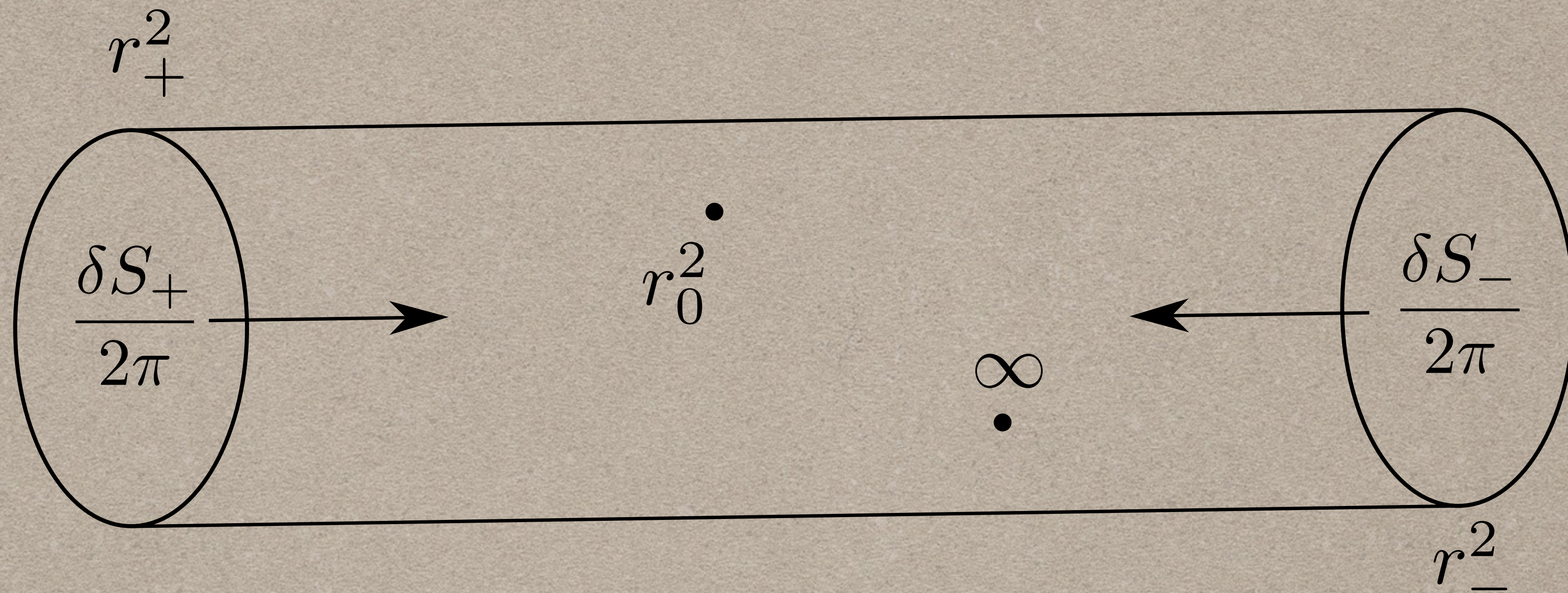

APPLICATIONS OF τ-FUNCTIONS

FROM THE CONSTRUCTION OF CONFORMAL MAPS TO BLACK HOLES

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PREAMBLE

Consider the 2nd order linear EDO:

$$y''(z) + p(z)y'(z) + q(z)y(z) = 0$$

Solutions are usually obtained as a series around a singular point

$$y_i^{\pm}(z) = (z - z_i)^{\alpha_i^{\pm}}(a_{0,i}^{\pm} + a_{1,i}^{\pm}(z - z_i) + a_{2,i}^{\pm}(z - z_i)^2 + \ldots)$$

However, providing the series is not really a "solution"...

Connection problem:
how different Frobenius solutions are related

$$\begin{pmatrix} y_i^+(z) & y_i^-(z) \end{pmatrix} = \begin{pmatrix} y_j^+(z) & y_j^-(z) \end{pmatrix} \begin{pmatrix} a_{ij} & b_{ij} \\ c_{ij} & d_{ij} \end{pmatrix}$$

Scattering problem, eigenvalue (quantization) problem

Typically both solutions have physical interpretation,
and are usually linked by a discrete symmetry,
like time reversal

Important subclass of problems:
Fuchsian case and confluent limits

THE ISOMONODROMY METHOD

Relation to flat holomorphic connections:

$$\frac{d}{dz}\begin{pmatrix} y(z) \\ y'(z) \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -q(z) & p(z) \end{pmatrix}\begin{pmatrix} y(z) \\ y'(z) \end{pmatrix}$$

Given a set of solutions $\Phi(z)$, *can define* $A(z) = [\partial_z \Phi(z)]\Phi^{-1}(z)$

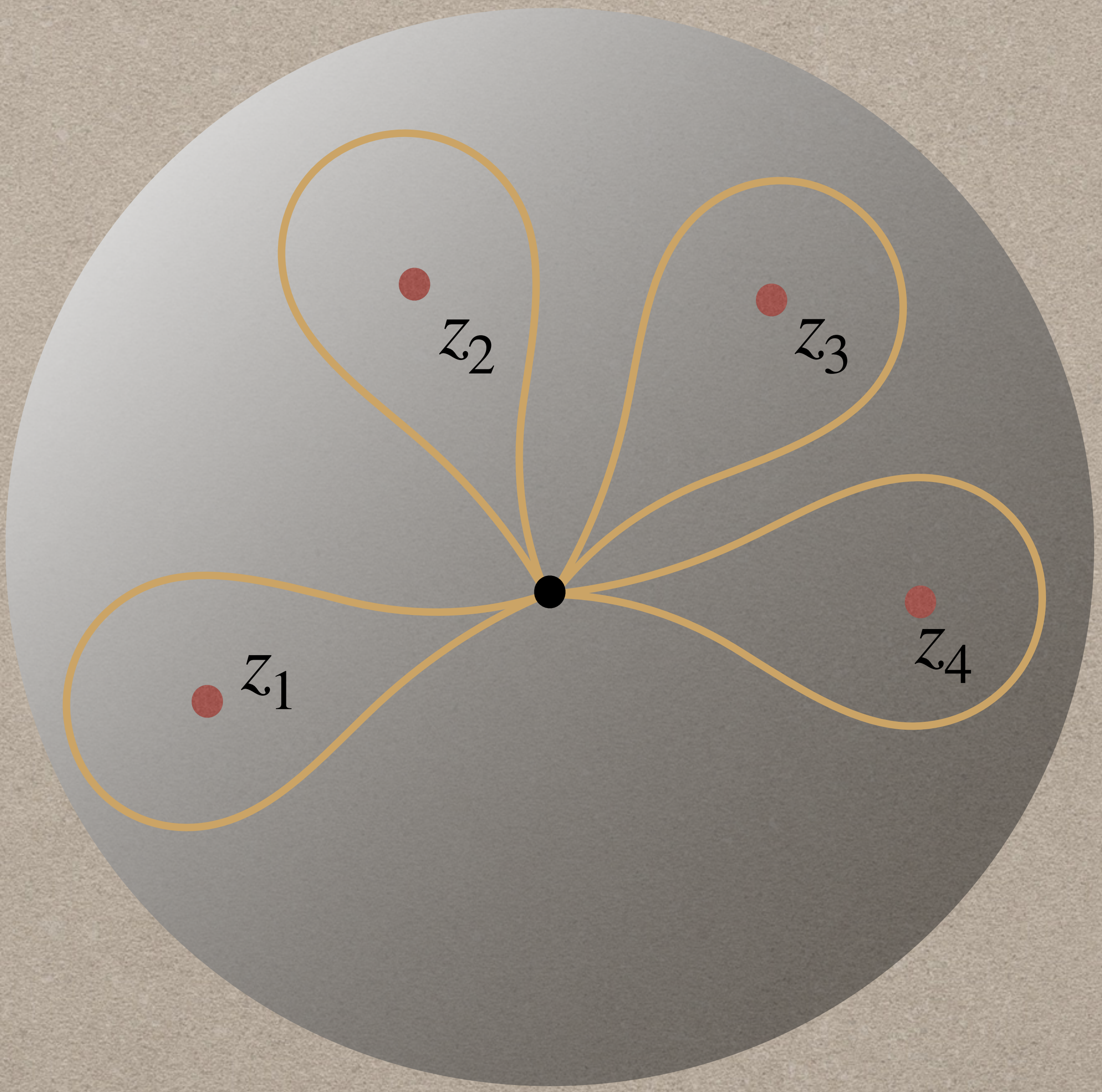
Converse not true:

$$y''(z) + p(z)y'(z) + q(z)y(z) = 0$$

$$p(z) = -\operatorname{Tr} A(z) - \frac{\partial_z A_{12}}{A_{12}} \qquad q(z) = \det A(z) - \partial_z A_{11} + A_{11}\frac{\partial_z A_{12}}{A_{12}}$$

An "oper"...

Gauge-invariant observables: holonomies and monodromies



$$M_i = \mathrm{P}\exp\left[\oint_{z_i} A(z)dz\right]$$

$$\mathrm{Tr}\, M_i = 2\cos \pi\theta_i$$

$$\mathrm{Tr}\, M_i M_j = 2\cos \pi\sigma_{ij}$$, etc.

"trace coordinates"

Holonomy = monodromy:

$$\Phi_i(z) = \begin{pmatrix} y_i^+(z) & y_i^-(z) \\ w_i^+(z) & w_i^-(z) \end{pmatrix} = \Psi_i(z) \begin{pmatrix} (z-z_i)^{\frac{1}{2}(\alpha_i+\theta_i)} & 0 \\ 0 & (z-z_i)^{\frac{1}{2}(\alpha_i-\theta_i)} \end{pmatrix}$$

$$\Phi_i(z_i + (z-z_i)e^{2\pi i}) = e^{\pi i \alpha}\Phi_i(z) M_i \qquad M_i = \begin{pmatrix} e^{\pi i \theta_i} & 0 \\ 0 & e^{-\pi i \theta_i} \end{pmatrix}$$

In a generic basis:

$$M_i = E_i \, e^{\pi i \theta_i \sigma^3} E_i^{-1}$$

connection matrix

Does the monodromy data determine the connection matrix?

Answer: for a wide class of equations, yes.

$$y^{+}(z) = T[y^{-}(z)]$$

translates to conditions on the connection matrix.

Transformation can be an involution between solutions, like time-reversal.

Example: scattering

$z = z_i$ $\quad\quad\quad$ $z = z_j$

$\Phi_i(z)$ $\quad\quad\quad$ $\Phi_j(z)$

"in" and "out" solutions related by time reversal

$$E_{ij} = \begin{pmatrix} \frac{1}{\mathcal{T}} & \frac{\mathcal{R}^*}{\mathcal{T}^*} \\ \frac{\mathcal{R}}{\mathcal{T}} & \frac{1}{\mathcal{T}^*} \end{pmatrix}$$

$$\mathrm{Tr}\, M_i M_j = 2\cos \pi\sigma_{ij}$$

$$|\mathcal{T}^2| = \left| \frac{\sin\frac{\pi}{2}(\theta_i + \theta_j - \sigma_{ij}) \sin\frac{\pi}{2}(\theta_i + \theta_j + \sigma_{ij})}{\sin \pi\theta_i \sin \pi\theta_j} \right|$$

1404.5188

w/ Novaes

Example 2: quantization (eigenvalue problem)

$$z = z_i \qquad\qquad\qquad\qquad z = z_j$$

$$\Phi_i(z) \qquad\qquad\qquad\qquad \Phi_j(z)$$

Need prescribed behavior at both singular points

$$E_{ij} = \begin{pmatrix} a_{ij} & 0 \\ c_{ij} & d_{ij} \end{pmatrix}$$

$$M_i M_j = M_j M_i \quad \longleftrightarrow \quad \sigma_{ij} = \theta_i + \theta_j + 2n, \quad n \in \mathbb{Z}$$

Example 3: classical Liouville theory

$$\Phi(z) = e^{\chi_L \sigma^-} e^{-\frac{1}{2}\phi_c \sigma^3} e^{\chi_R \sigma^+} = \begin{pmatrix} 1 & 0 \\ \chi_L & 1 \end{pmatrix} \begin{pmatrix} e^{-\frac{1}{2}\phi_c} & 0 \\ 0 & e^{\frac{1}{2}\phi_c} \end{pmatrix} \begin{pmatrix} 1 & \chi_R \\ 0 & 1 \end{pmatrix}$$

$$\mathscr{A}_z = [\partial_z \Phi(z)] \Phi^{-1}(z) \qquad \mathscr{A}_{12} = \mu \qquad \text{"oper" condition}$$

flatness is equivalent to Liouville equation

$$\partial_z \partial_{\bar{z}} \phi_c = \mu^2 e^{\phi_c}$$

General solution:

$$\phi_c(z, z^*) = \log\left(-\frac{2}{\mu^2} \frac{\partial_z \zeta \partial_{z^*} \zeta^*}{(\zeta(z) - \zeta^*(z^*))^2} \right)$$

Schwarzian parametrization of Liouville:

$$\{\zeta; z\} = \frac{1}{2}T(z)$$

Transformation to a linear equation

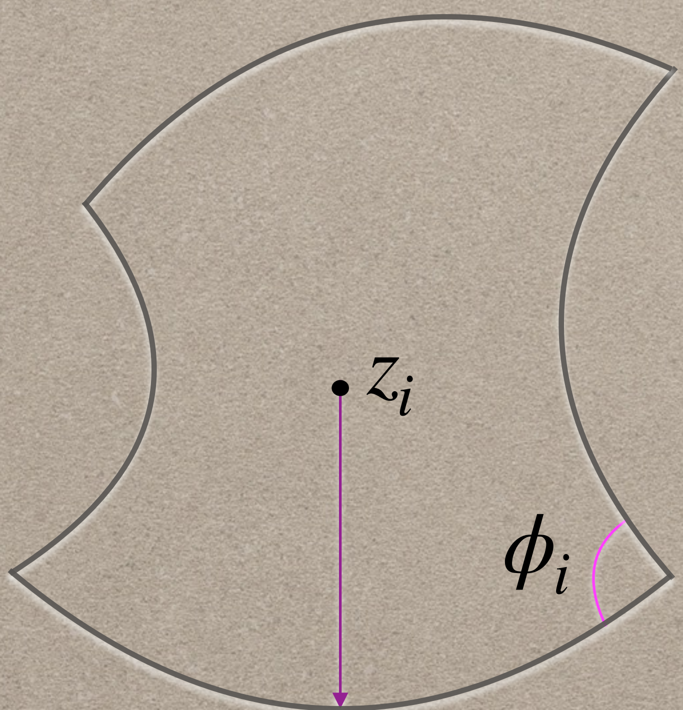
$$\zeta = \frac{y_+(z)}{y_-(z)} \qquad\qquad \frac{d^2 y_i}{dz^2} + T(z)y_i = 0$$

for a large class of domains – "polycircular arcs" – equation is determined by single and double poles:

$$T(z) = \sum_{i=1}^{N} \frac{\alpha_i}{(z - z_i)^2} + \frac{\beta_i}{z - z_i}$$

$\{\alpha_i, \beta_i, z_i\}$ *subject to Möbius invariance*

$$\sum_{i=1}^{N} \beta_i = 0, \qquad \sum_{i=1}^{N} (\beta_i z_i + \alpha_i) = 0, \qquad \sum_{i=1}^{N} (\beta_i z_i^2 + 2\alpha_i z_i) = 0$$



explicit representation of monodromy matrices

$$M_i = \frac{1}{r_i r_{i+1}} \begin{pmatrix} z_i \bar{z}_{i+1} + r_i^2 - |z_i|^2 & z_i(r_{i+1}^2 - |z_{i+1}|^2) - z_{i+1}(r_i^2 - |z_i|^2) \\ \bar{z}_{i+1} - \bar{z}_i & \bar{z}_i z_{i+1} + r_{i+1}^2 - |z_{i+1}|^2 \end{pmatrix}$$

