Yangian Symmetry and Integrability of Planar $\mathcal{N} = 4$ SYM

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Exactly Solvable Quantum Chains
IIP Natal
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work with A. Garus, M. Rosso (1803.06310, 1701.09162)
and with A. Garus (1804.09110).

Introduction and Overview

Aim:

Prove Yangian symmetry in integrable planar gauge theories.

Outline:

- Yangian Symmetry of Planar $\mathcal{N} = 4$ SYM
- Correlation Functions

General Assumptions:

- $\mathcal{N} = 4$ supersymmetric Yang–Mills theory
- Planar limit
- Most results also apply to ABJM
  ($\mathcal{N} = 6$ supersymmetric Chern–Simons theory)

I. Yangian Symmetry of Planar $\mathcal{N} = 4$ SYM

AdS/CFT Integrability

Integrability: Curious feature of planar $\mathcal{N} = 4$ SYM (and related);
enables efficient computations:

- planar spectrum of anomalous dimensions (finite λ)
- correlation functions of local operators
- colour-ordered scattering amplitudes
- null polygon Wilson loops
- planar loop integrands (integrals?)
- ...

What (precisely) is integrability? How to prove it?

- several ansätze or definitions in particular situations
- ...
- hidden symmetry enhancement:

$$\text{superconformal } \mathfrak{psu}(2,2|4) \to \text{Yangian } \mathrm{Y}[\mathfrak{psu}(2,2|4)]$$

Yangian Symmetry

"Symmetry" in what sense?

- Spectrum is not invariant (boundary conditions).
- Scattering amplitudes are IR divergent (massless particles).
- Null polygon Wilson loops are UV divergent.
- Smooth Maldacena–Wilson loops are finite and invariant.
- Symmetry for other observables less evident.
- Ordering principle, tools, . . .

Invariance of the action!

Complications:

- representation non-linear in fields,
- cyclic boundary conditions,
- implementation of planar limit,
- non-local properties,
- quantum anomalies?

Yangian Algebra

Defined in terms of level-zero and level-one generators J^A, $\widehat{\mathrm{J}}^A$:

Algebra Relations:

$$[\mathrm{J}^A, \mathrm{J}^B] \sim f_C^{AB} \mathrm{J}^C,$$
$$[\mathrm{J}^A, \widehat{\mathrm{J}}^B] \sim f_C^{AB} \widehat{\mathrm{J}}^C,$$
$$\big[\widehat{\mathrm{J}}^A, [\widehat{\mathrm{J}}^B, \mathrm{J}^C]\big] + \text{cyclic} \approx \{\mathrm{J}, \mathrm{J}, \mathrm{J}\}.$$

$\widehat{\mathrm{J}}$ in adjoint; satisfies Serre relation.

Coproduct:

$$\Delta \mathrm{J}^C \sim \mathrm{J}^C \otimes 1 + 1 \otimes \mathrm{J}^C,$$
$$\Delta \widehat{\mathrm{J}}^C \sim \widehat{\mathrm{J}}^C \otimes 1 + 1 \otimes \widehat{\mathrm{J}}^C + f_{AB}^C \mathrm{J}^A \otimes \mathrm{J}^B.$$

$\mathrm{J}/\widehat{\mathrm{J}}$ acts locally/bi-locally.

Level-one momentum (dual conformal) $\widehat{\mathrm{P}}$ easiest:

$$\Delta \widehat{\mathrm{P}} \sim \widehat{\mathrm{P}} \otimes 1 + 1 \otimes \widehat{\mathrm{P}} + \mathrm{P} \wedge \mathrm{D} + \mathrm{P} \wedge \mathrm{L} + \mathrm{Q} \wedge \bar{\mathrm{Q}}.$$

- based on super-Poincaré $(\mathrm{P}, \mathrm{L}, \mathrm{Q})$ and dilatation (D);
- can be defined in many (other, related) models.

