Quantum groups, Yang-Baxter maps and quasi-determinants

Zengo Tsuboi

Osaka City University Advanced Mathematical Institute (additional post member)

28 June 2018

Based on

- Z.T., 1708.06323 [Nucl.Phys.B 926(2018)200-238].

See also,

- V.Bazhanov, S.Sergeev, 1501.06984 [Nucl.Phys.B926(2018) 509-543],
- V.Bazhanov, S.Khoroshkin, S.Sergeev, Z.T. to appear (?)

Introduction

The Yang-Baxter map [Drinfeld 1990, Veselov 2000] is a map χ defined on a direct product of two sets

$$\mathcal{R}: \qquad \chi \times \chi \mapsto \chi \times \chi$$

Introduction

The Yang-Baxter map [Drinfeld 1990, Veselov 2000] is a map χ defined on a direct product of two sets

$$\mathcal{R}: \qquad \chi \times \chi \mapsto \chi \times \chi$$

and satisfies the set-thoretical Yang-Baxter equation (on $\chi \times \chi \times \chi$)

$$\mathcal{R}_{12} \circ \mathcal{R}_{13} \circ \mathcal{R}_{23} = \mathcal{R}_{23} \circ \mathcal{R}_{13} \circ \mathcal{R}_{12}.$$

Introduction

The Yang-Baxter map [Drinfeld 1990, Veselov 2000] is a map χ defined on a direct product of two sets

$$\mathcal{R}: \qquad \chi \times \chi \mapsto \chi \times \chi$$

and satisfies the set-thoretical Yang-Baxter equation (on $\chi \times \chi \times \chi$)

$$\mathcal{R}_{12} \circ \mathcal{R}_{13} \circ \mathcal{R}_{23} = \mathcal{R}_{23} \circ \mathcal{R}_{13} \circ \mathcal{R}_{12}.$$

This is related to discrete classical integrable systems.
(discrete Toda equation, etc.).

Introduction

Goal

From the point of view of quantum groups, classify all the Yang-Baxter maps and construct the maps explicitly.

Introduction

Goal

From the point of view of quantum groups, classify all the Yang-Baxter maps and construct the maps explicitly.

There exists quantum group $U_q(\mathfrak{g})$ for each Lie algebra $\mathfrak{g}$. Then the maps will be classified in terms of classification of the quantum groups.

Introduction

Method

For any quantum group $U_q(\mathfrak{g})$, there exists the universal R-matrix $\mathbf{R} \in U_q(\mathfrak{g}) \otimes U_q(\mathfrak{g})$ satisfying the Yang-Baxter equation

$$\mathbf{R}_{12}\mathbf{R}_{13}\mathbf{R}_{23} = \mathbf{R}_{23}\mathbf{R}_{13}\mathbf{R}_{12}.$$

Introduction

Method

For any quantum group $U_q(\mathfrak{g})$, there exists the universal R-matrix $\mathbf{R} \in U_q(\mathfrak{g}) \otimes U_q(\mathfrak{g})$ satisfying the Yang-Baxter equation

$$\mathbf{R}_{12}\mathbf{R}_{13}\mathbf{R}_{23} = \mathbf{R}_{23}\mathbf{R}_{13}\mathbf{R}_{12}.$$

The quantum Yang-Baxter map is define as an adjoint action of the universal R-matrix [Bazhanov-Sergeev 2015]:

$$\mathcal{R}: \qquad U_q(\mathfrak{g}) \otimes U_q(\mathfrak{g}) \mapsto U_q(\mathfrak{g}) \otimes U_q(\mathfrak{g})$$

$$\xi \mapsto \xi' = \mathbf{R}\xi\mathbf{R}^{-1}, \qquad \xi \in U_q(\mathfrak{g}) \otimes U_q(\mathfrak{g}).$$

Introduction

Method

For any quantum group $U_q(\mathfrak{g})$, there exists the universal R-matrix $\mathbf{R} \in U_q(\mathfrak{g}) \otimes U_q(\mathfrak{g})$ satisfying the Yang-Baxter equation

$$\mathbf{R}_{12}\mathbf{R}_{13}\mathbf{R}_{23} = \mathbf{R}_{23}\mathbf{R}_{13}\mathbf{R}_{12}.$$

The quantum Yang-Baxter map is define as an adjoint action of the universal R-matrix [Bazhanov-Sergeev 2015]:

$$\mathcal{R} : \qquad U_q(\mathfrak{g}) \otimes U_q(\mathfrak{g}) \mapsto U_q(\mathfrak{g}) \otimes U_q(\mathfrak{g})$$

$$\xi \mapsto \xi' = \mathbf{R}\xi\mathbf{R}^{-1}, \qquad \xi \in U_q(\mathfrak{g}) \otimes U_q(\mathfrak{g}).$$

The classical Yang-Baxter map is give by the quasi-classical limit

Quantum algebras $\mathcal{A} = U_q(\mathfrak{g})$

As an example, we consider $\mathfrak{g} = sl(n)$ or $gl(n)$.

Generators of $U_q(gl(n))$

$$\mathbf{E}_{i,i+1}, \mathbf{E}_{j+1,j}, \mathbf{E}_{k,k},$$

$(i, j \in \{1, 2, \ldots, n-1\},$
$k \in \{1, 2, \ldots, n\})$

Relations of $U_q(gl(n))$

$$[\mathbf{E}_{kk}, \mathbf{E}_{ij}] = (\delta_{ik} - \delta_{jk})\mathbf{E}_{ij},$$

$$[\mathbf{E}_{i,i+1}, \mathbf{E}_{j+1,j}] = \delta_{ij}(q - q^{-1})(q^{\mathbf{E}_{ii}-\mathbf{E}_{i+1,i+1}} - q^{-\mathbf{E}_{ii}+\mathbf{E}_{i+1,i+1}}),$$

$$[\mathbf{E}_{i,i+1}, \mathbf{E}_{j,j+1}] = [\mathbf{E}_{i+1,i}, \mathbf{E}_{j+1,j}] = 0 \quad \text{for} \quad |i-j| \geq 2,$$

Relations of $U_q(gl(n))$

$$[\mathbf{E}_{kk}, \mathbf{E}_{ij}] = (\delta_{ik} - \delta_{jk})\mathbf{E}_{ij},$$

$$[\mathbf{E}_{i,i+1}, \mathbf{E}_{j+1,j}] = \delta_{ij}(q - q^{-1})(q^{\mathbf{E}_{ii}-\mathbf{E}_{i+1,i+1}} - q^{-\mathbf{E}_{ii}+\mathbf{E}_{i+1,i+1}}),$$

$$[\mathbf{E}_{i,i+1}, \mathbf{E}_{j,j+1}] = [\mathbf{E}_{i+1,i}, \mathbf{E}_{j+1,j}] = 0 \quad \text{for} \quad |i-j| \geq 2,$$

and, for $i \in \{1, 2, \ldots, n-2\}$, the Serre relations

$$\mathbf{E}_{i,i+1}^2 \mathbf{E}_{i+1,i+2} - (q + q^{-1})\mathbf{E}_{i,i+1}\mathbf{E}_{i+1,i+2}\mathbf{E}_{i,i+1} + \mathbf{E}_{i+1,i+2}\mathbf{E}_{i,i+1}^2 = 0,$$

$\ldots\ldots$

Additional generators

For $i, j \in \{1, 2, \ldots, n\}$ and $i \neq j$, we define

$$\mathbf{E}_{ij} = (q - q^{-1})^{-1}(\mathbf{E}_{ik}\mathbf{E}_{kj} - q\mathbf{E}_{kj}\mathbf{E}_{ik}), \qquad \mathbf{E}_{ji} = \ldots,$$

$$i < k < j.$$

Additional generators

For $i, j \in \{1, 2, \ldots, n\}$ and $i \neq j$, we define

$$\mathbf{E}_{ij} = (q - q^{-1})^{-1}(\mathbf{E}_{ik}\mathbf{E}_{kj} - q\mathbf{E}_{kj}\mathbf{E}_{ik}), \qquad \mathbf{E}_{ji} = \ldots,$$
$$i < k < j.$$

$\mathbf{c} = \sum_{j=1}^{n} \mathbf{E}_{jj}$ is a central element of $\mathcal{A}$.

Additional generators

For $i, j \in \{1, 2, \ldots, n\}$ and $i \neq j$, we define

$$\mathbf{E}_{ij} = (q - q^{-1})^{-1}(\mathbf{E}_{ik}\mathbf{E}_{kj} - q\mathbf{E}_{kj}\mathbf{E}_{ik}), \qquad \mathbf{E}_{ji} = \ldots,$$
$$i < k < j.$$

$\mathbf{c} = \sum_{j=1}^{n} \mathbf{E}_{jj}$ is a central element of $\mathcal{A}$. The algebra $\mathcal{A}$ is isomorphic to $U_q(sl(n))$ under the condition $\mathbf{c} = 0$.

Additional generators

For $i, j \in \{1, 2, \dots, n\}$ and $i \neq j$, we define

$$\mathbf{E}_{ij} = (q - q^{-1})^{-1}(\mathbf{E}_{ik}\mathbf{E}_{kj} - q\mathbf{E}_{kj}\mathbf{E}_{ik}), \qquad \mathbf{E}_{ji} = \dots,$$
$$i < k < j.$$

$\mathbf{c} = \sum_{j=1}^{n} \mathbf{E}_{jj}$ is a central element of $\mathcal{A}$. The algebra $\mathcal{A}$ is isomorphic to $U_q(sl(n))$ under the condition $\mathbf{c} = 0$.

• Borel subalgebras

$\mathcal{B}_+$: generated by $\{\mathbf{E}_{ij}\}$

$\mathcal{B}_-$: generated by $\{\mathbf{E}_{ji}\}$ for $i \leq j$, $i, j \in \{1, 2, \dots, n\}$

Co-multiplication

The co-multiplication Δ is an algebra homomorphism from the algebra $\mathcal{A}$ to its tensor square

$$\Delta : \qquad \mathcal{A} \to \mathcal{A} \otimes \mathcal{A},$$

defined by

$$\Delta(\mathbf{E}_{i,i+1}) = \mathbf{E}_{i,i+1} \otimes q^{\mathbf{E}_{ii} - \mathbf{E}_{i+1,i+1}} + 1 \otimes \mathbf{E}_{i,i+1},$$

$$\Delta(\mathbf{E}_{i+1,i}) = \mathbf{E}_{i+1,i} \otimes 1 + q^{-\mathbf{E}_{ii} + \mathbf{E}_{i+1,i+1}} \otimes \mathbf{E}_{i+1,i}, \quad i \in \{1, 2, \dots, n-1\},$$

$$\Delta(\mathbf{E}_{kk}) = \mathbf{E}_{kk} \otimes 1 + 1 \otimes \mathbf{E}_{kk}, \quad k \in \{1, 2, \dots, n\}.$$

Co-multiplication

The co-multiplication Δ is an algebra homomorphism from the algebra $\mathcal{A}$ to its tensor square

$$\Delta : \qquad \mathcal{A} \to \mathcal{A} \otimes \mathcal{A},$$

defined by

$$\Delta(\mathbf{E}_{i,i+1}) = \mathbf{E}_{i,i+1} \otimes q^{\mathbf{E}_{ii} - \mathbf{E}_{i+1,i+1}} + 1 \otimes \mathbf{E}_{i,i+1},$$

$$\Delta(\mathbf{E}_{i+1,i}) = \mathbf{E}_{i+1,i} \otimes 1 + q^{-\mathbf{E}_{ii} + \mathbf{E}_{i+1,i+1}} \otimes \mathbf{E}_{i+1,i}, \quad i \in \{1, 2, \ldots, n-1\},$$

$$\Delta(\mathbf{E}_{kk}) = \mathbf{E}_{kk} \otimes 1 + 1 \otimes \mathbf{E}_{kk}, \quad k \in \{1, 2, \ldots, n\}.$$

We will also use the opposite co-multiplication Δ', defined by

$$\Delta' = \sigma \circ \Delta,$$

where $\sigma(\mathbf{a} \otimes \mathbf{b}) = \mathbf{b} \otimes \mathbf{a}$ for any $\mathbf{a}, \mathbf{b} \in \mathcal{A}$.

Universal R-matrix

The algebra $\mathcal{A}$ is a quasi-triangular Hopf algebra. Then there exists an element
$\mathbb{R} \in \mathcal{A} \otimes \mathcal{A}$, which satisfies

$$\Delta'(\mathbf{a})\,\mathbb{R} = \mathbb{R}\,\Delta(\mathbf{a}) \qquad \text{for all } \mathbf{a} \in \mathcal{A},$$

$$(\Delta \otimes 1)\,\mathbb{R} = \mathbb{R}_{13}\,\mathbb{R}_{23},$$

$$(1 \otimes \Delta)\,\mathbb{R} = \mathbb{R}_{13}\,\mathbb{R}_{12},$$

where $\mathbb{R}_{12} = \mathbb{R} \otimes 1$, $\mathbb{R}_{23} = 1 \otimes \mathbb{R}$ and $\mathbb{R}_{13} = (\sigma \otimes 1)\mathbb{R}_{23}$.