Hard problem: find accessory parameters in terms of monodromy data

Riemann-Hilbert problem (4-singular point case)

$$T(z) = \frac{\alpha_0}{z^2} + \frac{\alpha_1}{(z-1)^2} + \frac{\alpha_{t_0}}{(z-t_0)^2} + \frac{\alpha_\infty - \alpha_0 - \alpha_1 - \alpha_{t_0}}{z(z-1)} - \frac{t(t-1)K_0}{z(z-1)(z-t_0)}$$

Easy part: $\theta_i(4-\theta_i) = 4\alpha_i, \qquad \theta_i = (1-\phi_i)$

Hard part: $(t_0, K_0) = f(\{\theta_i, \sigma_{ij}\})$

task is made possible by the existence of a non-linear symmetry

Embed the ODE into a matrix system

$$\frac{d\Phi(z)}{dz} = A(z)\Phi(z) \qquad A(z) = \frac{A_0}{z} + \frac{A_1}{z-1} + \frac{A_t}{z-t}$$

See t as a deformation parameter

$$\mathscr{A}_z = A(z) \qquad \mathscr{A}_t = -\frac{A_t}{z-t}$$

Flatness of connection = Schlesinger equations

$$\frac{\partial A_0}{\partial t} = \frac{1}{t}[A_t, A_0], \qquad \frac{\partial A_1}{\partial t} = \frac{1}{t-1}[A_t, A_1], \qquad \frac{\partial A_t}{\partial t} = -\frac{1}{t}[A_t, A_0] - \frac{1}{t-1}[A_t, A_1]$$

Deformation by t changes EDO, but keeps monodromies

"isomonodromic deformations"

$$y'' + p(z,t)y' + q(z,t)y = 0,$$

$$p(z,t) = -\frac{\operatorname{Tr} A_0}{z} - \frac{\operatorname{Tr} A_1}{z-1} - \frac{\operatorname{Tr} A_t}{z-t} - \frac{A'_{12}}{A_{12}}$$

$$q(z,t) = \frac{\det A_0}{z^2} + \frac{\det A_1}{(z-1)^2} + \frac{\det A_t}{(z-t)^2} + \frac{\kappa}{z(z-1)} - \frac{t(t-1)H}{z(z-1)(z-t)} - A'_{11} + A_{11}\frac{A'_{12}}{A_{12}}$$

Extra singularities at the roots of

$$A_{12} = \frac{k(z-\lambda)}{z(z-1)(z-t)}$$

Hamiltonian system

$$H = \frac{1}{t}(\mathrm{Tr}\,A_0 A_t - \mathrm{Tr}\,A_0 \mathrm{Tr}\,A_t) + \frac{1}{t-1}(\mathrm{Tr}\,A_1 A_t - \mathrm{Tr}\,A_1 \mathrm{Tr}\,A_t)$$

$$H(\lambda,\mu,t) = \frac{\lambda(\lambda-1)(\lambda-t)}{t(t-1)}\left[\mu\left(\mu - \frac{\mathrm{Tr}\,A_0}{\lambda} - \frac{\mathrm{Tr}\,A_1}{\lambda-1} - \frac{\mathrm{Tr}\,A_t}{\lambda-t}\right) + \frac{\det A_0}{\lambda^2} + \frac{\det A_1}{(\lambda-1)^2} + \frac{\det A_t}{(\lambda-t)^2} + \frac{\kappa}{\lambda(\lambda-1)}\right]$$

$$\mu = A_{11}(z=\lambda) = \frac{[A_0]_{11}}{\lambda} + \frac{[A_1]_{11}}{\lambda-1} + \frac{[A_t]_{11}}{\lambda-t}$$

Garnier system:

$$\frac{d\lambda}{dt} = \{K,\lambda\}, \qquad \frac{d\mu}{dt} = \{K,\mu\} \qquad K = H + \frac{\lambda(\lambda-1)}{t(t-1)}\mu + \frac{\lambda-t}{t(t-1)}\kappa_1$$

$$\ddot{\lambda} = \frac{1}{2}\left(\frac{1}{\lambda} + \frac{1}{\lambda-1} + \frac{1}{\lambda-t}\right)\dot{\lambda}^2 - \left(\frac{1}{t} + \frac{1}{t-1} + \frac{1}{\lambda-t}\right)\dot{\lambda} + \frac{\lambda(\lambda-1)(\lambda-t)}{2t^2(1-t)^2}\left(\theta_\infty^2 - \theta_0^2\frac{t}{\lambda^2} + \theta_1^2\frac{t-1}{(\lambda-1)^2} + \left(1-\theta_t^2\right)\frac{t(t-1)}{(\lambda-t)^2}\right)$$

Jimbo, Miwa et al. *gave an asymptotic expression for the PVI tau function in terms of the monodromy invariants*

$$\frac{d}{dt}\log\tau(t,\{\theta_i\}) = \frac{1}{t}\mathrm{Tr}\,(A_0A_t) + \frac{1}{t-1}\mathrm{Tr}\,(A_1A_t)$$

for our purposes it is just a matter of choosing appropriate initial "merging" initial conditions for the Garnier system:

$$t(t-1)\frac{d}{dt}\log\tau(t,\{\theta_i\})\Big|_{t=t_0} = (t_0-1)\mathrm{Tr}\,A_0\mathrm{Tr}\,A_{t_0} + t_0\mathrm{Tr}\,A_1\mathrm{Tr}\,A_{t_0} + t_0(t_0-1)K_0$$

$$\frac{d}{dt}\left[t(t-1)\frac{d}{dt}\log\tau(t,\vec{\theta},\vec{\sigma})\right]\Bigg|_{t=t_0} = 2\det A_{t_0} - (\mathrm{Tr}\,A_{t_0})^2 - \mathrm{Tr}\,(A_\infty A_{t_0})$$

The second condition is strange in that it does not depend on the accessory parameters…

$$\frac{\partial \Phi^{\pm}}{\partial z}[\Phi^{\pm}]^{-1} = A^{\pm}(z) = L^{\pm}A(L^{\pm})^{-1} + \frac{\partial L^{\pm}}{\partial z}(L^{\pm})^{-1}$$
$$= \frac{A_0^{\pm}}{z} + \frac{A_1^{\pm}}{z-1} + \frac{1}{z-t}\begin{pmatrix}\theta_t \pm 1 & 0 \\ 0 & \beta\end{pmatrix}$$

Associated solutions, with shifted monodromies

$$\Phi^{+}(z) = L^{+}(z)\Phi(z) = \begin{pmatrix}1 & 0 \\ p^{+} & 1\end{pmatrix}\begin{pmatrix}z-t & 0 \\ 0 & 1\end{pmatrix}\begin{pmatrix}1 & q^{+} \\ 0 & 1\end{pmatrix}\Phi(z),$$
$$\Phi^{-}(z) = \frac{1}{z-t}\begin{pmatrix}1 & p^{-} \\ 0 & 1\end{pmatrix}\begin{pmatrix}1 & 0 \\ 0 & z-t\end{pmatrix}\begin{pmatrix}1 & 0 \\ q^{-} & 1\end{pmatrix}\Phi(z)$$