Spins in $\mathcal{N} = 4$ SYM

How to represent a (Yangian) symmetry algebra in $\mathcal{N} = 4$ SYM?

Spins:

- several flavours: $Z = A_\mu, \Psi_\alpha, \bar{\Psi}_\alpha, \Phi_m$.
- fields and derivatives: $Z(x) \to \partial_\mu Z(x), \partial_\mu \partial_\nu Z(x), \ldots$.
- $\mathrm{SU}(N)$ gauge theory: fields $Z = Z_{ij}$ are $N \times N$ matrices.

Superconformal/Level-Zero Representation:

- momentum generator P: $\mathrm{P}_\mu Z(x) \sim i\partial_\mu Z(x)$.
- gauge theory: gauge covariant representation

$$\mathrm{P}_\mu Z(x) \sim i D_\mu Z(x) = i\partial_\mu Z(x) - [A_\mu(x), Z(x)].$$

  $\to$ 'non-linear' representation!
- rotations L: $\mathrm{L}_{\mu\nu} Z \sim x_\mu D_\nu Z - x_\nu D_\mu Z$.
- supersymmetries Q: $\mathrm{Q}\Phi \sim \Psi$, $\mathrm{Q}\bar{\Psi} \sim D\Phi$, $\ldots$
- $\ldots$

Spin Chains

Recall: All fields Z are $N \times N$ matrices. Consider field monomials:

$$Z_1 Z_2 \ldots Z_n$$

- Product monomial is (covariant) $N \times N$ matrix.
- Ordering of fields matters (for sufficiently large N).
- Monomials of different length can be mixed (e.g. $\partial Z + i[A, Z]$).

Field polynomials: spin chain states of variable length!

Field polynomials relevant for various objects and observables in QFT:

- local operators $\mathcal{O}(x) = \operatorname{tr} Z_1(x) \ldots Z_n(x) + \ldots$,
- Wilson lines $W = \mathrm{P} \exp \int A = 1 + \int A + \frac{1}{2} \iint A_1 A_2 + \ldots$,
- colour-ordered correlators $F_n(x_1, \ldots, x_n) = \langle \operatorname{tr} Z_1(x_1) \ldots Z_n(x_n) \rangle$,
- action $\mathcal{S} = \int dx^4 \, \mathcal{L}(x) \sim \int dx^4 \, \operatorname{tr}(F^{\mu\nu} F_{\mu\nu}) + \ldots$.

Yangian Bi-local Representation

Superconformal action (level-zero Yangian): local insertion

$$\mathrm{J}^C(Z_1 \ldots Z_n) = \sum_{k=1}^{n} Z_1 \ldots \mathrm{J}^C Z_k \ldots Z_n.$$

Level-one Yangian action: bi-local insertion follows coproduct

$$\widehat{\mathrm{J}}^C(Z_1 \ldots Z_n) = f^C_{AB} \sum_{k<l=1}^{n} Z_1 \ldots \mathrm{J}^A Z_k \ldots \mathrm{J}^B Z_l \ldots Z_n$$
$$+ \sum_{k=1}^{n} Z_1 \ldots \widehat{\mathrm{J}}^C Z_k \ldots Z_n.$$

Issues:

- local term $\widehat{\mathrm{J}} Z_k$ as completion of bi-local terms;
- non-linear action of $\mathrm{J} Z_k$ and $\widehat{\mathrm{J}} Z_k$.

Equations of Motion

Application of $\widehat{\mathrm{J}}$ on the action needs extra care (cyclicity).
Consider the equations of motion first: [NB, Garus, Rosso 1701.09162]

$$\widehat{\mathrm{J}}(\text{e.o.m.}) \overset{?}{\sim} \text{e.o.m.}$$

Dirac equation is easiest:

$$D\cdot\Psi + [\Phi, \bar{\Psi}] = \partial\cdot\Psi + i[A, \Psi] + [\Phi, \bar{\Psi}] = 0.$$