Universal R-matrix

The algebra $\mathcal{A}$ is a quasi-triangular Hopf algebra. Then there exists an element
$\mathbb{R} \in \mathcal{A} \otimes \mathcal{A}$, which satisfies

$$\Delta'(\mathbf{a})\,\mathbb{R} = \mathbb{R}\,\Delta(\mathbf{a}) \qquad \text{for all } \mathbf{a} \in \mathcal{A},$$
$$(\Delta \otimes 1)\,\mathbb{R} = \mathbb{R}_{13}\,\mathbb{R}_{23},$$
$$(1 \otimes \Delta)\,\mathbb{R} = \mathbb{R}_{13}\,\mathbb{R}_{12},$$

where $\mathbb{R}_{12} = \mathbb{R} \otimes 1$, $\mathbb{R}_{23} = 1 \otimes \mathbb{R}$ and $\mathbb{R}_{13} = (\sigma \otimes 1)\mathbb{R}_{23}$.
The quantum Yang-Baxter equation follows from these.

$$\mathbb{R}_{12}\mathbb{R}_{13}\mathbb{R}_{23} = \mathbb{R}_{23}\mathbb{R}_{13}\mathbb{R}_{12}$$

Universal R-matrix

q-exponential function

$$\exp_q(x) = 1 + \sum_{k=1}^{\infty} \frac{x^k}{(k)_q!},$$

$$(k)_q! = (1)_q(2)_q \cdots (k)_q, \quad (k)_q = (1-q^k)/(1-q).$$

$\exp_q(x)^{-1} = \exp_{q^{-1}}(-x).$

Universal R-matrix

If we assume that the universal R-matrix has the form

$$\mathbb{R} = q^{\sum_{i=1}^{n} \mathbf{E}_{ii} \otimes \mathbf{E}_{ii}} \overline{\mathbb{R}},$$

where $\overline{\mathbb{R}} \in \mathcal{N}_+ \otimes \mathcal{N}_-$: $\mathcal{N}_+$ and $\mathcal{N}_-$ are nilpotent sub-algebras generated by $\{\mathbf{E}_{ij}\}$ and $\{\mathbf{E}_{ji}\}$ for $i < j$, $i, j \in \{1, 2, \ldots, n\}$ respectively.

Universal R-matrix

If we assume that the universal R-matrix has the form

$$\mathbb{R} = q^{\sum_{i=1}^{n} \mathbf{E}_{ii} \otimes \mathbf{E}_{ii}} \overline{\mathbb{R}},$$

where $\overline{\mathbb{R}} \in \mathcal{N}_+ \otimes \mathcal{N}_-$: $\mathcal{N}_+$ and $\mathcal{N}_-$ are nilpotent sub-algebras generated by $\{\mathbf{E}_{ij}\}$ and $\{\mathbf{E}_{ji}\}$ for $i < j$, $i, j \in \{1, 2, \ldots, n\}$ respectively.

Universal R-matrix

The universal is uniquely defined by [Kirillov-Reshetikhin, Rosso, Levendorskii-Soibelman,....]

$$\mathbb{R} = q^{\sum_{i=1}^{n} \mathbf{E}_{ii} \otimes \mathbf{E}_{ii}} \overrightarrow{\prod}_{i<j} \exp_{q^{-2}} \left((q - q^{-1})^{-1} \mathbf{E}_{ij} \otimes \mathbf{E}_{ji} \right),$$

where the product is taken over the reverse lexicographical order on (i, j): $(i_1, j_1) \prec (i_2, j_2)$ if $i_1 > i_2$, or $i_1 = i_2$ and $j_1 > j_2$.

Quantum Yang-Baxter map

Let $\mathbf{X} = \{\mathbf{E}_{ij}, q^{\mathbf{E}_{kk}}\}$, $i \neq j$ be the set of generators of $\mathcal{A}$ and $\mathbf{X}^{(a)}$ be the corresponding components in $\mathcal{A} \otimes \mathcal{A}$,

$$\mathbf{X}^{(1)} = \{\mathbf{x} \otimes 1 | \mathbf{x} \in \mathbf{X}\}, \qquad \mathbf{X}^{(2)} = \{1 \otimes \mathbf{x} | \mathbf{x} \in \mathbf{X}\}, \qquad \mathbf{X} = \{\mathbf{E}_{ij}, q^{\mathbf{E}_{kk}}\},$$

Quantum Yang-Baxter map

Let $\mathbf{X} = \{\mathbf{E}_{ij}, q^{\mathbf{E}_{kk}}\}$, $i \neq j$ be the set of generators of $\mathcal{A}$ and $\mathbf{X}^{(a)}$ be the corresponding components in $\mathcal{A} \otimes \mathcal{A}$,

$$\mathbf{X}^{(1)} = \{\mathbf{x} \otimes 1 | \mathbf{x} \in \mathbf{X}\}, \qquad \mathbf{X}^{(2)} = \{1 \otimes \mathbf{x} | \mathbf{x} \in \mathbf{X}\}, \qquad \mathbf{X} = \{\mathbf{E}_{ij}, q^{\mathbf{E}_{kk}}\},$$

Quantum Yang-Baxter map

$$\mathcal{R} : (\mathbf{X}^{(1)}, \mathbf{X}^{(2)}) \mapsto (\widetilde{\mathbf{X}}^{(1)}, \widetilde{\mathbf{X}}^{(2)}),$$

$$\widetilde{\mathbf{X}}^{(a)} = \mathbb{R}\mathbf{X}^{(a)}\mathbb{R}^{-1} = \left\{\mathbb{R}\mathbf{x}^{(a)}\mathbb{R}^{-1} | \mathbf{x}^{(a)} \in \mathbf{X}^{(a)}\right\}, \qquad a = 1, 2.$$

Quantum Yang-Baxter map

Let $\mathbf{X} = \{\mathbf{E}_{ij}, q^{\mathbf{E}_{kk}}\}$, $i \neq j$ be the set of generators of $\mathcal{A}$ and $\mathbf{X}^{(a)}$ be the corresponding components in $\mathcal{A} \otimes \mathcal{A}$,

$$\mathbf{X}^{(1)} = \{\mathbf{x} \otimes 1 | \mathbf{x} \in \mathbf{X}\}, \qquad \mathbf{X}^{(2)} = \{1 \otimes \mathbf{x} | \mathbf{x} \in \mathbf{X}\}, \qquad \mathbf{X} = \{\mathbf{E}_{ij}, q^{\mathbf{E}_{kk}}\},$$

Quantum Yang-Baxter map

$$\mathcal{R} : (\mathbf{X}^{(1)}, \mathbf{X}^{(2)}) \mapsto (\widetilde{\mathbf{X}}^{(1)}, \widetilde{\mathbf{X}}^{(2)}),$$

$$\widetilde{\mathbf{X}}^{(a)} = \mathbb{R}\mathbf{X}^{(a)}\mathbb{R}^{-1} = \left\{\mathbb{R}\mathbf{x}^{(a)}\mathbb{R}^{-1} | \mathbf{x}^{(a)} \in \mathbf{X}^{(a)}\right\}, \qquad a = 1, 2.$$

Note that any elements of $\widetilde{\mathbf{X}}^{(1)}$ commute with those of $\widetilde{\mathbf{X}}^{(2)}$. In addition, the algebra $\widetilde{\mathcal{A}}_a$ generated by the elements of the set $\widetilde{\mathbf{X}}^{(a)}$ is isomorphic to the algebra $\mathcal{A}$.

Quantum Yang-Baxter map

Let $\mathbf{X} = \{\mathbf{E}_{ij}, q^{\mathbf{E}_{kk}}\}$, $i \neq j$ be the set of generators of $\mathcal{A}$ and $\mathbf{X}^{(a)}$ be the corresponding components in $\mathcal{A} \otimes \mathcal{A}$,

$$\mathbf{X}^{(1)} = \{\mathbf{x} \otimes 1 | \mathbf{x} \in \mathbf{X}\}, \qquad \mathbf{X}^{(2)} = \{1 \otimes \mathbf{x} | \mathbf{x} \in \mathbf{X}\}, \qquad \mathbf{X} = \{\mathbf{E}_{ij}, q^{\mathbf{E}_{kk}}\},$$

Quantum Yang-Baxter map

$$\mathcal{R} : (\mathbf{X}^{(1)}, \mathbf{X}^{(2)}) \mapsto (\widetilde{\mathbf{X}}^{(1)}, \widetilde{\mathbf{X}}^{(2)}),$$

$$\widetilde{\mathbf{X}}^{(a)} = \mathbb{R}\mathbf{X}^{(a)}\mathbb{R}^{-1} = \left\{\mathbb{R}\mathbf{x}^{(a)}\mathbb{R}^{-1} | \mathbf{x}^{(a)} \in \mathbf{X}^{(a)}\right\}, \qquad a = 1, 2.$$

Note that any elements of $\widetilde{\mathbf{X}}^{(1)}$ commute with those of $\widetilde{\mathbf{X}}^{(2)}$. In addition, the algebra $\widetilde{\mathcal{A}}_a$ generated by the elements of the set $\widetilde{\mathbf{X}}^{(a)}$ is isomorphic to the algebra $\mathcal{A}$.

$\Longrightarrow$ The tensor product structure is preserved under the map.

Another universal R-matrix

One can prove that if $\mathbb{R}_{12} \in \mathcal{B}_{+} \otimes \mathcal{B}_{-}$ satisfies definition of the universal R-matrix, then

$$\mathbb{R}^{*}_{12} = \mathbb{R}^{-1}_{21} \in \mathcal{B}_{-} \otimes \mathcal{B}_{+},$$

also satisfies the def of the universal R-matrix.

Another universal R-matrix

One can prove that if $\mathbb{R}_{12} \in \mathcal{B}_+ \otimes \mathcal{B}_-$ satisfies definition of the universal R-matrix, then

$$\mathbb{R}^*_{12} = \mathbb{R}^{-1}_{21} \in \mathcal{B}_- \otimes \mathcal{B}_+,$$

also satisfies the def of the universal R-matrix.

$$\mathbb{R}_{12}\mathbb{R}_{13}\mathbb{R}_{23} = \mathbb{R}_{23}\mathbb{R}_{13}\mathbb{R}_{12},$$

$$\mathbb{R}^*_{12}\mathbb{R}_{13}\mathbb{R}_{23} = \mathbb{R}_{23}\mathbb{R}_{13}\mathbb{R}^*_{12},$$

$$\mathbb{R}^*_{12}\mathbb{R}^*_{13}\mathbb{R}_{23} = \mathbb{R}_{23}\mathbb{R}^*_{13}\mathbb{R}^*_{12}, \ldots.$$

L-operators

n-dimensional fundamental representation π of $\mathcal{A}$:
$\pi(\mathbf{E}_{kk}) = E_{kk}$, $\pi(\mathbf{E}_{ij}) = (q - q^{-1})E_{ij}$, for $i \neq j$.
(E_{ij}: $n \times n$ matrix unit.)

L-operators

n-dimensional fundamental representation π of $\mathcal{A}$:
$\pi(\mathbf{E}_{kk}) = E_{kk}$, $\pi(\mathbf{E}_{ij}) = (q - q^{-1})E_{ij}$, for $i \neq j$.
(E_{ij}: $n \times n$ matrix unit.)

$$\mathbb{L}^{-} = (\pi \otimes 1)\mathbb{R}^{*} \in \mathrm{End}(\mathbb{C}^{n}) \otimes \mathcal{B}_{+}$$

$$\mathbb{L}^{+} = (\pi \otimes 1)\mathbb{R} \in \mathrm{End}(\mathbb{C}^{n}) \otimes \mathcal{B}_{-}$$

L-operators

n-dimensional fundamental representation π of $\mathcal{A}$:
$\pi(\mathbf{E}_{kk}) = E_{kk}$, $\pi(\mathbf{E}_{ij}) = (q - q^{-1})E_{ij}$, for $i \neq j$.
(E_{ij}: $n \times n$ matrix unit.)

$$\mathbb{L}^{-} = (\pi \otimes 1)\mathbb{R}^{*} = \sum_{k=1}^{n} E_{kk} \otimes q^{-\mathbf{E}_{kk}} - \sum_{i<j} E_{ji} \otimes \mathbf{E}_{ij} q^{-\mathbf{E}_{ii}},$$

$$\mathbb{L}^{+} = (\pi \otimes 1)\mathbb{R} = \sum_{k=1}^{n} E_{kk} \otimes q^{\mathbf{E}_{kk}} + \sum_{i<j} E_{ij} \otimes q^{\mathbf{E}_{ii}} \mathbf{E}_{ji},$$

L-operators

n-dimensional fundamental representation π of $\mathcal{A}$:
$\pi(\mathbf{E}_{kk}) = E_{kk}$, $\pi(\mathbf{E}_{ij}) = (q - q^{-1})E_{ij}$, for $i \neq j$.
(E_{ij}: $n \times n$ matrix unit.)

$$\mathbb{L}^{-} = (\pi \otimes 1)\mathbb{R}^{*} = \sum_{k=1}^{n} E_{kk} \otimes q^{-\mathbf{E}_{kk}} - \sum_{i<j} E_{ji} \otimes \mathbf{E}_{ij} q^{-\mathbf{E}_{ii}},$$

$$\mathbb{L}^{+} = (\pi \otimes 1)\mathbb{R} = \sum_{k=1}^{n} E_{kk} \otimes q^{\mathbf{E}_{kk}} + \sum_{i<j} E_{ij} \otimes q^{\mathbf{E}_{ii}} \mathbf{E}_{ji},$$