"Toda equation"

$$\frac{d}{dt}t(t-1)\frac{d}{dt}\log\tau + \frac{1}{2}\theta_t(\theta_\infty + \theta_t - \theta_0 - \theta_1) = C\frac{\tau^+\tau^-}{\tau^2}$$

A careful analysis shows that, if we want to embed our ODE in the first line of the Fuchsian matrix system, the conditions are

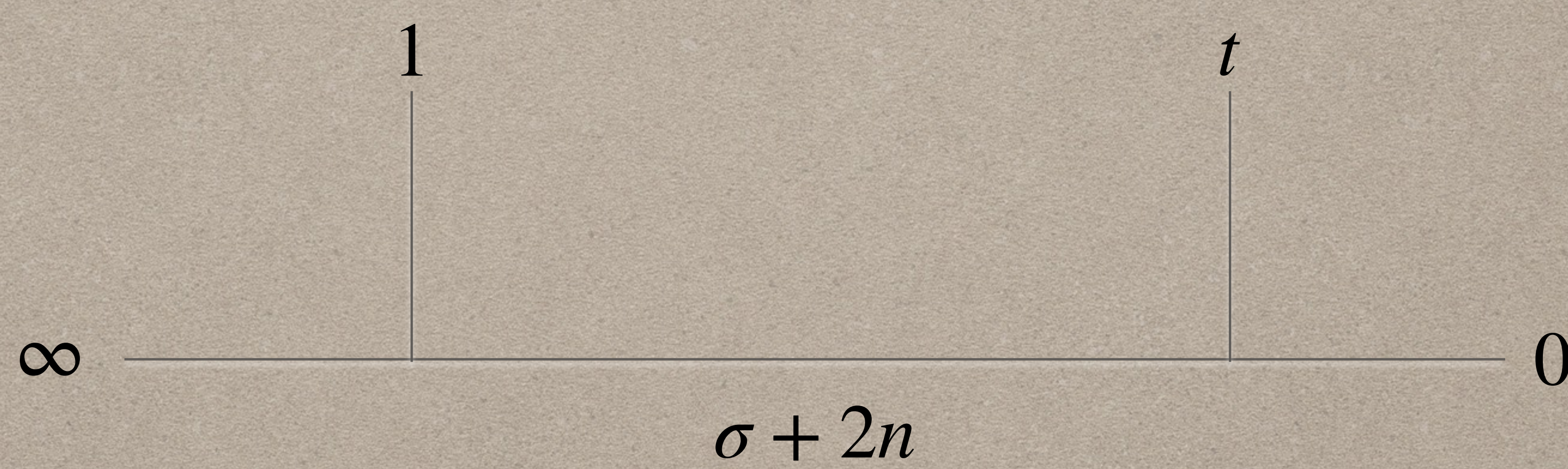
$$\left.\frac{d}{dt}\log\tau(t, \{\theta_0, \theta_1, \theta_t - 1, \theta_\infty\}, \{\sigma_{0t} - 1, \sigma_{1t} - 1\})\right|_{t=t_0} = K_0 + \frac{\theta_t - 1}{2}\left(\frac{\theta_0}{t_0} + \frac{\theta_1}{t_0 - 1}\right)$$

$$\tau(t_0, \{\theta_0, \theta_1, \theta_t, \theta_\infty\}, \{\sigma_{0t}, \sigma_{1t}\}) = 0$$

- zero of tau function = existence of solution of Fuchsian system with prescribed monodromy data;
- associated tau function = Hamilton-Jacobi solution of the PVI system;
- can find accessory parameters as solutions of transcendental equations
- can use numerical solutions of PVI, or asymptotic expansions given by GIL 2013 (Nekrasov functions) and GL 2016 (Fredholm determinant).

Nekrasov expansion: tau function as Virasoro conformal block

$$\tau(t, \{\theta_i\}) = \langle \Phi_\infty(\infty)\Phi_1(1)\mathscr{P}_\Delta \Phi_t(t)\Phi_0(0)\rangle$$



Zamolodchikov 89: expansion can be computed recursively

AGT (AFLT) 09-10: function can be computed by N=2 SYM SU(N) with matter (Nekrasov partition function)

GIL 12-13: Instanton function satisfies PVI for c=1

Double expansion near t=0

$$\tau(t) = \sum_{n\in\mathbb{Z}} \mathcal{N}^{\theta_\infty}_{\theta_1,\sigma+2n}\mathcal{N}^{\sigma+2n}_{\theta_t,\theta_0} s^n t^{\frac{1}{4}((\sigma+2n)^2-\theta_0^2-\theta_t^2)}(1-t)^{\frac{1}{2}\theta_0\theta_t}\sum_{\lambda,\mu\in\mathbb{Y}}\mathcal{B}_{\lambda,\mu}(\overrightarrow{\theta},\sigma+2n)t^{|\lambda|+|\mu|}$$

$$\mathcal{N}^{\theta_3}_{\theta_2,\theta_1} = \frac{\prod_{\epsilon=\pm} G(1+\frac{1}{2}(\theta_3+\epsilon(\theta_2+\theta_1)))G(1-\frac{1}{2}(\theta_3+\epsilon(\theta_2-\theta_1)))}{G(1-\theta_1)G(1-\theta_2)G(1+\theta_3)}$$

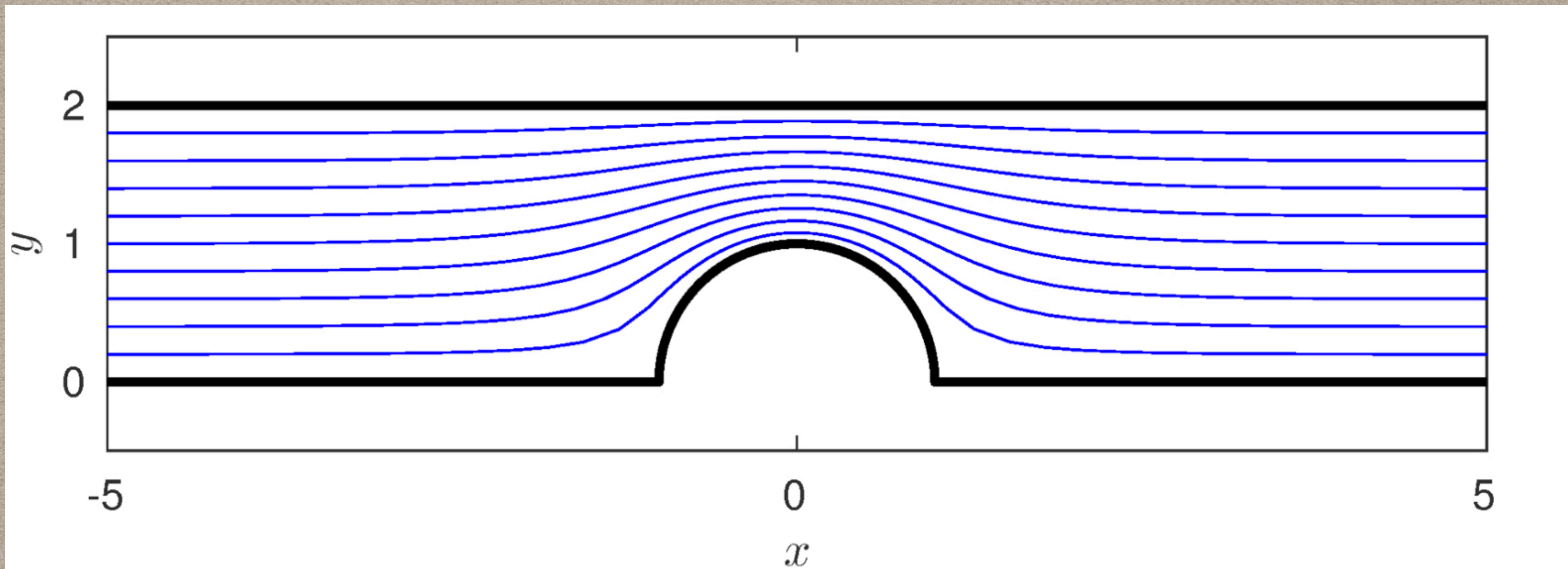
$$\mathcal{B}_{\lambda,\mu}(\overrightarrow{\theta},\sigma) = \prod_{(i,j)\in\lambda}\frac{((\theta_t+\sigma+2(i-j))^2-\theta_0^2)((\theta_1+\sigma+2(i-j))^2-\theta_\infty^2)}{16h_\lambda^2(i,j)(\lambda'_j-i+\mu_i-j+1+\sigma)^2}\times\prod_{(i,j)\in\mu}\frac{((\theta_t-\sigma+2(i-j))^2-\theta_0^2)((\theta_1-\sigma+2(i-j))^2-\theta_\infty^2)}{16h_\lambda^2(i,j)(\mu'_j-i+\lambda_i-j+1-\sigma)^2}$$