Bi-local action of $\widehat{\mathrm{P}}$ via coproduct $\Delta\widehat{\mathrm{P}} = \widehat{\mathrm{P}}_1 + \widehat{\mathrm{P}}_2 + \mathrm{J}^{(1)} \otimes \mathrm{J}^{(2)}$:

$$i\{\mathrm{J}^{(1)}A, \mathrm{J}^{(2)}\Psi\} + \{\mathrm{J}^{(1)}\Phi, \mathrm{J}^{(2)}\bar{\Psi}\} + D\cdot\widehat{\mathrm{P}}\Psi + i[\widehat{\mathrm{P}}A, \Psi] + [\Phi, \widehat{\mathrm{P}}\bar{\Psi}] \overset{!}{=} 0.$$

Defines local terms $\widehat{\mathrm{P}}Z$ in level-one action:

$$\widehat{\mathrm{P}}A \sim \{\Phi, \Phi\}, \qquad \widehat{\mathrm{P}}\Psi \sim \{\Phi, \Psi\}, \qquad \widehat{\mathrm{P}}\Phi = 0.$$

All terms cancel properly. Dirac equation Yangian-invariant!

Invariance of the Action

Aim: Show planar Yangian invariance of the action [NB, Garus, Rosso 1803.06310]

$$\widehat{\mathrm{J}}\,\mathcal{S} = 0.$$

Essential features of the action $\mathcal{S}$:

- single-trace, conformal, finite (disc, level zero, no anomalies?);
- cyclic, integrated, non-homogeneous polynomial

Task: Reconcile non-linear, bi-local representation with cyclicity.

Found definition for "$\widehat{\mathrm{J}}\,\mathcal{S}$" such that:

- $\widehat{\mathrm{J}}\,\mathcal{S} = 0$ for $\mathcal{N} = 4$ SYM and other planar integrable models
- $\widehat{\mathrm{J}}\,\mathcal{S} \neq 0$ for non-integrable models (plain $\mathcal{N} < 4$ SYM)

Invariance of the action shown for $\widehat{\mathrm{P}}$ (~ 1000 terms).

Proper definition of integrability!

Level-One Action on Cyclic Action

Expansion of non-linear action $\mathcal{S}$, generator J, representation $\mathrm{J}\mathcal{S}$:

$$\mathcal{S} = \sum_n \frac{1}{n}\,\mathcal{S}_{[n]}, \qquad \mathrm{J} = \sum_m \mathrm{J}_{[m]}, \qquad \mathrm{J}\mathcal{S} \simeq \sum_{n,m} \mathrm{J}_{[m],1}\mathcal{S}_{[n]}.$$

Proper definition for non-linear bi-local cyclic representation $\widehat{\mathrm{J}}\mathcal{S}$:

$$\begin{aligned}\widehat{\mathrm{J}}\mathcal{S} :\simeq{}& \sum_{n,m,l}\sum_{k=2}^{n} \frac{2k-n-2}{2(n+m+l)}\,\mathrm{J}^{(1)}_{[m],k+l}\mathrm{J}^{(2)}_{[l],1}\mathcal{S}_{[n]} \\ &+ \sum_{n,m,l}\sum_{k=1}^{l+1} \frac{2k-l-2}{n+m+l}\,\mathrm{J}^{(1)}_{[m],k}\mathrm{J}^{(2)}_{[l],1}\mathcal{S}_{[n]} + \sum_{n,m} \widehat{\mathrm{J}}_{[m],1}\mathcal{S}_{[n]}.\end{aligned}$$

Compare to double local action $\mathrm{J}^1\mathrm{J}^2\mathcal{S}$:

$$\mathrm{J}^1\mathrm{J}^2\mathcal{S} \simeq \sum_{n,m,l}\sum_{k=2}^{n} \mathrm{J}^1_{[m],k+l}\mathrm{J}^2_{[l],1}\mathcal{S}_{[n]} + \sum_{n,m,l}\sum_{k=1}^{l+1} \mathrm{J}^1_{[m],k}\mathrm{J}^2_{[l],1}\mathcal{S}_{[n]}.$$