$(\mathbb{L}^{-})_{kk}(\mathbb{L}^{+})_{kk} = 1$

R-matrices

Evaluating further the second space (quantum space) of these L-operators in the fundamental representation π, we obtain the block R-matrices

$$R^{-} = (1 \otimes \pi)\mathbb{L}^{-}$$

$$R^{+} = (1 \otimes \pi)\mathbb{L}^{+}$$

R-matrices

Evaluating further the second space (quantum space) of these L-operators in the fundamental representation π, we obtain the block R-matrices

$$R^{-} = (1 \otimes \pi)\mathbb{L}^{-} = \sum_{i,j} q^{-\delta_{ij}} E_{ii} \otimes E_{jj} - (q - q^{-1}) \sum_{i<j} E_{ji} \otimes E_{ij},$$

$$R^{+} = (1 \otimes \pi)\mathbb{L}^{+} = \sum_{i,j} q^{\delta_{ij}} E_{ii} \otimes E_{jj} + (q - q^{-1}) \sum_{i<j} E_{ij} \otimes E_{ji}.$$

L and R-operators with a spectral parameter

Then we define the spectral parameter dependent L-operator

$$\mathbb{L}(\lambda) = \lambda \mathbb{L}^{+} - \lambda^{-1} \mathbb{L}^{-}$$

and the R-matrix

$$R(\lambda) = \lambda R^{+} - \lambda^{-1} R^{-}.$$

Zero curvature representation

$$\mathbb{R}^*_{01}\mathbb{R}^*_{02}\mathbb{R}_{12} = \mathbb{R}_{12}\mathbb{R}^*_{02}\mathbb{R}^*_{01},$$

$$\mathbb{R}^*_{01}\mathbb{R}_{02}\mathbb{R}_{12} = \mathbb{R}_{12}\mathbb{R}_{02}\mathbb{R}^*_{01},$$

$$\mathbb{R}_{01}\mathbb{R}_{02}\mathbb{R}_{12} = \mathbb{R}_{12}\mathbb{R}_{02}\mathbb{R}_{01},$$

Zero curvature representation

$$\mathbb{R}^*_{01}\mathbb{R}^*_{02}\mathbb{R}_{12} = \mathbb{R}_{12}\mathbb{R}^*_{02}\mathbb{R}^*_{01} \quad \leftarrow \mathbb{R}^{-1}_{12},$$

$$\mathbb{R}^*_{01}\mathbb{R}_{02}\mathbb{R}_{12} = \mathbb{R}_{12}\mathbb{R}_{02}\mathbb{R}^*_{01},$$

$$\mathbb{R}_{01}\mathbb{R}_{02}\mathbb{R}_{12} = \mathbb{R}_{12}\mathbb{R}_{02}\mathbb{R}_{01},$$

Zero curvature representation

$$\mathbb{R}^*_{01}\mathbb{R}^*_{02} = (\mathbb{R}_{12}\mathbb{R}^*_{02}\mathbb{R}^{-1}_{12})(\mathbb{R}_{12}\mathbb{R}^*_{01}\mathbb{R}^{-1}_{12}),$$

$$\mathbb{R}^*_{01}\mathbb{R}_{02} = (\mathbb{R}_{12}\mathbb{R}_{02}\mathbb{R}^{-1}_{12})(\mathbb{R}_{12}\mathbb{R}^*_{01}\mathbb{R}^{-1}_{12})$$

$$\mathbb{R}_{01}\mathbb{R}_{02} = (\mathbb{R}_{12}\mathbb{R}_{02}\mathbb{R}^{-1}_{12})(\mathbb{R}_{12}\mathbb{R}_{01}\mathbb{R}^{-1}_{12}).$$

Zero curvature representation

$$\mathbb{R}^*_{01}\mathbb{R}^*_{02} = (\mathbb{R}_{12}\mathbb{R}^*_{02}\mathbb{R}^{-1}_{12})(\mathbb{R}_{12}\mathbb{R}^*_{01}\mathbb{R}^{-1}_{12}),$$

$$\mathbb{R}^*_{01}\mathbb{R}_{02} = (\mathbb{R}_{12}\mathbb{R}_{02}\mathbb{R}^{-1}_{12})(\mathbb{R}_{12}\mathbb{R}^*_{01}\mathbb{R}^{-1}_{12})$$

$$\mathbb{R}_{01}\mathbb{R}_{02} = (\mathbb{R}_{12}\mathbb{R}_{02}\mathbb{R}^{-1}_{12})(\mathbb{R}_{12}\mathbb{R}_{01}\mathbb{R}^{-1}_{12}).$$

Evaluating the first space of these (labeled by 0) in the fundamental representation π, we obtain

Zero curvature representation

$$\mathbb{R}^*_{01}\mathbb{R}^*_{02} = (\mathbb{R}_{12}\mathbb{R}^*_{02}\mathbb{R}^{-1}_{12})(\mathbb{R}_{12}\mathbb{R}^*_{01}\mathbb{R}^{-1}_{12}),$$

$$\mathbb{R}^*_{01}\mathbb{R}_{02} = (\mathbb{R}_{12}\mathbb{R}_{02}\mathbb{R}^{-1}_{12})(\mathbb{R}_{12}\mathbb{R}^*_{01}\mathbb{R}^{-1}_{12})$$

$$\mathbb{R}_{01}\mathbb{R}_{02} = (\mathbb{R}_{12}\mathbb{R}_{02}\mathbb{R}^{-1}_{12})(\mathbb{R}_{12}\mathbb{R}_{01}\mathbb{R}^{-1}_{12}).$$

Evaluating the first space of these (labeled by 0) in the fundamental representation π, we obtain

Zero curvature representation

$$\mathbb{L}^+_1\mathbb{L}^+_2 = \widetilde{\mathbb{L}}^+_2\widetilde{\mathbb{L}}^+_1, \quad \mathbb{L}^-_1\mathbb{L}^+_2 = \widetilde{\mathbb{L}}^+_2\widetilde{\mathbb{L}}^-_1, \quad \mathbb{L}^-_1\mathbb{L}^-_2 = \widetilde{\mathbb{L}}^-_2\widetilde{\mathbb{L}}^-_1,$$

where $\widetilde{\mathbb{L}}^{\pm}_{01} = \mathbb{R}_{12}\mathbb{L}^{\pm}_{01}\mathbb{R}^{-1}_{12}$, $\widetilde{\mathbb{L}}^{\pm}_{02} = \mathbb{R}_{12}\mathbb{L}^{\pm}_{02}\mathbb{R}^{-1}_{12}$ and we omit the space index 0.

Zero curvature representation

The zero curvature representation gives a rational map among generators for the case $U_q(sl(2))$ [Bazhanov-Sergeev 2015].

Zero curvature representation

The zero curvature representation gives a rational map among generators for the case $U_q(sl(2))$ [Bazhanov-Sergeev 2015].

"*In this paper we present detailed considerations of the above scheme on the example of the algebra $U_q(sl(2))$ leading to discrete Liouville equations, however the approach is rather general and can be applied to any quantized Lie algebra.* " [Bazhanov-Sergeev 2015]

Zero curvature representation

The zero curvature representation gives a rational map among generators for the case $U_q(sl(2))$ [Bazhanov-Sergeev 2015].

"*In this paper we present detailed considerations of the above scheme on the example of the algebra $U_q(sl(2))$ leading to discrete Liouville equations, however the approach is rather general and can be applied to any quantized Lie algebra.* " [Bazhanov-Sergeev 2015]

However, this optimistic idea soon run into difficulty if we consider $U_q(sl(3))$ case. Namely, square roots appear in the map for $U_q(sl(n))$, $n \geq 3$.

Zero curvature representation

The zero curvature representation gives a rational map among generators for the case $U_q(sl(2))$ [Bazhanov-Sergeev 2015].

"*In this paper we present detailed considerations of the above scheme on the example of the algebra $U_q(sl(2))$ leading to discrete Liouville equations, however the approach is rather general and can be applied to any quantized Lie algebra.* " [Bazhanov-Sergeev 2015]

However, this optimistic idea soon run into difficulty if we consider $U_q(sl(3))$ case. Namely, square roots appear in the map for $U_q(sl(n))$, $n \geq 3$.
To overcome this difficulty, we will make a change of variables by twisting the universal R-matrix.

Twisting universal R-matrices

Twisting [Drinfeld, Reshetikhin]

If $\mathbf{F} \in \mathcal{A} \otimes \mathcal{A}$ satisfies

$$(\Delta \otimes 1)\,\mathbf{F} = \mathbf{F}_{13}\,\mathbf{F}_{23},$$

$$(1 \otimes \Delta)\,\mathbf{F} = \mathbf{F}_{13}\,\mathbf{F}_{12},$$

$$\mathbf{F}_{12}\mathbf{F}_{13}\mathbf{F}_{23} = \mathbf{F}_{23}\mathbf{F}_{13}\mathbf{F}_{12},$$

Twisting universal R-matrices

Twisting [Drinfeld, Reshetikhin]

If $\mathbf{F} \in \mathcal{A} \otimes \mathcal{A}$ satisfies

$$(\Delta \otimes 1)\,\mathbf{F} = \mathbf{F}_{13}\,\mathbf{F}_{23},$$

$$(1 \otimes \Delta)\,\mathbf{F} = \mathbf{F}_{13}\,\mathbf{F}_{12},$$

$$\mathbf{F}_{12}\mathbf{F}_{13}\mathbf{F}_{23} = \mathbf{F}_{23}\mathbf{F}_{13}\mathbf{F}_{12},$$

then gauge transformed universal R-matrices

$$\mathbf{R} = \mathbf{F}_{21}\mathbb{R}\mathbf{F}_{12}^{-1} q^{\mathbf{c}\otimes\mathbf{c}}, \qquad \mathbf{R}^{*} = \mathbf{F}_{21}\mathbb{R}^{*}\mathbf{F}_{12}^{-1} q^{\mathbf{c}\otimes\mathbf{c}}$$

satisfy the defining relations for the universal R-matrix for the gauge transformed co-multiplication $\Delta^{F}(a) = \mathbf{F}\Delta(a)\mathbf{F}^{-1}, a \in \mathcal{A}$.

Twisting universal R-matrices

Twisting [Drinfeld, Reshetikhin]

If $\mathbf{F} \in \mathcal{A} \otimes \mathcal{A}$ satisfies

$$(\Delta \otimes 1)\,\mathbf{F} = \mathbf{F}_{13}\,\mathbf{F}_{23},$$

$$(1 \otimes \Delta)\,\mathbf{F} = \mathbf{F}_{13}\,\mathbf{F}_{12},$$

$$\mathbf{F}_{12}\mathbf{F}_{13}\mathbf{F}_{23} = \mathbf{F}_{23}\mathbf{F}_{13}\mathbf{F}_{12},$$

then gauge transformed universal R-matrices

$$\mathbf{R} = \mathbf{F}_{21}\mathbb{R}\mathbf{F}_{12}^{-1}q^{\mathbf{c}\otimes\mathbf{c}}, \qquad \mathbf{R}^{*} = \mathbf{F}_{21}\mathbb{R}^{*}\mathbf{F}_{12}^{-1}q^{\mathbf{c}\otimes\mathbf{c}}$$

satisfy the defining relations for the universal R-matrix for the gauge transformed co-multiplication $\Delta^{F}(a) = \mathbf{F}\Delta(a)\mathbf{F}^{-1}, a \in \mathcal{A}$.

$\mathbf{R}$ and $\mathbf{R}^{*}$ satisfy the same Yang-Baxter relations as $\mathbb{R}$ and $\mathbb{R}^{*}$.

Twisting the universal R-matrices

$$\mathbf{F}_{12} = q^{\sum_{i=1}^{n} \omega_{i-1} \otimes \mathbf{E}_{ii}},$$
$$\omega_i = \mathbf{E}_{11} + \cdots + \mathbf{E}_{ii},$$

$\omega_0 = 0$, $\omega_n = \mathbf{c}$.

Twisting L-operators

The gauge transformed **L**-operators are defined by evaluating the gauge transformed universal R-matrices.

$$\mathbf{L}^{-} = (\pi \otimes 1)(\mathbf{R}^{*}) = \sum_{k=1}^{n} E_{kk} \otimes q^{2\omega_{k-1}} - \sum_{i<j} E_{ji} \otimes q^{\omega_{i-1}+\omega_{j-1}} \mathbf{E}_{ij},$$

$$\mathbf{L}^{+} = (\pi \otimes 1)(\mathbf{R}) = \sum_{k=1}^{n} E_{kk} \otimes q^{2\omega_{k}} + \sum_{i<j} E_{ij} \otimes q^{\omega_{i}+\omega_{j}} \mathbf{E}_{ji},$$

$\omega_i = \mathbf{E}_{11} + \cdots + \mathbf{E}_{ii}$.