$$s = \frac{(w_{1t}-2p_{1t}-p_{0t}p_{01})-(w_{01}-2p_{01}-p_{0t}p_{1t})\exp(\pi i\sigma_{0t})}{(2\cos\pi(\theta_t-\sigma_{0t})-p_0)(2\cos\pi(\theta_1-\sigma_{0t})-p_\infty)}$$

$$p_i = 2\cos\pi\theta_i,\quad p_{ij} = 2\cos\pi\sigma_{ij}, w_{0t} = p_0p_t+p_1p_\infty,\quad w_{1t} = p_1p_t+p_0p_\infty,\quad w_{01} = p_0p_1+p_tp_\infty\,.$$

GIL2013

Numerical tests for computing accessory parameters



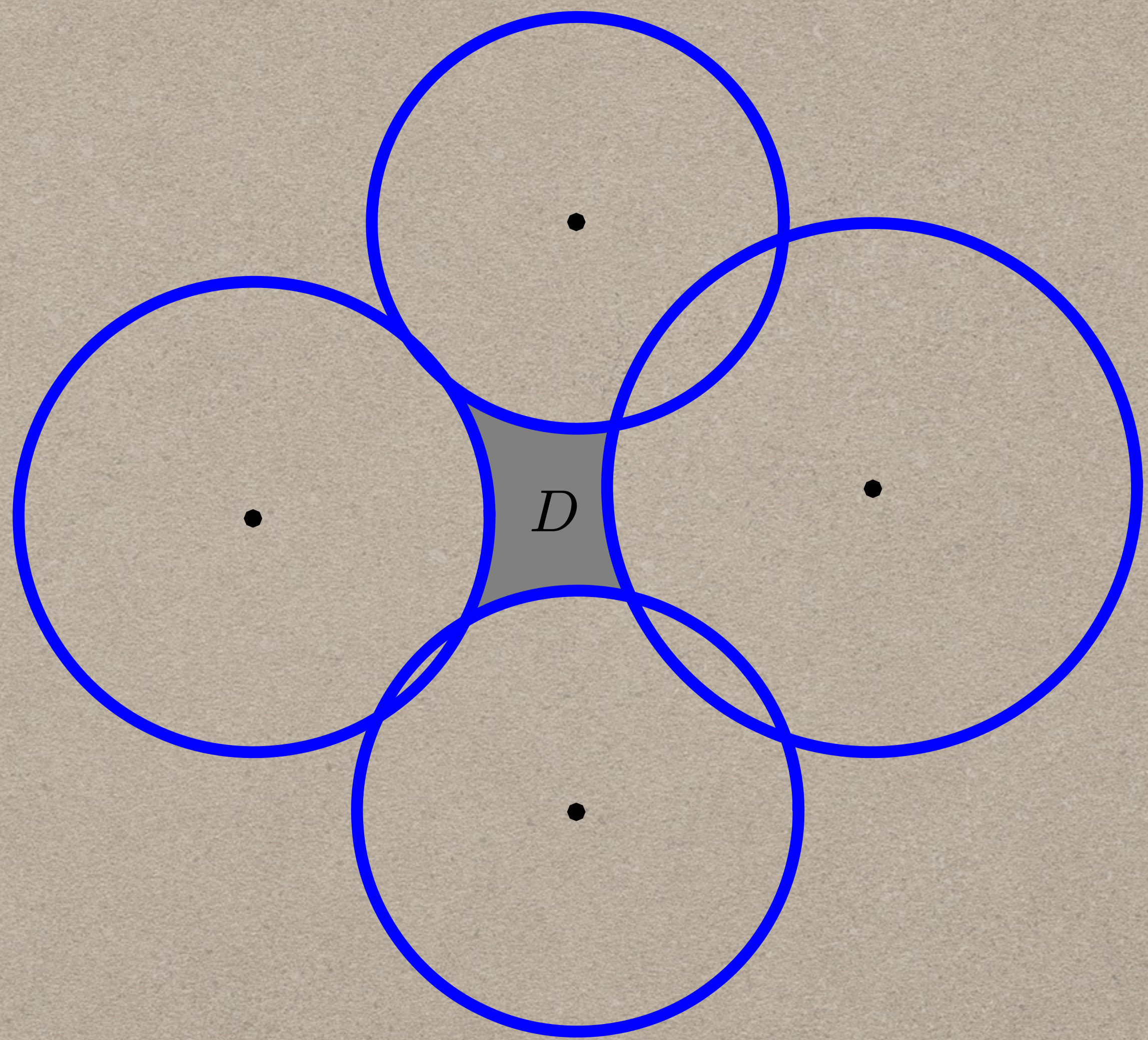
$$h = -\cos \pi\sigma$$

$$t_0^{1-\sigma} \simeq \frac{1+\sin(\pi\sigma)}{1-\sin(\pi\sigma)} \frac{\Gamma^4(\frac{1}{4}+\frac{\sigma}{2})}{\Gamma^4(\frac{5}{4}-\frac{\sigma}{2})} \frac{\Gamma^2(1-\sigma)}{\Gamma^2(\sigma-1)} \qquad K_0 \simeq \frac{(\sigma-1)^2-(\theta_0+\theta_{t_0}-1)^2}{4t_0}$$

$$h = 2 \Rightarrow t_0 \simeq 3.905353\times 10^{-4}, \quad K_0 \simeq -2.725292\times 10^2$$

Fast convergence. At worst as bad as naive numerics.