Potential Yangian Anomalies

More elegant proof: Consider classical anomaly term

$$\mathcal{A}^\mu := \widehat{\mathrm{P}}^\mu \mathcal{S} \overset{?}{=} 0.$$

From level-one algebra $[\mathrm{J}, \widehat{\mathrm{J}}] \sim \widehat{\mathrm{J}}$ we know

$$\mathrm{P}\,\mathcal{A}^\mu = \mathrm{Q}\,\mathcal{A}^\mu = 0, \qquad \mathcal{A}^\mu \text{ is a vector of dimension 1.}$$

Therefore $\mathcal{A}^\mu = \int dx^4\, \mathcal{O}^\mu$ with local operator $\mathcal{O}^\mu$:

- dimension-5 vector operator
- top component of supermultiplet

However: top components of long multiplets at dimension ≥ 10.
No suitable short supermultiplets. No classical anomaly terms!

Even better: level-one bonus symmetry $\widehat{\mathrm{B}} \sim \mathrm{Q} \wedge \mathrm{S}$:

$$\mathcal{B} = \widehat{\mathrm{B}}\,\mathcal{S}; \qquad \mathrm{P}\,\mathcal{B} = \mathrm{L}\,\mathcal{B} = \mathrm{R}\,\mathcal{B} = \mathrm{D}\,\mathcal{B} = 0, \quad \mathrm{CP}\,\mathcal{B} = -\mathcal{B}.$$

No CP-odd dimension-4 scalar operator $\mathcal{O}$ with $\mathrm{B} = \int dx^4\, \mathcal{O}$!

Yangian Symmetry in Quantum Theory

Yangian symmetry in classical action shown! Implications for QFT?
Noether: Conserved currents/charges? Bi-local representation?!

Consider general correlators of fields:

$$F_{1\ldots n}(x_1,\ldots,x_n) := \bigl\langle Z_1(x_1)\ldots Z_n(x_n)\bigr\rangle.$$

Ward–Takahashi identities for $F_{1\ldots n}(x_1,\ldots,x_n)$!

$$\begin{aligned}
\mathrm{J}\langle\ldots\rangle &= \sum\nolimits_k \bigl\langle Z_1(x_1)\ldots \mathrm{J}Z_k(x_k)\ldots Z_n(x_n)\bigr\rangle \overset{!}{=} 0,\\
\widehat{\mathrm{J}}\langle\ldots\rangle &= \sum\nolimits_{k<l} \bigl\langle Z_1(x_1)\ldots \mathrm{J}Z_k(x_k)\ldots \mathrm{J}Z_l(x_l)\ldots Z_n(x_n)\bigr\rangle\\
&\quad + \sum\nolimits_k \bigl\langle Z_1(x_1)\ldots \widehat{\mathrm{J}}Z_k(x_k)\ldots Z_n(x_n)\bigr\rangle \overset{!}{=} 0.
\end{aligned}$$

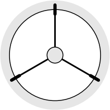
Complication: $\mathcal{N}=4$ SYM is gauge theory.
• gauge fixing • unphysical d.o.f. • Yangian closes onto gauge.

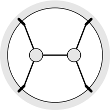
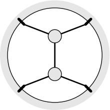
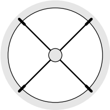
II. Correlation Functions

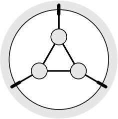
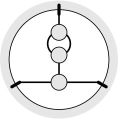
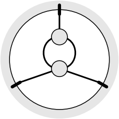
Correlators of Fields

Test Slavnov–Taylor identities for some correlators:

$\langle \mathrm{tr}\, Z_1 Z_2 \rangle =$  ,

$\langle \mathrm{tr}\, Z_1 Z_2 Z_3 \rangle = i$  ,

$\langle \mathrm{tr}\, Z_1 Z_2 Z_3 Z_4 \rangle = -$  $-$  $+\, i$  ,

$\langle \mathrm{tr}\, Z_1 Z_2 Z_3 \rangle_{(1)} = -i$  $-\, i$  $-$  .