Zero curvature representation for twisted L-operators

The zero-curvature representation for the twisted L-operators has the same form as the one for the original L-operators

$$\mathbf{L}_1^+\mathbf{L}_2^+ = \widetilde{\mathbf{L}}_2^+\widetilde{\mathbf{L}}_1^+, \quad \mathbf{L}_1^-\mathbf{L}_2^+ = \widetilde{\mathbf{L}}_2^+\widetilde{\mathbf{L}}_1^-, \quad \mathbf{L}_1^-\mathbf{L}_2^- = \widetilde{\mathbf{L}}_2^-\widetilde{\mathbf{L}}_1^-.$$

Zero curvature representation for twisted L-operators

The zero-curvature representation for the twisted L-operators has the same form as the one for the original L-operators

$$\mathbf{L}_1^+\mathbf{L}_2^+ = \widetilde{\mathbf{L}}_2^+\widetilde{\mathbf{L}}_1^+, \quad \mathbf{L}_1^-\mathbf{L}_2^+ = \widetilde{\mathbf{L}}_2^+\widetilde{\mathbf{L}}_1^-, \quad \mathbf{L}_1^-\mathbf{L}_2^- = \widetilde{\mathbf{L}}_2^-\widetilde{\mathbf{L}}_1^-.$$

The set of generators $\mathbf{X}^{(a)} = \{\mathbf{L}_{ij}^{+(a)}, \mathbf{L}_{ji}^{-(a)}\}_{i \le j}$ for the Yang-Baxter map $\mathcal{R} : (\mathbf{X}^{(1)}, \mathbf{X}^{(2)}) \to (\widetilde{\mathbf{X}}^{(1)}, \widetilde{\mathbf{X}}^{(2)})$, ($\widetilde{\mathbf{X}}^{(a)} = \mathbf{R}_{12}\mathbf{X}^{(a)}\mathbf{R}_{12}^{-1}$) is different:

$$\mathbf{L}_{ij}^+ = \mathbf{u}_i^{\frac{1}{2}}\mathbf{u}_j^{\frac{1}{2}}\mathbf{E}_{ji}, \qquad \mathbf{L}_{ji}^- = -\mathbf{u}_{i-1}^{\frac{1}{2}}\mathbf{u}_{j-1}^{\frac{1}{2}}\mathbf{E}_{ij} \quad \text{for} \quad i<j,$$

$$\mathbf{L}_{kk}^+ = \mathbf{u}_k \qquad \mathbf{L}_{kk}^- = \mathbf{u}_{k-1}.$$

$(\mathbf{u}_k := q^{2\omega_k} = q^{2(\mathbf{E}_{11}+\cdots+\mathbf{E}_{kk})})$,

Zero curvature representation for twisted L-operators

The zero-curvature representation for the twisted L-operators has the same form as the one for the original L-operators

$$\mathbf{L}_1^+\mathbf{L}_2^+ = \widetilde{\mathbf{L}}_2^+\widetilde{\mathbf{L}}_1^+, \quad \mathbf{L}_1^-\mathbf{L}_2^+ = \widetilde{\mathbf{L}}_2^+\widetilde{\mathbf{L}}_1^-, \quad \mathbf{L}_1^-\mathbf{L}_2^- = \widetilde{\mathbf{L}}_2^-\widetilde{\mathbf{L}}_1^-.$$

The set of generators $\mathbf{X}^{(a)} = \{\mathbf{L}_{ij}^{+(a)}, \mathbf{L}_{ji}^{-(a)}\}_{i\leq j}$ for the Yang-Baxter map $\mathcal{R} : (\mathbf{X}^{(1)}, \mathbf{X}^{(2)}) \to (\widetilde{\mathbf{X}}^{(1)}, \widetilde{\mathbf{X}}^{(2)})$, $(\widetilde{\mathbf{X}}^{(a)} = \mathbf{R}_{12}\mathbf{X}^{(a)}\mathbf{R}_{12}^{-1})$ is different:

$$\mathbf{L}_{ij}^+ = \mathbf{u}_i^{\frac{1}{2}}\mathbf{u}_j^{\frac{1}{2}}\mathbf{E}_{ji}, \qquad \mathbf{L}_{ji}^- = -\mathbf{u}_{i-1}^{\frac{1}{2}}\mathbf{u}_{j-1}^{\frac{1}{2}}\mathbf{E}_{ij} \quad \text{for} \quad i<j,$$

$$\mathbf{L}_{kk}^+ = \mathbf{u}_k \qquad \mathbf{L}_{kk}^- = \mathbf{u}_{k-1}.$$

$(\mathbf{u}_k := q^{2\omega_k} = q^{2(\mathbf{E}_{11}+\cdots+\mathbf{E}_{kk})})$, $\mathbf{L}_{kk}^+\mathbf{L}_{kk}^- \neq 1$

Solving the zero curvature representation

$$\mathbf{L}_1^+\mathbf{L}_2^+ = \widetilde{\mathbf{L}}_2^+\widetilde{\mathbf{L}}_1^+, \quad \mathbf{L}_1^-\mathbf{L}_2^+ = \widetilde{\mathbf{L}}_2^+\widetilde{\mathbf{L}}_1^-, \quad \mathbf{L}_1^-\mathbf{L}_2^- = \widetilde{\mathbf{L}}_2^-\widetilde{\mathbf{L}}_1^-,$$

To do: write the matrix elements of $\widetilde{\mathbf{L}}_a^\pm = (\widetilde{\mathbf{L}}_{ij}^{\pm(a)})_{i,j=1}^n$ in terms of matrix elements of $\mathbf{L}_1^+, \mathbf{L}_2^+, \mathbf{L}_1^-, \mathbf{L}_2^-$.

Solving the zero curvature representation

$$\mathbf{L}_1^+\mathbf{L}_2^+ = \widetilde{\mathbf{L}}_2^+\widetilde{\mathbf{L}}_1^+, \quad \mathbf{L}_1^-\mathbf{L}_2^+ = \widetilde{\mathbf{L}}_2^+\widetilde{\mathbf{L}}_1^-, \quad \mathbf{L}_1^-\mathbf{L}_2^- = \widetilde{\mathbf{L}}_2^-\widetilde{\mathbf{L}}_1^-,$$

To do: write the matrix elements of $\widetilde{\mathbf{L}}_a^\pm = (\widetilde{\mathbf{L}}_{ij}^{\pm(a)})_{i,j=1}^n$ in terms of matrix elements of $\mathbf{L}_1^+, \mathbf{L}_2^+, \mathbf{L}_1^-, \mathbf{L}_2^-$.

$$\begin{pmatrix} 1 & 0 & 0 \\ \mathbf{L}_{21}^{-(1)} & \mathbf{u}_1^{(1)} & 0 \\ \mathbf{L}_{31}^{-(1)} & \mathbf{L}_{32}^{-(1)} & \mathbf{u}_2^{(1)} \end{pmatrix} \begin{pmatrix} \mathbf{u}_1^{(2)} & \mathbf{L}_{12}^{+(2)} & \mathbf{L}_{13}^{+(2)} \\ 0 & \mathbf{u}_2^{(2)} & \mathbf{L}_{23}^{+(2)} \\ 0 & 0 & 1 \end{pmatrix}$$
$$= \begin{pmatrix} \widetilde{\mathbf{u}}_1^{(2)} & \widetilde{\mathbf{L}}_{12}^{+(2)} & \widetilde{\mathbf{L}}_{13}^{+(2)} \\ 0 & \widetilde{\mathbf{u}}_2^{(2)} & \widetilde{\mathbf{L}}_{23}^{+(2)} \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ \widetilde{\mathbf{L}}_{21}^{-(1)} & \widetilde{\mathbf{u}}_1^{(1)} & 0 \\ \widetilde{\mathbf{L}}_{31}^{-(1)} & \widetilde{\mathbf{L}}_{32}^{-(1)} & \widetilde{\mathbf{u}}_2^{(1)} \end{pmatrix},$$

Solving the zero curvature representation

$$\mathbf{L}_1^+\mathbf{L}_2^+ = \widetilde{\mathbf{L}}_2^+\widetilde{\mathbf{L}}_1^+, \quad \mathbf{L}_1^-\mathbf{L}_2^+ = \widetilde{\mathbf{L}}_2^+\widetilde{\mathbf{L}}_1^-, \quad \mathbf{L}_1^-\mathbf{L}_2^- = \widetilde{\mathbf{L}}_2^-\widetilde{\mathbf{L}}_1^-,$$

To do: write the matrix elements of $\widetilde{\mathbf{L}}_a^\pm = (\widetilde{\mathbf{L}}_{ij}^{\pm(a)})_{i,j=1}^n$ in terms of matrix elements of $\mathbf{L}_1^+, \mathbf{L}_2^+, \mathbf{L}_1^-, \mathbf{L}_2^-$.

$$\begin{pmatrix} \mathcal{E}_{11} & \mathcal{E}_{12} & \mathcal{E}_{13} \\ 0 & \mathcal{E}_{22} & \mathcal{E}_{23} \\ 0 & 0 & \mathcal{E}_{33} \end{pmatrix} \begin{pmatrix} \mathcal{F}_{11} & 0 & 0 \\ \mathcal{F}_{21} & \mathcal{F}_{22} & 0 \\ \mathcal{F}_{31} & \mathcal{F}_{32} & \mathcal{F}_{33} \end{pmatrix} =$$

$$= \begin{pmatrix} 1 & 0 & 0 \\ \mathbf{L}_{21}^{-(1)} & \mathbf{u}_1^{(1)} & 0 \\ \mathbf{L}_{31}^{-(1)} & \mathbf{L}_{32}^{-(1)} & \mathbf{u}_2^{(1)} \end{pmatrix} \begin{pmatrix} \mathbf{u}_1^{(2)} & \mathbf{L}_{12}^{+(2)} & \mathbf{L}_{13}^{+(2)} \\ 0 & \mathbf{u}_2^{(2)} & \mathbf{L}_{23}^{+(2)} \\ 0 & 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} \widetilde{\mathbf{u}}_1^{(2)} & \widetilde{\mathbf{L}}_{12}^{+(2)} & \widetilde{\mathbf{L}}_{13}^{+(2)} \\ 0 & \widetilde{\mathbf{u}}_2^{(2)} & \widetilde{\mathbf{L}}_{23}^{+(2)} \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ \widetilde{\mathbf{L}}_{21}^{-(1)} & \widetilde{\mathbf{u}}_1^{(1)} & 0 \\ \widetilde{\mathbf{L}}_{31}^{-(1)} & \widetilde{\mathbf{L}}_{32}^{-(1)} & \widetilde{\mathbf{u}}_2^{(1)} \end{pmatrix},$$

Solution

$$\widetilde{\mathbf{u}}_i^{(1)} = \overrightarrow{\prod}_{k=1}^{i} \mathbb{H}_k^{-1} \mathbf{u}_k^{(1)} \mathbf{u}_k^{(2)},$$

$$\widetilde{\mathbf{u}}_i^{(2)} = \mathbb{H}_i \overleftarrow{\prod}_{k=1}^{i-1} (\mathbf{u}_k^{(1)} \mathbf{u}_k^{(2)})^{-1} \mathbb{H}_k \qquad \text{for} \quad 1 \leq i \leq n,$$

$$\widetilde{\mathbf{L}}_{ij}^{-(1)} = \left(\overrightarrow{\prod}_{k=1}^{i-1} \mathbb{H}_k^{-1} \mathbf{u}_k^{(1)} \mathbf{u}_k^{(2)} \right) \mathbb{F}_{ij} \qquad \text{for} \quad 1 \leq j < i \leq n,$$

$$\widetilde{\mathbf{L}}_{ij}^{+(2)} = \mathbb{E}_{ij} \mathbb{H}_j \overleftarrow{\prod}_{k=1}^{j-1} (\mathbf{u}_k^{(1)} \mathbf{u}_k^{(2)})^{-1} \mathbb{H}_k \quad \text{for} \qquad 1 \leq i < j \leq n.$$

Solution

$$\widetilde{\mathbf{u}}_i^{(1)} = \overrightarrow{\prod}_{k=1}^{i} \mathbb{H}_k^{-1} \mathbf{u}_k^{(1)} \mathbf{u}_k^{(2)},$$

$$\widetilde{\mathbf{u}}_i^{(2)} = \mathbb{H}_i \overleftarrow{\prod}_{k=1}^{i-1} (\mathbf{u}_k^{(1)} \mathbf{u}_k^{(2)})^{-1} \mathbb{H}_k \qquad \text{for} \quad 1 \le i \le n,$$

$$\widetilde{\mathbf{L}}_{ij}^{-(1)} = \left(\overrightarrow{\prod}_{k=1}^{i-1} \mathbb{H}_k^{-1} \mathbf{u}_k^{(1)} \mathbf{u}_k^{(2)} \right) \mathbb{F}_{ij} \qquad \text{for} \quad 1 \le j < i \le n,$$

$$\widetilde{\mathbf{L}}_{ij}^{+(2)} = \mathbb{E}_{ij} \mathbb{H}_j \overleftarrow{\prod}_{k=1}^{j-1} (\mathbf{u}_k^{(1)} \mathbf{u}_k^{(2)})^{-1} \mathbb{H}_k \quad \text{for} \qquad 1 \le i < j \le n.$$

quasi-determinants for the matrix $\mathbf{J} = \mathbf{L}_1^{-} \mathbf{L}_2^{+}$,

$$\mathbb{H}_i = |\mathbf{J}_{i,\dots,n}^{i,\dots,n}|_{ii}, \quad \mathbb{E}_{ij} = |\mathbf{J}_{j,j+1,\dots,n}^{i,j+1,\dots,n}|_{ij} |\mathbf{J}_{j,\dots,n}^{j,\dots,n}|_{jj}^{-1},$$
$$\mathbb{F}_{ji} = |\mathbf{J}_{j,\dots,n}^{j,\dots,n}|_{jj}^{-1} |\mathbf{J}_{i,j+1,\dots,n}^{j,j+1,\dots,n}|_{ji}.$$

Solution

$$\mathbf{L}_1^+\mathbf{L}_2^+ = \widetilde{\mathbf{L}}_2^+\widetilde{\mathbf{L}}_1^+, \quad \mathbf{L}_1^-\mathbf{L}_2^+ = \widetilde{\mathbf{L}}_2^+\widetilde{\mathbf{L}}_1^-, \quad \mathbf{L}_1^-\mathbf{L}_2^- = \widetilde{\mathbf{L}}_2^-\widetilde{\mathbf{L}}_1^-,$$

The other solutions $\widetilde{\mathbf{L}}_1^+, \widetilde{\mathbf{L}}_2^-$ can be obtained by substituting $\widetilde{\mathbf{L}}_2^+, \widetilde{\mathbf{L}}_1^-$ into the first and the third zero curvature relations.