w/ Anselmo, Nelson, Crowdy



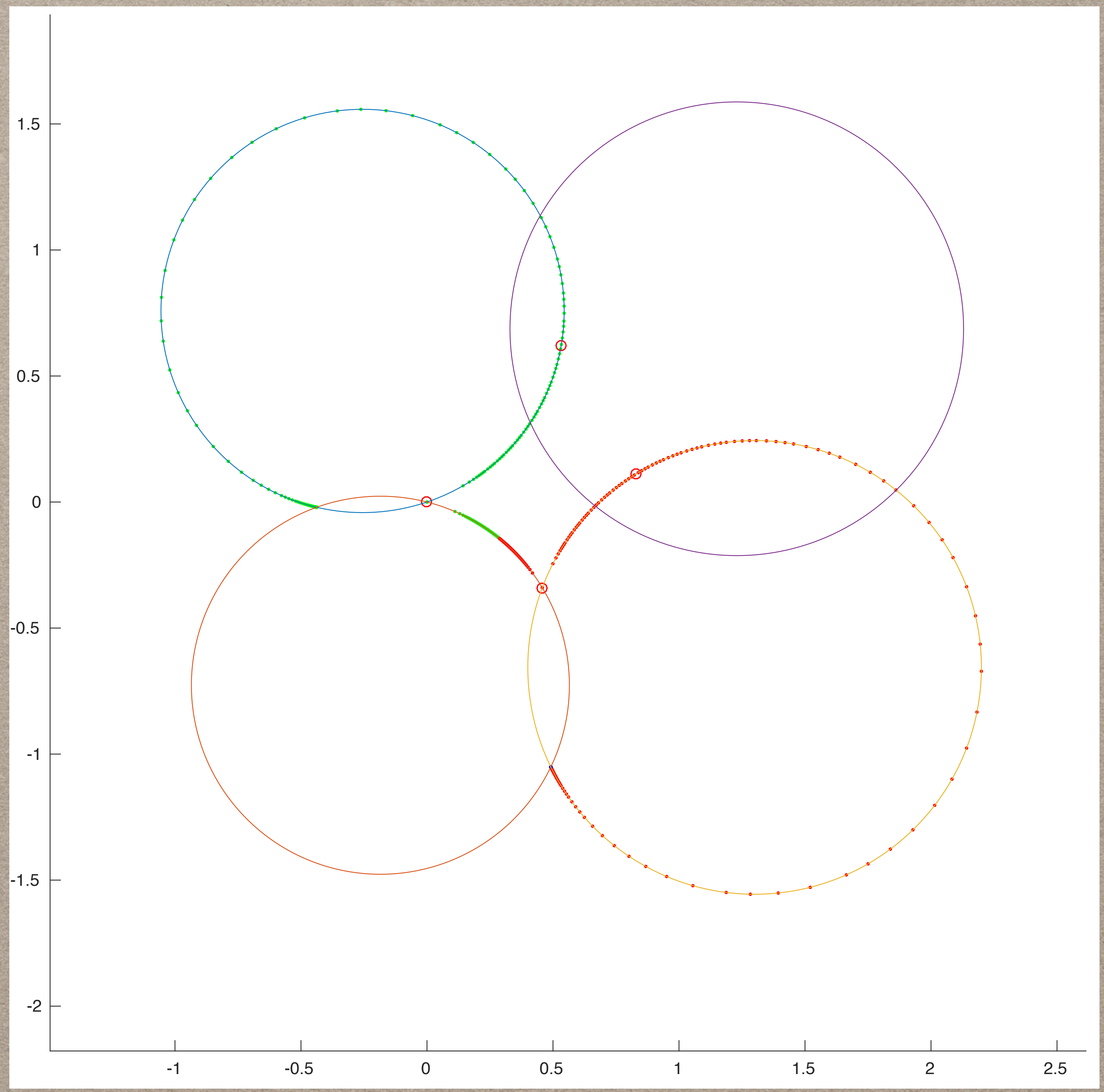
θ_0	0.1827991846	σ_{0t_0}	$1 - 0.4304546489i$
θ_{t_0}	0.2869823004	σ_{1t_0}	$1 - 0.5385684561i$
θ_1	0.3673544015	σ_{01}	$0.9631297769 + 0.7221017400i$
θ_{∞_0}	0.0853271421	$J(\theta_i, \sigma_{ij})$	0

	New method	Howell's method
K_0	-0.4364792362	-0.4365168488
t_0	0.2086468690	0.2086251630

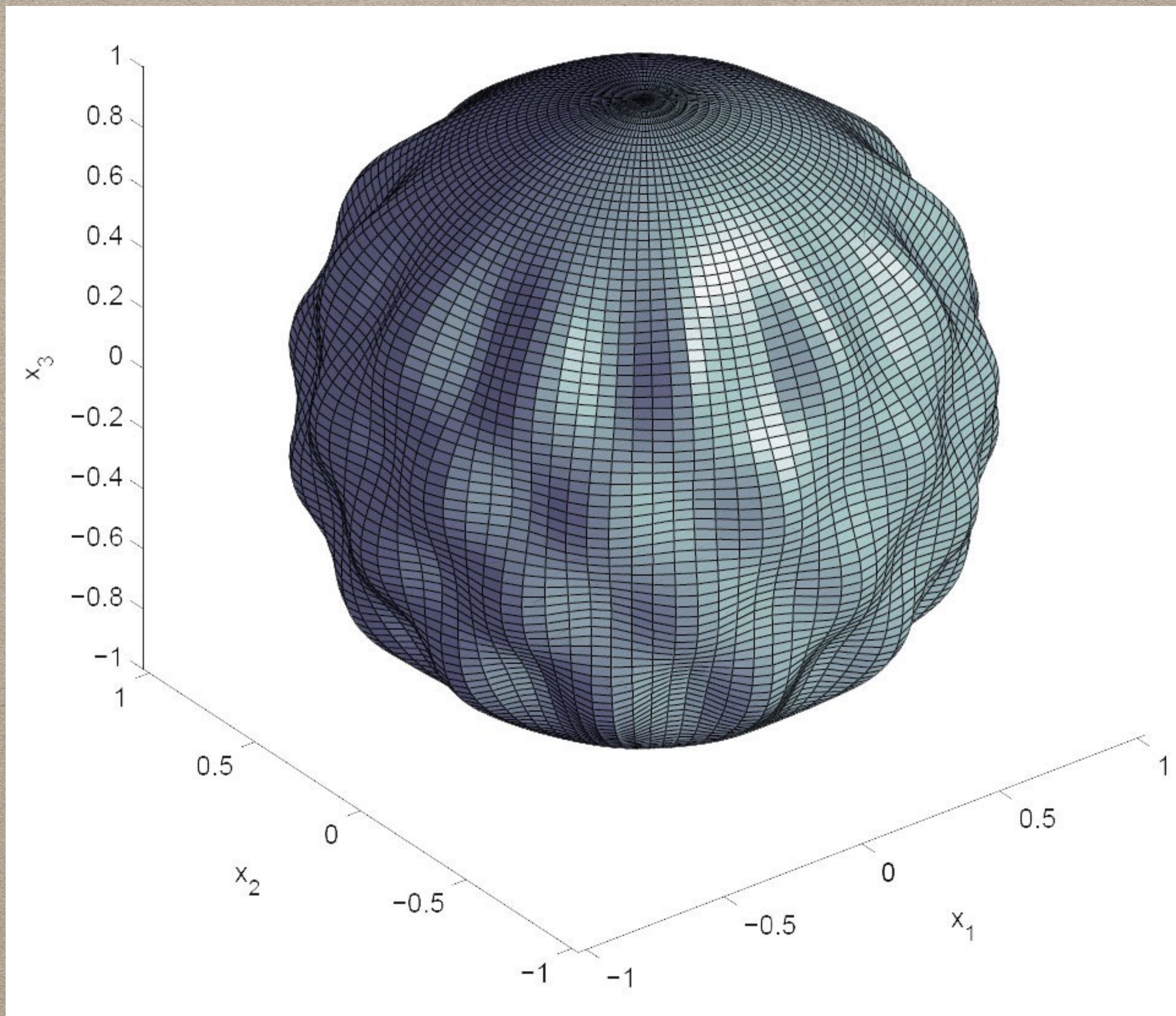
At this point we'd like to conjecture that the tau function would have only one zero in the [0,1] interval, but…

$$t_0 \simeq 1.0706 \times 10^{-7}$$

"isomonodromic region"