- restrict to planar / colour-ordered contributions;
- off-shell: no complications due to mass shell condition;

Yangian Symmetry of Propagator

Level-one generators almost annihilate gauge propagator $\langle A_1 A_2 \rangle$

$$\widehat{\mathrm{J}}^C \big\langle A_1\, A_2 \big\rangle = f^C_{AB} \big\langle \mathrm{J}^A A_1\, \mathrm{J}^B A_2 \big\rangle = d_1 d_2 R^C_{12}.$$

Proof: consider instead $\langle dA_1\, A_2 \rangle$; "integration by parts" on $\mathrm{J} F_1$

$$\begin{aligned} \widehat{\mathrm{J}}^C \big\langle dA_1 A_2 \big\rangle &= f^C_{AB} \big\langle \mathrm{J}^A F_1\, \mathrm{J}^B A_2 \big\rangle \\ &= f^C_{AB} \mathrm{J}^A \big\langle F_1\, \mathrm{J}^B A_2 \big\rangle - f^C_{AB} \big\langle F_1\, \mathrm{J}^A \mathrm{J}^B A_2 \big\rangle \\ &= f^C_{AB} \mathrm{J}^A \big\langle F_1\, (\mathrm{J}^B X \cdot F)_2 \big\rangle - \tfrac{1}{2} f^C_{AB} \big\langle F_1\, [\mathrm{J}^A, \mathrm{J}^B] A_2 \big\rangle = 0. \end{aligned}$$

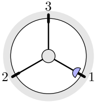
- first term zero due to conformal symmetry,
- second due to $f^C_{AB} f^{AB}_D = f^C_{AB}\, \mathrm{J}^A X^M\, \mathrm{J}^B X^N\, F_{MN} = 0$ ($\mathcal{N} = 4!$).

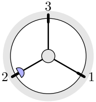
Action must be double total derivative: $\widehat{\mathrm{J}}^C \langle A_1 A_2 \rangle = d_1 d_2 R^C_{12}$.

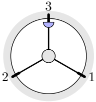
Conformal Symmetry of 3-Point Function

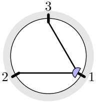
Start simple: tree-level conformal invariance at 3 points

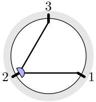
$\mathrm{J}\big\langle \operatorname{tr} Z_1 Z_2 Z_3 \big\rangle$

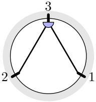
$= \quad i$ 

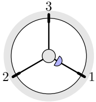
$+\, i$ 

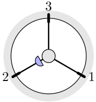
$+\, i$ 

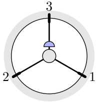
$+$ 

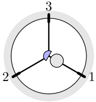
$+$ 

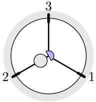
$+$ 

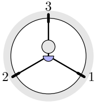
$= -i$ 

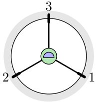
$-\, i$ 

$-\, i$ 

$-\, i$ 

$-\, i$ 

$-\, i$ 

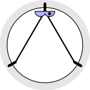
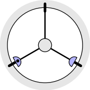
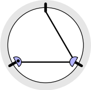
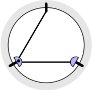
$= -i$ 

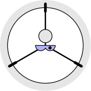
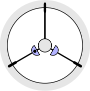
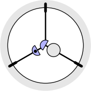
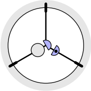
$= 0.$

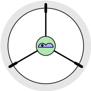
Invariance of action implies invariance of correlator.
Also confirmed invariance for properly gauge-fixed correlator.