Solution

$$\mathbf{L}_1^+\mathbf{L}_2^+ = \widetilde{\mathbf{L}}_2^+\widetilde{\mathbf{L}}_1^+, \quad \mathbf{L}_1^-\mathbf{L}_2^+ = \widetilde{\mathbf{L}}_2^+\widetilde{\mathbf{L}}_1^-, \quad \mathbf{L}_1^-\mathbf{L}_2^- = \widetilde{\mathbf{L}}_2^-\widetilde{\mathbf{L}}_1^-,$$

The other solutions $\widetilde{\mathbf{L}}_1^+, \widetilde{\mathbf{L}}_2^-$ can be obtained by substituting $\widetilde{\mathbf{L}}_2^+, \widetilde{\mathbf{L}}_1^-$ into the first and the third zero curvature relations.

$$\widetilde{\mathbf{L}}_1^+ = (\widetilde{\mathbf{L}}_2^+)^{-1}\mathbf{L}_1^+\mathbf{L}_2^+. \qquad \widetilde{\mathbf{L}}_2^- = \mathbf{L}_1^-\mathbf{L}_2^-(\widetilde{\mathbf{L}}_1^-)^{-1}.$$

quasi-determinants [Gelfand, Retakh]

• $N \times N$ matrix whose matrix elements a_{ij} are elements of an associative algebra (not necessary commutative algebra):

$$A = A^{1,2,\dots,N}_{1,2,\dots,N} = \begin{pmatrix} a_{1,1} & a_{1,2} & \cdots & a_{1,N} \\ a_{2,1} & a_{2,2} & \cdots & a_{2,N} \\ \vdots & \vdots & \ddots & \vdots \\ a_{N,1} & a_{N,2} & \cdots & a_{N,N} \end{pmatrix},$$

quasi-determinants [Gelfand, Retakh]

- $N \times N$ matrix whose matrix elements a_{ij} are elements of an associative algebra (not necessary commutative algebra):

$$A = A^{1,2,\ldots,N}_{1,2,\ldots,N} = \begin{pmatrix} a_{1,1} & a_{1,2} & \cdots & a_{1,N} \\ a_{2,1} & a_{2,2} & \cdots & a_{2,N} \\ \vdots & \vdots & \ddots & \vdots \\ a_{N,1} & a_{N,2} & \cdots & a_{N,N} \end{pmatrix},$$

- $m \times n$ sub matrix:

$$A^{i_1,i_2,\ldots,i_m}_{j_1,j_1,\ldots,j_n} = \begin{pmatrix} a_{i_1,j_1} & a_{i_1,j_2} & \cdots & a_{i_1,j_n} \\ a_{i_2,j_1} & a_{i_2,j_2} & \cdots & a_{i_2,j_n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{i_m,j_1} & a_{i_m,j_2} & \cdots & a_{i_m,j_n} \end{pmatrix},$$

$\{i_1, i_2, \ldots, i_m\}, \{j_1, j_2, \ldots, j_n\} \subset \mathcal{I} = \{1, 2, \ldots, N\}.$

quasi-determinants [Gelfand, Retakh]

In general, there are N^2 quasi-determinants for a $N \times N$ matrix A.

quasi-determinants [Gelfand, Retakh]

In general, there are N^2 quasi-determinants for a $N \times N$ matrix A.
(i, j)-quasi-determinant of A is denoted as $|A|_{ij}$.

quasi-determinants [Gelfand, Retakh]

In general, there are N^2 quasi-determinants for a $N \times N$ matrix A.
(i, j)-quasi-determinant of A is denoted as $|A|_{ij}$.

In case all the quasi-determinants of A are not zero, the inverse matrix of A can be expressed in terms of them:

$$A^{-1} = (|A|_{ji}^{-1})_{1 \le i,j \le N}.$$

quasi-determinants [Gelfand, Retakh]

In general, there are N^2 quasi-determinants for a $N \times N$ matrix A. (i,j)-quasi-determinant of A is denoted as $|A|_{ij}$.

In case all the quasi-determinants of A are not zero, the inverse matrix of A can be expressed in terms of them:

$$A^{-1} = (|A|_{ji}^{-1})_{1 \le i,j \le N}.$$

If all the matrix elements of A are commutative, then they reduce to $|A|_{ij} = (-1)^{i+j} \det A / \det A^{1,\dots,\hat{i},\dots N}_{1,\dots,\hat{j},\dots N}$.

quasi-determinants [Gelfand, Retakh]

In general, there are N^2 quasi-determinants for a $N \times N$ matrix A.
(i, j)-quasi-determinant of A is denoted as $|A|_{ij}$.

In case all the quasi-determinants of A are not zero, the inverse matrix of A can be expressed in terms of them:

$$A^{-1} = (|A|_{ji}^{-1})_{1 \le i,j \le N}.$$

If all the matrix elements of A are commutative, then they reduce to $|A|_{ij} = (-1)^{i+j} \det A / \det A^{1,\dots,\hat{i},\dots N}_{1,\dots,\hat{j},\dots N}$.

$\Longrightarrow$ Quasi-determinants are non-commutative analogues of ratios of determinants (rather than non-commutative analogues of determinants).

quasi-determinants [Gelfand, Retakh]

• For a 1×1-sub-matrix $A^i_j = (a_{ij})$ of $A = A^{1,2,\dots,N}_{1,2,\dots,N}$,
(i, j)-th quasi-determinant is defined by $|A^i_j|_{ij} = a_{ij}$.

quasi-determinants [Gelfand, Retakh]

• For a 1×1-sub-matrix $A^{i}_{j} = (a_{ij})$ of $A = A^{1,2,\dots,N}_{1,2,\dots,N}$, (i,j)-th quasi-determinant is defined by $|A^{i}_{j}|_{ij} = a_{ij}$.

• (i,j)-th quasi-determinant of the submatrix $A^{i_1,i_2,\dots,i_m}_{j_1,j_2,\dots,j_m}$ of A is recursively defined by

$$|A^{i_1,\dots,i_m}_{j_1,\dots,j_m}|_{ij} = a_{ij} - \sum_{\substack{k\in\{j_1,j_2,\dots,j_m\}\setminus\{j\},\\ l\in\{i_1,i_2,\dots,i_m\}\setminus\{i\}}} a_{ik}\left(|A^{i_1,\dots,\hat{i},\dots,i_m}_{j_1,\dots,\hat{j},\dots,j_m}|_{lk}\right)^{-1} a_{lj},$$

where $\{i_1,i_2,\dots,i_m\},\{j_1,j_2,\dots,j_m\} \subset \{1,\dots,N\}$; $m \geq 2$; $i \in \{i_1,i_2,\dots,i_m\}$, $j \in \{j_1,j_2,\dots,j_m\}$.

quasi-Plücker coordinates [Gelfand, Retakh]

- Left quasi-Plücker coordinates of $m \times N$ matrix $A^{1,2,\dots,m}_{1,2,\dots,N}$

$$q^{j_1,j_2,\dots,j_{m-1}}_{ij}(A^{1,2,\dots,m}_{1,2,\dots,N}) = (|A^{1,2,\dots,m}_{i,j_1,\dots,j_{m-1}}|_{si})^{-1}|A^{1,2,\dots,m}_{j,j_1,\dots,j_{m-1}}|_{sj},$$

$s \in \{1,2,\dots,m\}$; $m < N$, $i,j,j_1,j_2,\dots,j_{m-1} \in \{1,2,\dots,N\}$, $i \notin \{j_1,j_2,\dots,j_{m-1}\}$. This does not depend on s.

quasi-Plücker coordinates [Gelfand, Retakh]

- Left quasi-Plücker coordinates of $m \times N$ matrix $A^{1,2,\dots,m}_{1,2,\dots,N}$

$$q^{j_1,j_2,\dots,j_{m-1}}_{ij}(A^{1,2,\dots,m}_{1,2,\dots,N}) = (|A^{1,2,\dots,m}_{i,j_1,\dots,j_{m-1}}|_{si})^{-1}|A^{1,2,\dots,m}_{j,j_1,\dots,j_{m-1}}|_{sj},$$

$s \in \{1,2,\dots,m\}$; $m < N$, $i,j,j_1,j_2,\dots,j_{m-1} \in \{1,2,\dots,N\}$, $i \notin \{j_1,j_2,\dots,j_{m-1}\}$. This does not depend on s.

- Commutative case (ratios of Plücker coordinates)

$$q^{j_1,j_2,\dots,j_{m-1}}_{ij}(A^{1,2,\dots,m}_{1,2,\dots,N}) = \det(A^{1,2,\dots,m}_{i,j_1,\dots,j_{m-1}})^{-1}\det(A^{1,2,\dots,m}_{j,j_1,\dots,j_{m-1}}).$$

quasi-Plücker coordinates [Gelfand, Retakh]

- Left quasi-Plücker coordinates of $m \times N$ matrix $A^{1,2,\dots,m}_{1,2,\dots,N}$

$$q^{j_1,j_2,\dots,j_{m-1}}_{ij}(A^{1,2,\dots,m}_{1,2,\dots,N}) = (|A^{1,2,\dots,m}_{i,j_1,\dots,j_{m-1}}|_{si})^{-1}|A^{1,2,\dots,m}_{j,j_1,\dots,j_{m-1}}|_{sj},$$

$s \in \{1,2,\dots,m\}$; $m < N$, $i,j,j_1,j_2,\dots,j_{m-1} \in \{1,2,\dots,N\}$, $i \notin \{j_1,j_2,\dots,j_{m-1}\}$. This does not depend on s.

- Commutative case (ratios of Plücker coordinates)

$$q^{j_1,j_2,\dots,j_{m-1}}_{ij}(A^{1,2,\dots,m}_{1,2,\dots,N}) = \det(A^{1,2,\dots,m}_{i,j_1,\dots,j_{m-1}})^{-1}\det(A^{1,2,\dots,m}_{j,j_1,\dots,j_{m-1}}).$$

- Right quasi-Plücker coordinates

quasi-Plücker coordinates [Gelfand, Retakh]

Quasi-Plücker coordinates also satisfy quasi-Plücker relations, which reduce to Plücker relations in case all the matrix elements are commutative.

quasi-Plücker coordinates [Gelfand, Retakh]

Quasi-Plücker coordinates also satisfy quasi-Plücker relations, which reduce to Plücker relations in case all the matrix elements are commutative.

$\Longrightarrow$ useful in non-commutative soliton theory:
non-Abelian Hirota-Miwa equation, non-Abelian Toda equation, etc.

