\,T_k$$

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Important property: $$W^{-1}\,F_{ij}\,W = \frac{1}{e}\,\varepsilon_{ijk}\,\frac{\hat{r}_k}{r^2}\,T_R \qquad T_R \equiv (\hat{r}\cdot T)_{\text{at } x_R}$$

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Charge operator:

$$Q_S = e^{i\,e\,\alpha\,\int_{S^2_\infty} d\sigma\,d\tau\,W^{-1}\,F_{ij}\,W\,\frac{dx^i}{d\sigma}\,\frac{dx^j}{d\tau}} = e^{-i\,e\,\alpha\,\int_{S^2_\infty} d\vec{\Sigma}\cdot\vec{B}^R} = e^{i\,4\,\pi\,\alpha\,T_R}$$

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Integral Bianchi identity implies: $Q_S^{(\alpha=1)} = e^{-i\,e\,\int_{S^2_\infty} d\vec{\Sigma}\cdot\vec{B}^R} = \mathbb{1}$

# Wu-Yang and ’t Hooft-Polyakov monopoles

At spatial infinity:

$$A_i = -\frac{1}{e}\,\varepsilon_{ijk}\,\frac{\hat{r}_j}{r}\,T_k \qquad F_{ij} = \frac{1}{e}\,\varepsilon_{ijk}\,\frac{\hat{r}_k}{r^2}\,\hat{r}\cdot T \qquad [T_i\,,\,T_j] = i\,\varepsilon_{ijk}\,T_k$$

Important property: $$W^{-1}\,F_{ij}\,W = \frac{1}{e}\,\varepsilon_{ijk}\,\frac{\hat{r}_k}{r^2}\,T_R \qquad T_R \equiv (\hat{r}\cdot T)_{\text{at } x_R}$$

Charge operator:

$$Q_S = e^{i\,e\,\alpha\,\int_{S^2_\infty} d\sigma\,d\tau\,W^{-1}\,F_{ij}\,W\,\frac{dx^i}{d\sigma}\,\frac{dx^j}{d\tau}} = e^{-i\,e\,\alpha\,\int_{S^2_\infty} d\vec{\Sigma}\cdot\vec{B}^R} = e^{i\,4\,\pi\,\alpha\,T_R}$$

Integral Bianchi identity implies: $$Q_S^{(\alpha=1)} = e^{-i\,e\,\int_{S^2_\infty} d\vec{\Sigma}\cdot\vec{B}^R} = \mathbb{1}$$

eigenvalues of $\int_{S^2_\infty} d\vec{\Sigma}\cdot\vec{B}^R = \dfrac{2\,\pi\,n}{e}$

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eigenvalues of $\int_{S^2_\infty} d\vec{\Sigma}\cdot\vec{B}^R = \frac{2\,\pi\,n}{e}$

Text book charges: $\qquad Q_{\text{old}} = \int_{S^2_\infty} d\sigma\,d\tau\,F_{ij}\,\frac{dx^i}{d\sigma}\,\frac{dx^j}{d\tau} = 0$

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Look for theories in $d+1$ dimensions where the equations of motion take the integral form

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Examples:
1) Integrable theories in $1+1$ dimensions (soliton theories)
2) Chern-Simons theories in $2+1$ dimensions
3) Yang-Mills in $2+1$ and $3+1$ dimensions

# *Thank You*

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