Yangian Symmetry of 3-Point Function

Yangian action on correlator of 3 fields at tree level

$\widehat{\mathrm{J}}\langle \mathrm{tr}\, Z_1 Z_2 Z_3 \rangle \simeq 3$  $+\, i$  $+$  $+$ 

$\simeq -3i$  $+\, i$  $+\, i$  $-\, i$ 

$\simeq -i$  $= 0.$

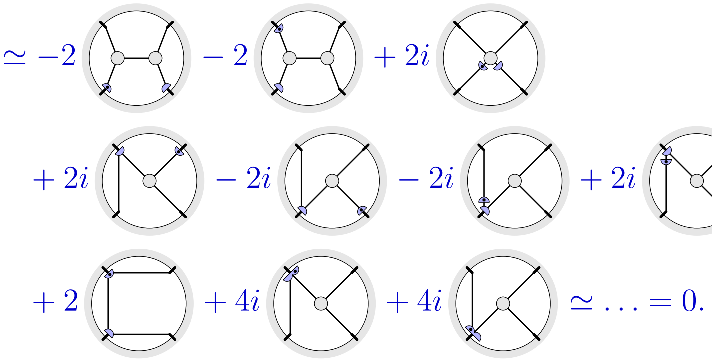
Invariance based on:

- conformal invariance of propagator and 3-vertex,
- Yangian invariance of 3-vertex.

Also showed $\mathrm{Q} \wedge \mathrm{J}$ invariance of gauge-fixed correlator.

Yangian Symmetry of 4-Point Function

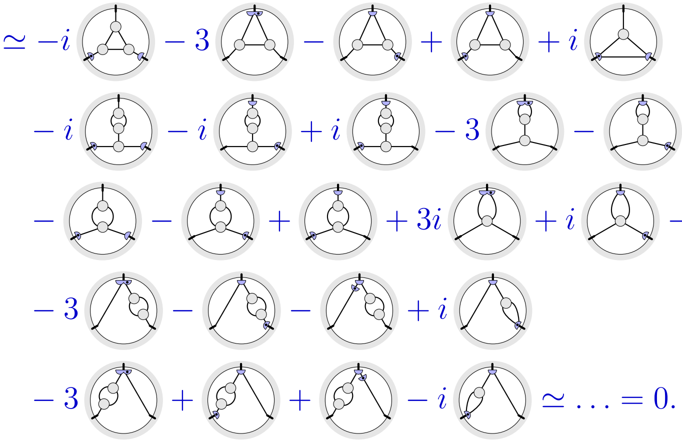
Yangian action on tree-level correlator of 4 fields $\widehat{\mathrm{J}}\langle \operatorname{tr} Z_1 Z_2 Z_3 Z_4 \rangle$



- conformal invariance of propagator, 3-vertex and 4-vertex,
- Yangian invariance of 3-vertex and 4-vertex,
- commutativity of constituents $[\mathrm{J}^{(1)}, \mathrm{J}^{(2)}] = 0$.

3-Function at One Loop

Yangian action on one-loop correlator of 3 fields $\widehat{\mathrm{J}}\langle \mathrm{tr}\, Z_1 Z_2 Z_3 \rangle_{(1)}$



Invariance shown modulo gauge fixing and divergences.

Anomalies?

Classical symmetries may suffer from quantum anomalies:

- No established framework for anomalies of non-local symmetries (in colour-space not necessarily in spacetime).
- Violation of (non-local) current? Cohomological origin?

Potential anomaly terms:

- quantum analysis similar to classical one?
- consider gauge fixing ...
- consider regularisation ...

However:

- Not an issue for Wilson loop expectation value at one loop.
- Integrability "works" at finite coupling: no anomaly expected?

III. Conclusions

Conclusions

Yangian Symmetry of Planar $\mathcal{N} = 4$ SYM:

- classical action of planar $\mathcal{N} = 4$ SYM Yangian invariant
- model classically integrable (same for ABJM)

Correlation Functions

- Ward–Takahashi/Slavnov–Taylor identities tested
- No quantum anomalies to be expected?!

Outlook: Apply to scattering amplitudes (LSZ), Wilson loops, ...
Derive algebraic integrability methods?!