The solution of the zero-curvature relation can be rewritten in terms of quasi-Plücker coordinates of a block matrix:

$$\mathbf{M} = \begin{pmatrix} \mathbf{0} & \mathbf{L}^{-(1)}\mathbf{L}^{-(2)} \\ \mathbf{L}^{+(1)}\mathbf{L}^{+(2)} & \mathbf{L}^{-(1)}\mathbf{L}^{+(2)} \end{pmatrix} = \begin{pmatrix} \mathbf{0} & \mathbf{L}^{-(1)}\mathbf{L}^{-(2)} \\ \mathbf{L}^{+(1)}\mathbf{L}^{+(2)} & \mathbf{J} \end{pmatrix},$$

The solution of the zero-curvature relation can be rewritten in terms of quasi-Plücker coordinates of a block matrix:

$$\mathbf{M}=\begin{pmatrix} \mathbf{0} & \mathbf{L}^{-(1)}\mathbf{L}^{-(2)} \\ \mathbf{L}^{+(1)}\mathbf{L}^{+(2)} & \mathbf{L}^{-(1)}\mathbf{L}^{+(2)} \end{pmatrix}=\begin{pmatrix} \mathbf{0} & \mathbf{L}^{-(1)}\mathbf{L}^{-(2)} \\ \mathbf{L}^{+(1)}\mathbf{L}^{+(2)} & \mathbf{J} \end{pmatrix},$$

Define a sub-matrix

$$\mathbf{M}^{\bar{i}_1,\bar{i}_2,\dots,\bar{i}_a,k_1,k_2,\dots,k_c}_{\bar{j}_1,\bar{j}_2,\dots,\bar{j}_b,l_1,l_2,\dots,l_d}=\begin{pmatrix} \mathbf{0} & (\mathbf{L}^{-(1)}\mathbf{L}^{-(2)})^{\bar{i}_1,\bar{i}_2,\dots,\bar{i}_a}_{l_1,l_2,\dots,l_d} \\ (\mathbf{L}^{+(1)}\mathbf{L}^{+(2)})^{k_1,k_2,\dots,k_c}_{\bar{j}_1,\bar{j}_2,\dots,\bar{j}_b} & (\mathbf{L}^{-(1)}\mathbf{L}^{+(2)})^{k_1,k_2,\dots,k_c}_{l_1,l_2,\dots,l_d} \end{pmatrix},$$

$$\bar{i}_1,\bar{i}_2,\dots,\bar{i}_a,k_1,k_2,\dots,k_c,\bar{j}_1,\bar{j}_2,\dots,\bar{j}_b,l_1,l_2,\dots,l_d\in\{1,2,\dots,n\}.$$

For $1 \leq i \leq j \leq n$,

$$\widetilde{\mathbf{L}}^{+(1)}_{ij} = \left(\overrightarrow{\prod}_{k=1}^{i-1} q^{k+1,k+2,\dots,n}_{k\,\bar{k}}(\mathbf{M}^{k,k+1,\dots,n}_{\bar{1},\bar{2},\dots,\bar{n},1,2,\dots,n}) \right) q^{i+1,i+2,\dots,n}_{i\,\bar{j}}(\mathbf{M}^{i,i+1,\dots,n}_{\bar{1},\bar{2},\dots,\bar{n},1,2,\dots,n}),$$

(and similar formulas for $\widetilde{\mathbf{L}}^{-(1)}_{ji}, \widetilde{\mathbf{L}}^{+(2)}_{ij}, \widetilde{\mathbf{L}}^{-(2)}_{ji}$) solve the zero-curvature relation.

For $1 \leq i \leq j \leq n$,

$$\widetilde{\mathbf{L}}_{ij}^{+(1)} = \left(\overrightarrow{\prod}_{k=1}^{i-1} q_{k\,\bar{k}}^{k+1,k+2,\ldots,n}(\mathbf{M}_{\bar{1},\bar{2},\ldots,\bar{n},1,2,\ldots,n}^{k,k+1,\ldots,n}) \right) q_{i\,\bar{j}}^{i+1,i+2,\ldots,n}(\mathbf{M}_{\bar{1},\bar{2},\ldots,\bar{n},1,2,\ldots,n}^{i,i+1,\ldots,n}),$$

(and similar formulas for $\widetilde{\mathbf{L}}_{ji}^{-(1)}, \widetilde{\mathbf{L}}_{ij}^{+(2)}, \widetilde{\mathbf{L}}_{ji}^{-(2)}$) solve the zero-curvature relation.

A solution of a set theoretical (quantum) Yang-Baxter equation is obtained in terms of quasi-Plücker coordinates over a matrix composed of L-operators.

Heisenberg-Weyl realization (Minimal representation)

The Heisenberg-Weyl algebra $\mathcal{W}_q$

$$\mathbf{u}_i \mathbf{w}_j = q^{2\delta_{ij}} \mathbf{w}_j \mathbf{u}_i, \qquad \mathbf{u}_i \mathbf{u}_j = \mathbf{u}_j \mathbf{u}_i, \qquad \mathbf{w}_i \mathbf{w}_j = \mathbf{w}_j \mathbf{w}_i.$$

Heisenberg-Weyl realization (Minimal representation)

The Heisenberg-Weyl algebra $\mathcal{W}_q$

$$\mathbf{u}_i \mathbf{w}_j = q^{2\delta_{ij}} \mathbf{w}_j \mathbf{u}_i, \qquad \mathbf{u}_i \mathbf{u}_j = \mathbf{u}_j \mathbf{u}_i, \qquad \mathbf{w}_i \mathbf{w}_j = \mathbf{w}_j \mathbf{w}_i.$$

Homomorphism from $U_q(sl(n))$ to $\mathcal{W}_q$ (minimal rep.)

$$\mathbf{L}^{+}_{i,i} = \mathbf{u}_i, \qquad \mathbf{L}^{+}_{i,j} = \mathbf{w}_i^{-1} \mathbf{w}_{i+1}^{-1} \cdots \mathbf{w}_{j-1}^{-1} (\mathbf{u}_j - \kappa \mathbf{u}_{j-1}),$$

$$\mathbf{L}^{-}_{i,i} = \mathbf{u}_{i-1}, \qquad \mathbf{L}^{-}_{j,i} = \kappa^{-1} \mathbf{w}_i \mathbf{w}_{i+1} \cdots \mathbf{w}_{j-1} (-\mathbf{u}_i + \kappa \mathbf{u}_{i-1}), \quad i < j,$$

where $\kappa \in \mathbb{C}$.

Heisenberg-Weyl realization (Minimal representation)

The Heisenberg-Weyl algebra $\mathcal{W}_q$

$$\mathbf{u}_i \mathbf{w}_j = q^{2\delta_{ij}} \mathbf{w}_j \mathbf{u}_i, \qquad \mathbf{u}_i \mathbf{u}_j = \mathbf{u}_j \mathbf{u}_i, \qquad \mathbf{w}_i \mathbf{w}_j = \mathbf{w}_j \mathbf{w}_i.$$

Homomorphism from $U_q(sl(n))$ to $\mathcal{W}_q$ (minimal rep.)

$$\mathbf{L}^{+}_{i,i} = \mathbf{u}_i, \qquad \mathbf{L}^{+}_{i,j} = \mathbf{w}_i^{-1} \mathbf{w}_{i+1}^{-1} \cdots \mathbf{w}_{j-1}^{-1} (\mathbf{u}_j - \kappa \mathbf{u}_{j-1}),$$

$$\mathbf{L}^{-}_{i,i} = \mathbf{u}_{i-1}, \qquad \mathbf{L}^{-}_{j,i} = \kappa^{-1} \mathbf{w}_i \mathbf{w}_{i+1} \cdots \mathbf{w}_{j-1} (-\mathbf{u}_i + \kappa \mathbf{u}_{i-1}), \quad i < j,$$

where $\kappa \in \mathbb{C}$.

This realizes a representation which has neither a highest weight nor a lowest weight.

Asymptoric representation

For $\xi \in \mathbb{C} \setminus \{0\}$,

$$\tau_\xi : \quad \mathbf{u}_i \to \mathbf{u}_i, \qquad \mathbf{w}_i \to \xi \mathbf{w}_i$$

gives an automorphism of $\mathcal{W}_q$.

Asymptoric representation

For $\xi \in \mathbb{C} \setminus \{0\}$,

$$\tau_\xi : \quad \mathbf{u}_i \to \mathbf{u}_i, \qquad \mathbf{w}_i \to \xi \mathbf{w}_i$$

gives an automorphism of $\mathcal{W}_q$. Taking note on this fact, we will take the limits $\kappa \to 0, \infty$.

Asymptoric representation

For $\xi \in \mathbb{C} \setminus \{0\}$,

$$\tau_\xi : \quad \mathbf{u}_i \to \mathbf{u}_i, \qquad \mathbf{w}_i \to \xi \mathbf{w}_i$$

gives an automorphism of $\mathcal{W}_q$. Taking note on this fact, we will take the limits $\kappa \to 0, \infty$.
$\mathbf{L}^{+,0}_{i,j} = \lim_{\kappa \to 0} \mathbf{L}^{+}_{i,j}$:

$$\mathbf{L}^{+,0}_{i,i} = \mathbf{u}_i, \qquad \mathbf{L}^{+,0}_{i,j} = \mathbf{w}_i^{-1} \mathbf{w}_{i+1}^{-1} \cdots \mathbf{w}_{j-1}^{-1} \mathbf{u}_j, \quad i < j.$$

Asymptoric representation

For $\xi \in \mathbb{C} \setminus \{0\}$,

$$\tau_\xi: \quad \mathbf{u}_i \to \mathbf{u}_i, \qquad \mathbf{w}_i \to \xi \mathbf{w}_i$$

gives an automorphism of $\mathcal{W}_q$. Taking note on this fact, we will take the limits $\kappa \to 0, \infty$.

$\mathbf{L}^{+,0}_{i,j} = \lim_{\kappa \to 0} \mathbf{L}^{+}_{i,j}$:

$$\mathbf{L}^{+,0}_{i,i} = \mathbf{u}_i, \qquad \mathbf{L}^{+,0}_{i,j} = \mathbf{w}_i^{-1}\mathbf{w}_{i+1}^{-1} \cdots \mathbf{w}_{j-1}^{-1}\mathbf{u}_j, \quad i < j.$$

$\mathbf{L}^{+,\infty}_{i,j} = \lim_{\kappa \to \infty} \tau_\kappa(\mathbf{L}^{+}_{i,j})$:

$$\mathbf{L}^{+,\infty}_{i,i} = \mathbf{u}_i, \qquad \mathbf{L}^{+,\infty}_{i,i+1} = -\mathbf{w}_i^{-1}\mathbf{u}_i, \qquad \text{otherwise} \quad \mathbf{L}^{+,\infty}_{i,j} = 0.$$

Factorization of L-operators

Factorization of $\mathbf{L}^{+}$ for minimal rep.

$$\mathbf{L}_1^{+,0}\, \tau_{\kappa^{-1}}(\mathbf{L}_2^{+,\infty}) = \mathbf{U}^{+}\, \mathbf{L}^{+},$$

Factorization of L-operators

Factorization of $\mathbf{L}^{+}$ for minimal rep.

$$\mathbf{L}_1^{+,0}\,\tau_{\kappa^{-1}}(\mathbf{L}_2^{+,\infty}) = \mathbf{U}^{+}\,\mathbf{L}^{+},$$

$$(\kappa \to 0)(\kappa \to \infty) = (\text{diagonal})(\text{minimal rep.})$$

Factorization of L-operators

Factorization of $\mathbf{L}^{+}$ for minimal rep.

$$\mathbf{L}_1^{+,0}\,\tau_{\kappa^{-1}}(\mathbf{L}_2^{+,\infty}) = \mathbf{U}^{+}\,\mathbf{L}^{+},$$

$$(\kappa \to 0)(\kappa \to \infty) = \text{(diagonal)(minimal rep.)}$$

Factorization of $\mathbf{L}^{-}$ for minimal rep.

$$\mathbf{L}_1^{-,\infty}\,\tau_{\kappa^{-1}}(\mathbf{L}_2^{-,0}) = \mathbf{U}^{-}\,\mathbf{L}^{-},$$

$$(\kappa \to \infty)(\kappa \to 0) = \text{(diagonal)(minimal rep.)}$$

Factorization of an R-operator

$\mathbf{L}^{\pm,0}$ and $\mathbf{L}^{\pm,\infty}$ give homomorphisms from $\mathcal{B}_{\mp}$ to $\mathcal{W}_q$.

Factorization of an R-operator

$\mathbf{L}^{\pm,0}$ and $\mathbf{L}^{\pm,\infty}$ give homomorphisms from $\mathcal{B}_{\mp}$ to $\mathcal{W}_q$.

Evaluate the universal R-matrix $\mathbf{R} \in \mathcal{B}_+ \otimes \mathcal{B}_-$ by these homomorphisms.

Factorization of an R-operator

$\mathbf{L}^{\pm,0}$ and $\mathbf{L}^{\pm,\infty}$ give homomorphisms from $\mathcal{B}_{\mp}$ to $\mathcal{W}_q$.

Evaluate the universal R-matrix $\mathbf{R} \in \mathcal{B}_+ \otimes \mathcal{B}_-$ by these homomorphisms.

Factorization of the universal R-matrix for minimal rep.

$$\mathbf{R}_{13}^{\mathrm{min},\mathrm{min}} = (\text{'trivial' R})\, \mathbf{R}_{14}^{0,0} \mathbf{R}_{13}^{0,\infty} \mathbf{R}_{24}^{\infty,0} \mathbf{R}_{23}^{\infty,\infty} (\text{'trivial' R})$$

[cf. affine case: Meneghelli-Teschner 2015]

Discrete quantum evolution system [cf. Bazhanov-Sergeev 2015]

Quantum Yang-Baxter map gives an automorphism

$$\mathcal{R}: \quad \mathcal{A}_1 \otimes \mathcal{A}_2 \mapsto \mathbf{R}(\mathcal{A}_1 \otimes \mathcal{A}_2)\mathbf{R}^{-1} \simeq \mathcal{A}_1 \otimes \mathcal{A}_2 \qquad (\mathcal{A}_i \simeq \mathcal{A}).$$

Based on this map, we define a discrete quantum evolution system for the algebra of observables

$$\mathcal{O} = \mathcal{A}_1 \otimes \mathcal{A}_2 \otimes \cdots \otimes \mathcal{A}_{2N-1} \otimes \mathcal{A}_{2N}, \qquad N \geq 1.$$

Discrete quantum evolution system [cf. Bazhanov-Sergeev 2015]

Quantum Yang-Baxter map gives an automorphism

$$\mathcal{R}: \quad \mathcal{A}_1 \otimes \mathcal{A}_2 \mapsto \mathbf{R}(\mathcal{A}_1 \otimes \mathcal{A}_2)\mathbf{R}^{-1} \simeq \mathcal{A}_1 \otimes \mathcal{A}_2 \qquad (\mathcal{A}_i \simeq \mathcal{A}).$$

Based on this map, we define a discrete quantum evolution system for the algebra of observables

$$\mathcal{O} = \mathcal{A}_1 \otimes \mathcal{A}_2 \otimes \cdots \otimes \mathcal{A}_{2N-1} \otimes \mathcal{A}_{2N}, \qquad N \geq 1.$$

$$\check{\mathcal{R}}_{ij} = \sigma_{ij} \circ \mathcal{R}_{ij}, \quad \mathcal{S}: (\mathbf{X}^{(1)}, \mathbf{X}^{(2)}, \ldots, \mathbf{X}^{(2N)}) \mapsto (\mathbf{X}^{(2)}, \mathbf{X}^{(3)}, \ldots, \mathbf{X}^{(1)}),$$

$\mathbf{X}^{(i)}$: set of the generators of $\mathcal{A}_i$.