Example 2: Kerr-AdS5 black hole: resonances (quasi-normal modes)



w/ Amado, Pallante

Fredholm determinant: rel. b/w tau-function and CR-operator

$$\tau(t) = \text{const.} \cdot t^{\frac{1}{4}(\sigma^2-\theta_0^2-\theta_t^2)}(1-t)^{-\frac{1}{2}\theta_t\theta_1}\det(\mathbf{1}-AD)$$

$$(Ag)(z) = \oint_{\mathscr{C}} \frac{dz'}{2\pi i} A(z,z')g(z'), \quad (Dg)(z) = \oint_{\mathscr{C}} \frac{dz'}{2\pi i} D(z,z')g(z'), \quad g(z') = \begin{pmatrix} f_+(z) \\ f_-(z) \end{pmatrix}$$

$$A(z,z') = \frac{\Psi(\theta_1,\theta_\infty,\sigma;z)\Psi^{-1}(\theta_1,\theta_\infty,\sigma;z') - \mathbf{1}}{z-z'}$$

$$D(z,z') = \Phi(t)\frac{\mathbf{1} - \Psi(\theta_t,\theta_0,-\sigma;t/z)\Psi^{-1}(\theta_t,\theta_0,-\sigma;t/z')}{z-z'}\Phi^{-1}(t)$$

$$\Psi(\theta_1,\theta_2,\theta_3;z) = \begin{pmatrix} \phi(\theta_1,\theta_2,\theta_3;z) & \chi(\theta_1,\theta_2,\theta_3;z) \\ \chi(\theta_1,\theta_2,-\theta_3;z) & \phi(\theta_1,\theta_2,-\theta_3,z) \end{pmatrix}, \qquad \Phi(t) = \begin{pmatrix} t^{-\sigma/2}\tilde{s}^{1/2} & 0 \\ 0 & t^{\sigma/2}\tilde{s}^{-1/2} \end{pmatrix},$$

$$\phi(\theta_1,\theta_2,\theta_3;z) = {}_2F_1(\tfrac{1}{2}(\theta_1+\theta_2+\theta_3), \tfrac{1}{2}(\theta_1-\theta_2+\theta_3); \theta_3; z)$$

$$\chi(\theta_1,\theta_2,\theta_3;z) = \frac{\theta_2^2-(\theta_1+\theta_3)^2}{4\theta_3(1+\theta_3)} z\, {}_2F_1(1+\tfrac{1}{2}(\theta_1+\theta_2+\theta_3), 1+\tfrac{1}{2}(\theta_1-\theta_2+\theta_3); 2+\theta_3; z)\,.$$

GL2016

Scalar field in Kerr-AdS5: separable wave equation

Angular equation: Heun with 4 regular singular points

$$u = 0, \quad u = 1, \quad u = u_0 = \frac{a_2^2 - a_1^2}{a_2^2 - 1}, \quad u = \infty,$$

$$\alpha_0 = \pm \frac{m_1}{2}, \quad \alpha_1 = \frac{1}{2}\left(2 \pm \sqrt{4 + \mu^2}\right), \quad \alpha_{u_0} = \pm \frac{m_2}{2}, \quad \alpha_\infty = \pm \frac{1}{2}(\omega + a_1 m_1 + a_2 m_2)\,.$$

$$4u_0(u_0 - 1)Q_0 = -\frac{\omega^2 + a_1^2\mu^2 - \lambda}{a_2^2 - 1} - u_0\left[(m_2 - \Delta + 1)^2 - m_2^2 - 1\right] - (u_0 - 1)\left[(1 - m_1 - m_2)^2 - \beta^2 - 1\right]$$

Eigenvalues: $\quad \sigma_{0u_0}(m_1, m_2, \beta, \Delta, u_0, \lambda_\ell) = m_1 + m_2 + 2j, \qquad j \in \mathbb{Z}$

Separation constant

$$\lambda_\ell \simeq \ell(\ell+2) - 2\omega\left(a_1 m_1 + a_2 m_2\right) - \left(a_1 m_1 + a_2 m_2\right)^2 + \frac{a_1^2 + a_2^2}{2}\left(\beta^2 + \mu^2 - \ell(\ell+2)\right) + \frac{\left(a_2^2 - a_1^2\right)\left(m_2^2 - m_1^2\right)}{2\ell(\ell+2)}\left(\beta^2 - \mu^2 - (\ell^2 + 2\ell + 4)\right) + \mathcal{O}((a_2^2 - a_1^2)^2)$$

Check with AdS spheroidal literature

Radial equation: also Heun's with 4 regular singular points

$$z_0 = (r_+^2 - r_-^2)/(r_+^2 - r_0^2)$$

$$\theta_k = \pm \frac{i}{2\pi}\left(\frac{\omega - m_1\Omega_{k,a} - m_2\Omega_{k,b}}{T_k}\right), \quad \theta_\infty = 2 - \Delta,$$

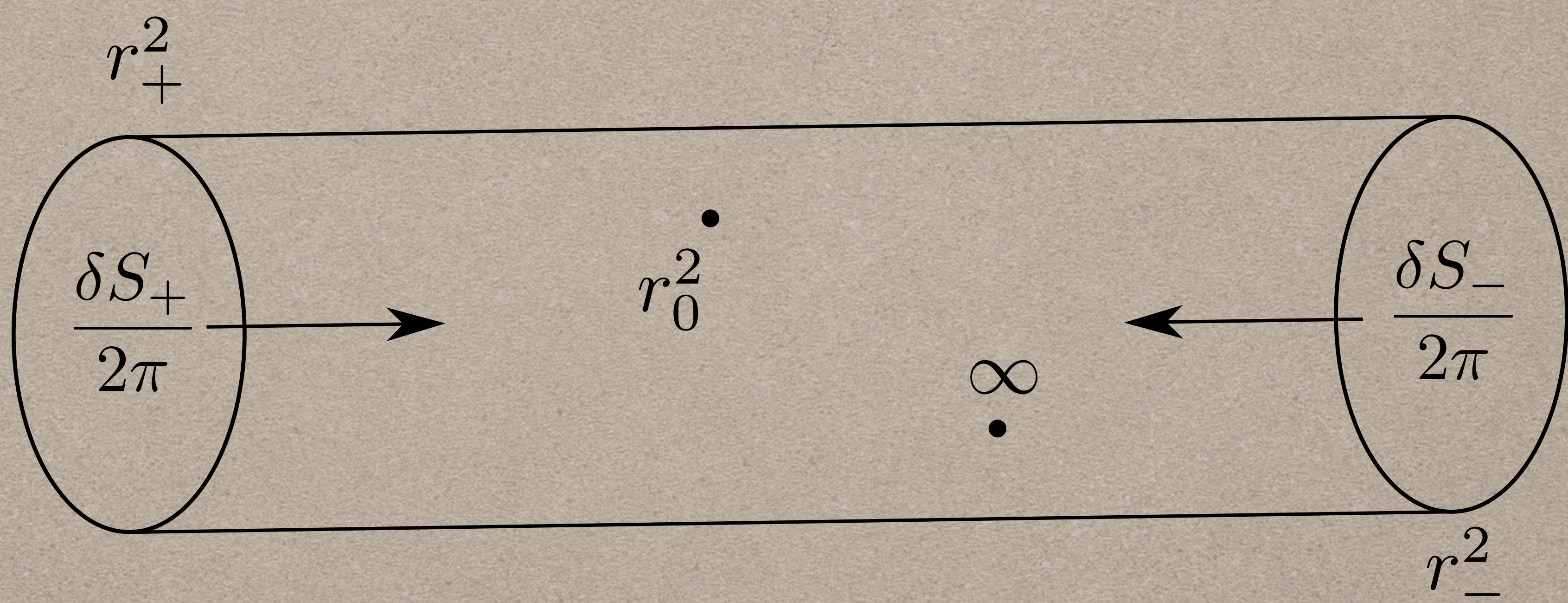
$$4z_0(z_0 - 1)K_0 = -\frac{\lambda + \mu^2 r_-^2 - \omega^2}{r_+^2 - r_0^2} - (z_0 - 1)[(\theta_- + \theta_+ - 1)^2 - \theta_0^2 - 1] - z_0\left[\left(\theta_+ - \Delta + 1\right)^2 - \theta_+^2 - 1\right]$$

No energy flux at outer horizon and at infinity: quantization

$$\sigma_{1z_0}(\theta_k, \Delta, z_0, \omega_n, \lambda_\ell) = \theta_+ + \Delta + 2n - 2, \qquad n \in \mathbb{Z}\,.$$

Now, need to use tau function to find frequencies…

Liouville representation:



- Unitary!
- Entropy intake = Liouville momentum;
- Extra singular point not clear.