Discrete quantum evolution system [cf. Bazhanov-Sergeev 2015]

Quantum Yang-Baxter map gives an automorphism

$$\mathcal{R}: \quad \mathcal{A}_1 \otimes \mathcal{A}_2 \mapsto \mathbf{R}(\mathcal{A}_1 \otimes \mathcal{A}_2)\mathbf{R}^{-1} \simeq \mathcal{A}_1 \otimes \mathcal{A}_2 \qquad (\mathcal{A}_i \simeq \mathcal{A}).$$

Based on this map, we define a discrete quantum evolution system for the algebra of observables

$$\mathcal{O} = \mathcal{A}_1 \otimes \mathcal{A}_2 \otimes \cdots \otimes \mathcal{A}_{2N-1} \otimes \mathcal{A}_{2N}, \qquad N \geq 1.$$

$$\check{\mathcal{R}}_{ij} = \sigma_{ij} \circ \mathcal{R}_{ij}, \quad \mathcal{S}: (\mathbf{X}^{(1)}, \mathbf{X}^{(2)}, \ldots, \mathbf{X}^{(2N)}) \mapsto (\mathbf{X}^{(2)}, \mathbf{X}^{(3)}, \ldots, \mathbf{X}^{(1)}),$$

$\mathbf{X}^{(i)}$: set of the generators of $\mathcal{A}_i$.

The operator

$$\mathcal{U} = \mathcal{S} \circ \left(\check{\mathcal{R}}_{12} \circ \check{\mathcal{R}}_{34} \circ \cdots \circ \check{\mathcal{R}}_{2n-1,2n}\right)$$

gives one step of discrete time evolution $(t \to t+1)$, which is an automorphism of $\mathcal{O}$: $\mathcal{U}(\mathcal{O}) \simeq \mathcal{O}$.

Commuting integrals of motion

Transfer matrices are generating function of integrals of motion.

$$\mathbf{T}(\lambda) = \mathrm{Tr}_0 \left(\mathbf{L}_{01}(\lambda) \mathbf{L}^{+}_{02} \cdots \mathbf{L}_{0,2N-1}(\lambda) \mathbf{L}^{+}_{0,2N} \right),$$

Commuting integrals of motion

Transfer matrices are generating function of integrals of motion.

$$\mathbf{T}(\lambda) = \mathrm{Tr}_0 \left(\mathbf{L}_{01}(\lambda)\mathbf{L}^{+}_{02} \cdots \mathbf{L}_{0,2N-1}(\lambda)\mathbf{L}^{+}_{0,2N} \right),$$

$$\mathcal{U}(\mathbf{T}(\lambda)) = \mathbf{T}(\lambda),$$

Commuting integrals of motion

Transfer matrices are generating function of integrals of motion.

$$\mathbf{T}(\lambda) = \mathrm{Tr}_0 \left(\mathbf{L}_{01}(\lambda)\mathbf{L}^{+}_{02} \cdots \mathbf{L}_{0,2N-1}(\lambda)\mathbf{L}^{+}_{0,2N} \right),$$
$$\overline{\mathbf{T}}(\lambda) = \mathrm{Tr}_0 \left(\mathbf{L}^{-}_{01}\mathbf{L}_{02}(\lambda) \cdots \mathbf{L}^{-}_{0,2N-1}\mathbf{L}_{0,2N}(\lambda) \right)$$

$$\mathcal{U}(\mathbf{T}(\lambda)) = \mathbf{T}(\lambda), \qquad \mathcal{U}(\overline{\mathbf{T}}(\lambda)) = \overline{\mathbf{T}}(\lambda)$$

Commuting integrals of motion

Transfer matrices are generating function of integrals of motion.

$$\mathbf{T}(\lambda) = \mathrm{Tr}_0 \left(\mathbf{L}_{01}(\lambda)\mathbf{L}^{+}_{02} \cdots \mathbf{L}_{0,2N-1}(\lambda)\mathbf{L}^{+}_{0,2N} \right),$$
$$\overline{\mathbf{T}}(\lambda) = \mathrm{Tr}_0 \left(\mathbf{L}^{-}_{01}\mathbf{L}_{02}(\lambda) \cdots \mathbf{L}^{-}_{0,2N-1}\mathbf{L}_{0,2N}(\lambda) \right)$$

$$\mathcal{U}(\mathbf{T}(\lambda)) = \mathbf{T}(\lambda), \qquad \mathcal{U}(\overline{\mathbf{T}}(\lambda)) = \overline{\mathbf{T}}(\lambda)$$

$$\mathbf{T}(\lambda) = \lambda^{N} \sum_{j=0}^{N} \lambda^{-2j} \mathbf{G}_j, \qquad \overline{\mathbf{T}}(\lambda) = \lambda^{-N} \sum_{j=0}^{N} \lambda^{2j} \overline{\mathbf{G}}_j.$$

$$[\mathbf{G}_i, \mathbf{G}_j] = [\mathbf{G}_i, \overline{\mathbf{G}}_j] = [\overline{\mathbf{G}}_i, \overline{\mathbf{G}}_j] = 0.$$

Quasi-classical limit

Quasi-classical limit

$$q = e^{\pi i \mathrm{b}^2}, \qquad \mathrm{b} \to 0,$$

Quasi-classical limit

Quasi-classical limit

$$q = e^{\pi i \mathrm{b}^2}, \qquad \mathrm{b} \to 0,$$

$U_q(gl(n)) \to \mathcal{P}(gl(n))$

$$q^{\mathbf{E}_{ii}} \to k_i, \quad \text{and} \quad \mathbf{E}_{ij} \to e_{ij} \quad \text{for} \quad i \neq j,$$

Quasi-classical limit

Quasi-classical limit

$$q = e^{\pi i \mathrm{b}^2}, \qquad \mathrm{b} \to 0,$$

$U_q(gl(n)) \to \mathcal{P}(gl(n))$

$$q^{\mathbf{E}_{ii}} \to k_i, \quad \text{and} \quad \mathbf{E}_{ij} \to e_{ij} \quad \text{for} \quad i \neq j,$$

Poisson brakets,

$$[\ ,\ ] \to 2\pi i \mathrm{b}^2 \{\ ,\ \}, \qquad \mathrm{b} \to 0,$$

Poisson algebra $\mathcal{P}(gl(n))$

$$\{k_l, e_{ij}\} = \frac{\delta_{il} - \delta_{jl}}{2} e_{ij} k_l, \qquad \{k_i, k_j\} = 0,$$

$$\{e_{i,i+1}, e_{j+1,j}\} = \delta_{ij}(k_i k_{i+1}^{-1} - k_i^{-1} k_{i+1}),$$

$$\{e_{i,i+1}, e_{j,j+1}\} = \{e_{i+1,i}, e_{j+1,j}\} = 0 \quad \text{for} \quad |i-j| \geq 2,$$

Poisson algebra $\mathcal{P}(gl(n))$

$$\{k_l, e_{ij}\} = \frac{\delta_{il} - \delta_{jl}}{2} e_{ij} k_l, \qquad \{k_i, k_j\} = 0,$$

$$\{e_{i,i+1}, e_{j+1,j}\} = \delta_{ij}(k_i k_{i+1}^{-1} - k_i^{-1} k_{i+1}),$$

$$\{e_{i,i+1}, e_{j,j+1}\} = \{e_{i+1,i}, e_{j+1,j}\} = 0 \quad \text{for} \quad |i-j| \geq 2,$$

Serre relations,

$$\{e_{i,i+1}, \{e_{i,i+1}, e_{i+1,i+2}\}\} - \frac{1}{4} e_{i,i+1}^2 e_{i+1,i+2} = 0, \ldots$$

Other generators,

$$e_{ij} = \{e_{ik}, e_{kj}\} - \frac{1}{2} e_{kj} e_{ik}, \qquad e_{ji} = \ldots,$$

$$i < k < j.$$

Quasi-classical expansion of the universal R-matrix

The universal R-matrix is singular in the limit $\mathrm{b} \to 0$.

$$\mathbf{R} = \prod_{i<j} (1 - e_{ij} \otimes e_{ji})^{-\frac{1}{2}}$$

$$\times \exp\left(\frac{1}{i\pi \mathrm{b}^2} \left(2 \sum_{i \geq j} \log k_i \otimes \log k_j + \frac{1}{2} \sum_{i<j} \mathrm{Li}_2(e_{ij} \otimes e_{ji}) \right) \right) (1 + O(\mathrm{b}^2)),$$

$$\mathrm{Li}_2(x) = -\int_0^x \frac{\log(1-t)}{t} \mathrm{d}t.$$

Classical Yang-Baxter map

Although the quasi-classical limit of the universal R-matrix becomes singular, its adjoint action $\xi \in \mathcal{A} \otimes \mathcal{A} \to \mathbf{R}\xi\mathbf{R}^{-1} \in \mathcal{A} \otimes \mathcal{A}$ is well defined. Thus the $q \to 1$ limit of the quantum Yang-Baxter map is well defined.

$$\overline{\mathcal{R}} = \lim_{q \to 1} \mathcal{R}$$

Zero-curvature representation

The zero-curvature representation for the classical case has the same as the quantum case.

$$\boldsymbol{\ell}_1^+ \, \boldsymbol{\ell}_2^+ = \widetilde{\boldsymbol{\ell}}_2^+ \, \widetilde{\boldsymbol{\ell}}_1^+ \,, \qquad \boldsymbol{\ell}_1^- \, \boldsymbol{\ell}_2^+ = \widetilde{\boldsymbol{\ell}}_2^+ \, \widetilde{\boldsymbol{\ell}}_1^- \,, \qquad \boldsymbol{\ell}_1^- \, \boldsymbol{\ell}_2^- = \widetilde{\boldsymbol{\ell}}_2^- \, \widetilde{\boldsymbol{\ell}}_1^- \,,$$

However, the matrix elements of the L-operators $\boldsymbol{\ell}_a^{\pm}$ are commutative.

Solution of the zero-curvature representation

One can obtain the solution by taking the limit $q \to 1$. In particular, the solution is written in terms of ratios of product of minor determinants (Plücker coordinates) of a single matrix.

$$\begin{pmatrix} 0 & \ell_1^- \ell_2^- \\ \ell_1^+ \ell_2^+ & \ell_1^- \ell_2^+ \end{pmatrix} = \begin{pmatrix} 0 & \ell_1^- \ell_2^- \\ \ell_1^+ \ell_2^+ & \mathsf{J} \end{pmatrix}.$$

Example for $\mathcal{P}(sl(3))$

$$\widetilde{u}_1^{(1)} = \begin{vmatrix} J_{22} & J_{23} \\ J_{32} & J_{33} \end{vmatrix} (u_2^{(1)} u_2^{(2)})^{-1}, \qquad \widetilde{u}_2^{(1)} = J_{33},$$

$$\widetilde{\ell}_{12}^{(1)} = \begin{vmatrix} (\ell_1^+ \ell_2^+)_{12} & J_{12} & J_{13} \\ (\ell_1^+ \ell_2^+)_{22} & J_{22} & J_{23} \\ 0 & J_{32} & J_{33} \end{vmatrix} (u_1^{(1)} u_1^{(2)} u_2^{(1)} u_2^{(2)})^{-1}, \qquad \widetilde{\ell}_{23}^{(1)} = \begin{vmatrix} (\ell_1^+ \ell_2^+)_{23} & J_{23} \\ 1 & J_{33} \end{vmatrix} (u_2^{(1)} u_2^{(2)})^{-1},$$

$$\widetilde{\ell}_{13}^{(1)} = \begin{vmatrix} (\ell_1^+ \ell_2^+)_{13} & J_{12} & J_{13} \\ (\ell_1^+ \ell_2^+)_{23} & J_{22} & J_{23} \\ 1 & J_{32} & J_{33} \end{vmatrix} (u_1^{(1)} u_1^{(2)} u_2^{(1)} u_2^{(2)})^{-1}, \qquad \widetilde{\ell}_{31}^{(1)} = J_{31}, \qquad \widetilde{\ell}_{32}^{(1)} = J_{32},$$

$$\widetilde{\ell}_{21}^{(1)} = \begin{vmatrix} J_{21} & J_{23} \\ J_{31} & J_{33} \end{vmatrix} (u_2^{(1)} u_2^{(2)})^{-1}, \quad \widetilde{u}_1^{(2)} = \frac{u_1^{(1)} u_1^{(2)} u_2^{(1)} u_2^{(2)}}{\begin{vmatrix} J_{22} & J_{23} \\ J_{32} & J_{33} \end{vmatrix}}, \qquad \widetilde{u}_2^{(2)} = \frac{u_2^{(1)} u_2^{(2)}}{J_{33}},$$

$$\widetilde{\ell}_{12}^{(2)} = \frac{\begin{vmatrix} J_{12} & J_{13} \\ J_{32} & J_{33} \end{vmatrix} u_2^{(1)} u_2^{(2)}}{\begin{vmatrix} J_{22} & J_{23} \\ J_{32} & J_{33} \end{vmatrix} J_{33}}, \qquad \widetilde{\ell}_{13}^{(2)} = \frac{J_{13}}{J_{33}}, \qquad \widetilde{\ell}_{23}^{(2)} = \frac{J_{23}}{J_{33}}, \qquad \widetilde{\ell}_{21}^{(2)} = \frac{\begin{vmatrix} (\ell_1^- \ell_2^-)_{21} & (\ell_1^- \ell_2^-)_{22} & 0 \\ J_{21} & J_{22} & J_{23} \\ J_{31} & J_{32} & J_{33} \end{vmatrix}}{\begin{vmatrix} J_{22} & J_{23} \\ J_{32} & J_{33} \end{vmatrix}},$$

$$\widetilde{\ell}_{31}^{(2)} = \frac{\begin{vmatrix} (\ell_1^- \ell_2^-)_{31} & (\ell_1^- \ell_2^-)_{32} & (\ell_1^- \ell_2^-)_{33} \\ J_{21} & J_{22} & J_{23} \\ J_{31} & J_{32} & J_{33} \end{vmatrix}}{\begin{vmatrix} J_{22} & J_{23} \\ J_{32} & J_{33} \end{vmatrix}}, \qquad \widetilde{\ell}_{32}^{(2)} = \frac{\begin{vmatrix} (\ell_1^- \ell_2^-)_{32} & (\ell_1^- \ell_2^-)_{33} \\ J_{32} & J_{33} \end{vmatrix} u_2^{(1)} u_2^{(2)}}{J_{33} \begin{vmatrix} J_{22} & J_{23} \\ J_{32} & J_{33} \end{vmatrix}}.$$

Quasi-classical limit for the minimal representatioin

The Heisenberg-Weyl algebra $\mathcal{W}_q$ reduces to the classical Heisenberg-Weyl algebra $\mathcal{W}$ in the quasi-classical limit.

$$\{u_i, w_j\} = \delta_{ij} w_j u_i, \qquad \{u_i, u_j\} = \{w_i, w_j\} = 0.$$

Quasi-classical limit for the minimal representatioin

The Heisenberg-Weyl algebra $\mathcal{W}_q$ reduces to the classical Heisenberg-Weyl algebra $\mathcal{W}$ in the quasi-classical limit.

$$\{u_i, w_j\} = \delta_{ij} w_j u_i, \qquad \{u_i, u_j\} = \{w_i, w_j\} = 0.$$

Minimal representation

(homomorphism from $\mathcal{P}(sl(n))$ to $\mathcal{W}$.)

$$\ell^+_{i,i} = u_i, \qquad \ell^+_{i,j} = w_i^{-1} w_{i+1}^{-1} \cdots w_{j-1}^{-1}(u_j - \kappa u_{j-1}),$$

$$\ell^-_{i,i} = u_{i-1}, \qquad \ell^-_{j,i} = \kappa^{-1} w_i w_{i+1} \cdots w_{j-1}(-u_i + \kappa u_{i-1}), \quad i < j.$$

Solution of the zero-curvature relation for classical minimal rep

For instance, for $n = 3$ case, we explicitly obtain

$$\widetilde{u}_1^{(1)} = \Big(\kappa_1 w_2^{(2)}(\kappa_1 u_1^{(1)} u_2^{(2)} w_1^{(2)} - w_1^{(1)}(\kappa_1 - u_1^{(1)})(\kappa_2 u_1^{(2)} - u_2^{(2)})) - \kappa_2 u_1^{(2)} w_1^{(1)} w_2^{(1)}(\kappa_1 - u_1^{(1)})(\kappa_2 u_2^{(2)} - 1)\Big)(\kappa_1^2 u_2^{(2)} w_1^{(2)} w_2^{(2)})^{-1},$$

(and similar relations for $\widetilde{u}_2^{(1)}, \widetilde{u}_1^{(2)}, \widetilde{u}_2^{(2)}, \widetilde{w}_1^{(1)}, \widetilde{w}_2^{(1)}, \widetilde{w}_1^{(2)}, \widetilde{w}_2^{(2)}$)

Solution of the zero-curvature relation for classical minimal rep

Rewriting this type of formula, we obtain the following relations for $\mathcal{P}(sl(n))$.

$$u_i^{(1)} = \kappa_1 \left(\prod_{k=1}^{i-1} \frac{\widetilde{w}_k^{(2)}}{w_k^{(2)}} \right) \frac{w_i^{(1)} - \widetilde{w}_i^{(2)}}{w_i^{(1)} - \kappa_1 w_i^{(2)}},$$

$$\text{(and similar eqs. for } u_i^{(2)}, \widetilde{u}_i^{(1)}, \widetilde{u}_i^{(2)}), \qquad i \in \{1, 2, \dots, n-1\}.$$

Symplectic form

Under these relations, the following function

$$\Phi = \sum_{i=1}^{n-1} \sum_{a=1}^{2} \left(\log \widetilde{u}_i^{(a)} \mathrm{d} \log \widetilde{w}_i^{(a)} - \log u_i^{(a)} \mathrm{d} \log w_i^{(a)} \right)$$

becomes a closed form:

$$\mathrm{d}\Phi = \sum_{i=1}^{n-1} \sum_{a=1}^{2} \left(\mathrm{d} \log \widetilde{u}_i^{(a)} \wedge \mathrm{d} \log \widetilde{w}_i^{(a)} - \mathrm{d} \log u_i^{(a)} \wedge \mathrm{d} \log w_i^{(a)} \right) = 0.$$

Symplectic form

Under these relations, the following function

$$\Phi = \sum_{i=1}^{n-1} \sum_{a=1}^{2} \left(\log \widetilde{u}_i^{(a)} \mathrm{d} \log \widetilde{w}_i^{(a)} - \log u_i^{(a)} \mathrm{d} \log w_i^{(a)} \right)$$

becomes a closed form:

$$\mathrm{d}\Phi = \sum_{i=1}^{n-1} \sum_{a=1}^{2} \left(\mathrm{d} \log \widetilde{u}_i^{(a)} \wedge \mathrm{d} \log \widetilde{w}_i^{(a)} - \mathrm{d} \log u_i^{(a)} \wedge \mathrm{d} \log w_i^{(a)} \right) = 0.$$

This is also an exact form ($\Phi = \mathrm{d}\mathcal{L}$):

$$\mathcal{L} = 2 \sum_{k<i} \log \frac{\widetilde{w}_i^{(1)}}{w_i^{(1)}} \log \frac{\widetilde{w}_k^{(2)}}{w_k^{(2)}} + \sum_{i=1}^{n-1} \log \frac{\widetilde{w}_i^{(1)}}{\kappa_2 w_i^{(1)}} \log \frac{\widetilde{w}_i^{(2)}}{w_i^{(2)}}$$
$$+ \sum_{i=1}^{n-1} \left\{ -\mathrm{Li}_2 \left(\frac{\kappa_2 \widetilde{w}_i^{(1)}}{\widetilde{w}_i^{(2)}} \right) + \mathrm{Li}_2 \left(\frac{\kappa_2 \widetilde{w}_i^{(1)}}{\kappa_1 w_i^{(2)}} \right) + \cdots - \mathrm{Li}_2 \left(\frac{\kappa_1 w_i^{(2)}}{\kappa_2 \widetilde{w}_i^{(1)}} \right) \right\}.$$

Discrete solition equations for $\mathcal{P}(sl(n))$

We consider the map on $\mathcal{O} = \mathcal{A}_1 \otimes \mathcal{A}_2 \otimes \cdots \otimes \mathcal{A}_{2N-1} \otimes \mathcal{A}_{2N}$ for $q \to 1$
($u_i^{2m+1,t+1} = \mathcal{U}(u_i^{2m-1,t})$, $u_i^{2m,t+1} = \mathcal{U}(u_i^{2m,t})$, $m = 1, \ldots, N$; $i = 1, \ldots, n-1$).

$$u_i^{2m-1,t} = \kappa_1 \left(\prod_{k=1}^{i-1} \frac{w_k^{2m,t+1}}{w_k^{2m,t}} \right) \frac{w_i^{2m-1,t} - w_i^{2m,t+1}}{w_i^{2m-1,t} - \kappa_1 w_i^{2m,t}}$$

(and similar eqs. for $u_i^{2m,t}, u_i^{2m+1,t+1}, u_i^{2m,t+1}$).

Discrete solition equations for $\mathcal{P}(sl(n))$

The consistency condition produces the following equations

$$\left(\prod_{k=1}^{i-1} \frac{w_k^{2m+2,t+2}}{w_k^{2m+2,t+1}}\right) \frac{w_i^{2m+1,t+1} - w_i^{2m+2,t+2}}{w_i^{2m+1,t+1} - \kappa_1 w_i^{2m+2,t+1}} = \left(\prod_{k=1}^{i-1} \frac{w_k^{2m,t+1}}{w_k^{2m,t}}\right) \frac{w_i^{2m+1,t+1} - \kappa_2^{-1} w_i^{2m,t+1}}{w_i^{2m+1,t+1} - \kappa_1 \kappa_2^{-1} w_i^{2m,t}},$$

(and one similar eq.)

These equations reduce to a discrete Liouville equation for $\mathcal{P}(sl(2))$ [Bazhanov-Sergeev 2015].

Discrete solition equations for $\mathcal{P}(sl(n))$

The consistency condition produces the following equations

$$\left(\prod_{k=1}^{i-1} \frac{w_k^{2m+2,t+2}}{w_k^{2m+2,t+1}}\right) \frac{w_i^{2m+1,t+1} - w_i^{2m+2,t+2}}{w_i^{2m+1,t+1} - \kappa_1 w_i^{2m+2,t+1}} = \left(\prod_{k=1}^{i-1} \frac{w_k^{2m,t+1}}{w_k^{2m,t}}\right) \frac{w_i^{2m+1,t+1} - \kappa_2^{-1} w_i^{2m,t+1}}{w_i^{2m+1,t+1} - \kappa_1 \kappa_2^{-1} w_i^{2m,t}},$$

(and one similar eq.)

These equations reduce to a discrete Liouville equation for $\mathcal{P}(sl(2))$ [Bazhanov-Sergeev 2015].

We expected that $\mathcal{P}(sl(n))$ case corresponds to discrete Toda field equations. However, the equations seem to be something more complicated.

Summary

- Quantum Yang-Baxter maps are defined in terms of adjoint action of the univeral R-matrix [Bazhanov-Sergeev 2015].

Summary

- Quantum Yang-Baxter maps are defined in terms of adjoint action of the univeral R-matrix [Bazhanov-Sergeev 2015].
- Solving the zero-curvature representation, we obtained the quantum Yang-Baxter map for $U_q(sl(n))$. It is expressed as a product of quasi-Plücker coordinates over a matrix (written in terms of L-operators, which are image of the universal R-matrix). Twisting of the universal R-matrix was essential for the rationality of the map.

Summary

- Quantum Yang-Baxter maps are defined in terms of adjoint action of the univeral R-matrix [Bazhanov-Sergeev 2015].
- Solving the zero-curvature representation, we obtained the quantum Yang-Baxter map for $U_q(sl(n))$. It is expressed as a product of quasi-Plücker coordinates over a matrix (written in terms of L-operators, which are image of the universal R-matrix). Twisting of the universal R-matrix was essential for the rationality of the map.
- Classical Yang-Baxter maps are derived through the quasiclassical limit.

Summary

- Quantum Yang-Baxter maps are defined in terms of adjoint action of the univeral R-matrix [Bazhanov-Sergeev 2015].
- Solving the zero-curvature representation, we obtained the quantum Yang-Baxter map for $U_q(sl(n))$. It is expressed as a product of quasi-Plücker coordinates over a matrix (written in terms of L-operators, which are image of the universal R-matrix). Twisting of the universal R-matrix was essential for the rationality of the map.
- Classical Yang-Baxter maps are derived through the quasiclassical limit.
- Discrete integrable systems (soliton equations) follow from Yang-Baxter maps.

Summary

- Quantum Yang-Baxter maps are defined in terms of adjoint action of the univeral R-matrix [Bazhanov-Sergeev 2015].
- Solving the zero-curvature representation, we obtained the quantum Yang-Baxter map for $U_q(sl(n))$. It is expressed as a product of quasi-Plücker coordinates over a matrix (written in terms of L-operators, which are image of the universal R-matrix). Twisting of the universal R-matrix was essential for the rationality of the map.
- Classical Yang-Baxter maps are derived through the quasiclassical limit.
- Discrete integrable systems (soliton equations) follow from Yang-Baxter maps.
- Conjecture [Bazhanov-Sergeev 2015]: all the discrete integrable equations could be derived in this way.