Schwarzschild (zero angular momenta)

r_+	z_0	ω_0
0.005	2.49988×10^{-5}	$3.9998498731325748 - 1.5044808171834238 \times 10^{-6}i$
0.01	9.99800×10^{-5}	$3.9993983005189682 - 1.2123793015872442 \times 10^{-5}i$
0.05	2.48756×10^{-3}	$3.9844293869590734 - 1.7525974895168137 \times 10^{-3}i$
0.1	9.80392×10^{-3}	$3.9355764849860639 - 1.7970664179740506 \times 10^{-2}i$
0.2	3.70370×10^{-2}	$3.7906778316981978 - 0.1667439940917780i$
0.4	0.121212	$3.7173879743704008 - 0.7462495474087164i$
0.6	0.209302	$3.8914015767067012 - 1.3656095289384492i$

Table 1. First quasi-normal modes ω_0 for Schwarzschild-AdS$_5$ for massless scalar field and some values of r_+. The results were obtained using the Fredholm determinant expansion for the τ-function with $N = 16$.

r_+	Frobenius	QEP
0.005	$3.9998498731325743 - 1.5044808171845522 \times 10^{-6}i$	$3.9998483860043481 - 2.8895543908757586 \times 10^{-5}i$
0.01	$3.9993983005189876 - 1.2123793015712405 \times 10^{-5}i$	$3.9993981402971502 - 2.3439366987252536 \times 10^{-5}i$
0.05	$3.9844293869590911 - 1.7525974895155961 \times 10^{-3}i$	$3.9844293921364538 - 1.7526437924554161 \times 10^{-3}i$
0.1	$3.9355764849860673 - 1.7970664179739766 \times 10^{-2}i$	$3.9355763694852816 - 1.7970671629389028 \times 10^{-2}i$
0.2	$3.7906778316982394 - 0.1667439940917505i$	$3.7906771832980760 - 0.1667441392742093i$
0.4	$3.7173879743704317 - 0.7462495474087220i$	$3.7173988607936563 - 0.7462476412816416i$
0.6	$3.8914015767126869 - 1.3656095289361863i$	$3.8913340701538795 - 1.3656086881322822i$

Table 2. The same quasi-normal modes frequencies ω_0 computed using numerical matching from Frobenius solutions (with 15 terms) and the Quadratic Eigenvalue Problem (with 120 point-lattice).

Generic ($a_1 \simeq a_2$) Kerr case

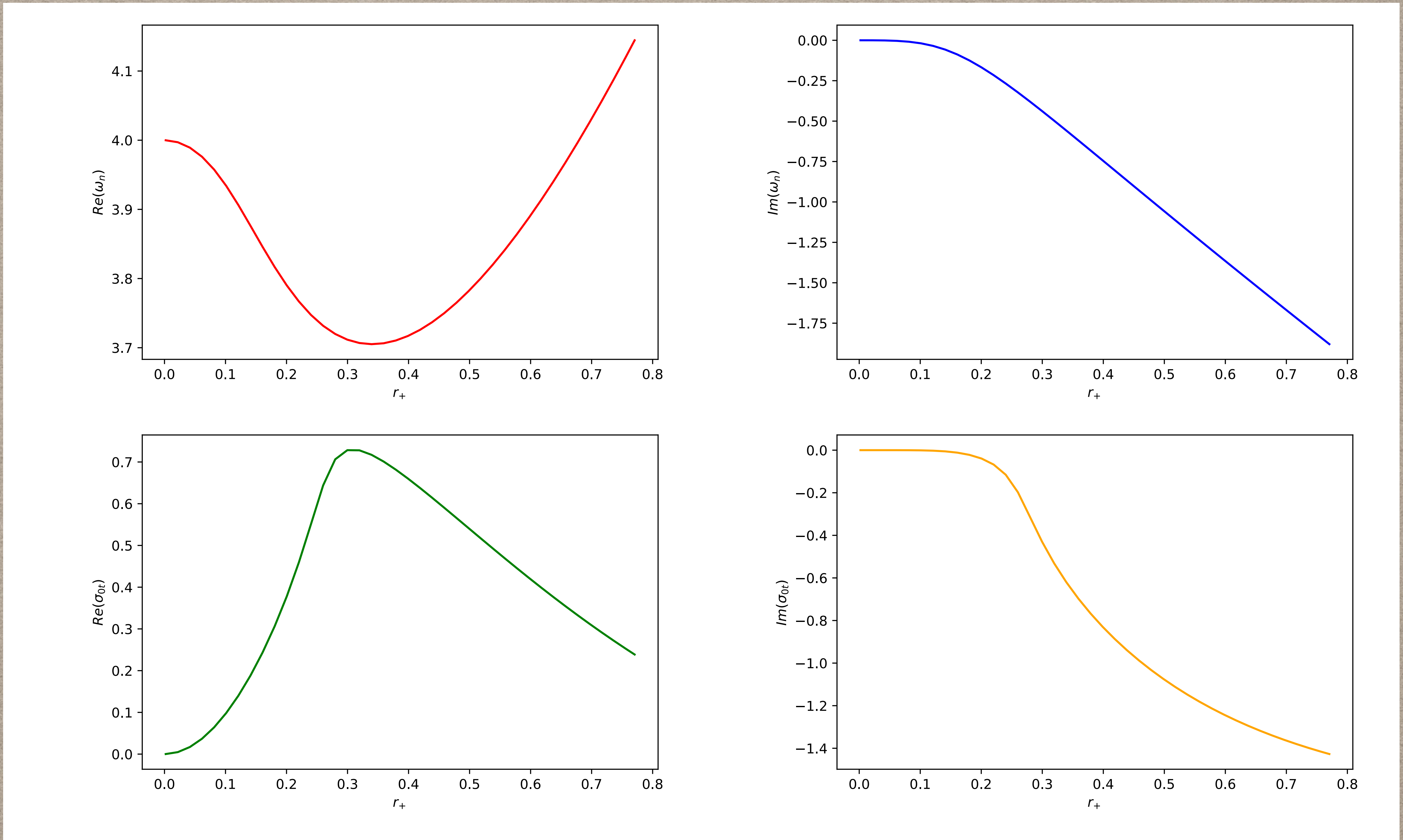


Figure 1. In the first row, the dependence of the real and imaginary parts of the first quasi-normal mode frequency ω_0 for small Kerr-AdS$_5$ black holes ($a_1 = 0.002, a_2 = 0.00199, \mu = 7.99 \times 10^{-8}$). In the second, the dependence of the composite parameter σ_{0t}.

- Work in progress!
- Numerics: Fredholm O(N^4) vs. Nekrasov O(e^N);
- Connection problem for Heun's equation seems solved (at least for the subclass);
- New significance for zeros of tau function;
- Higher number of points, real zeros, recursion relations, etc…

- Best shot at finding superradiant mode (TBA)
- AdS/CFT applications? Interpretation of conformal blocks?
- Radial vs. angular systems: thermodynamical interpretation?
- higher modes?
- Confluent limit also works, gives Painlevé V
- Useful for quantization of the Rabi model (w/ Queiroz, Almeida)
- Useful for usual Kerr (d=4). Astrophysical applications?

Thank you!

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