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Ising Anyons in Frustration-Free Majorana-Dimer Models

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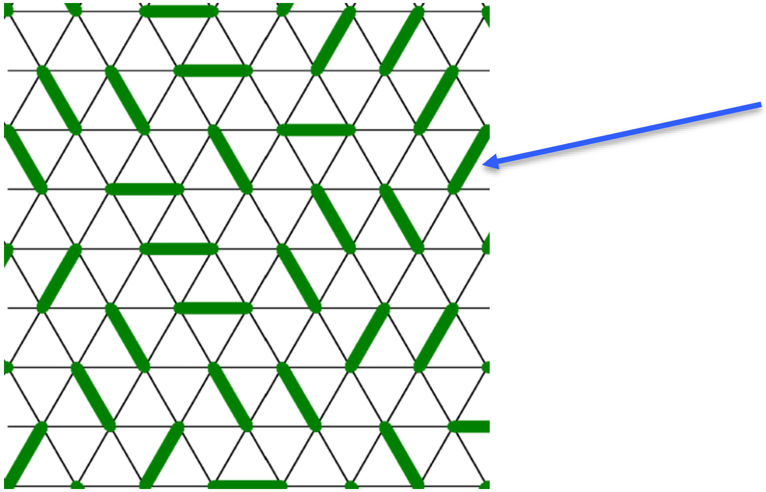
PRB 94, 115127 (2016)

Related work: Tarantino & Fidkowski,
PRB 94, 115115 (2016)

Dimer models

- Historically: description of resonating valence bond (RVB) phases in (doped) antiferromagnets (*Anderson; Rokhsar & Kivelson; Read & Sachdev; Fradkin & Kivelson*)



$$\frac{1}{\sqrt{2}}\left(|\uparrow\downarrow\rangle - |\downarrow\uparrow\rangle\right)$$

=

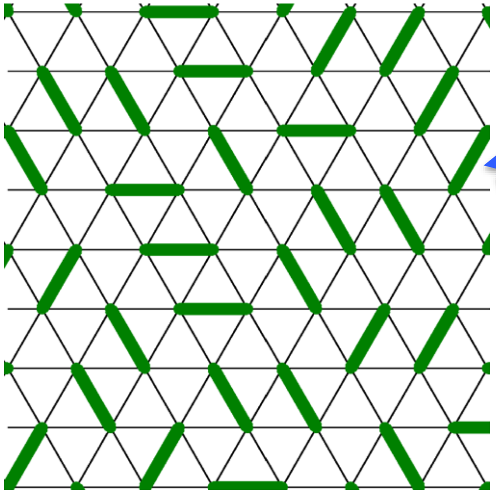
Dimer

- Orthogonality of dimer configurations leads to exactly solvable models:
  - Triangular lattice Rokhsar-Kivelson model: solvable model for a Z_2 spin liquid (*Moessner & Sondhi*)

Picture credit: Chris Herdman

Fermion dimer models

New solvable models when replacing spins with fermions?



~~Spins S_i on sites~~

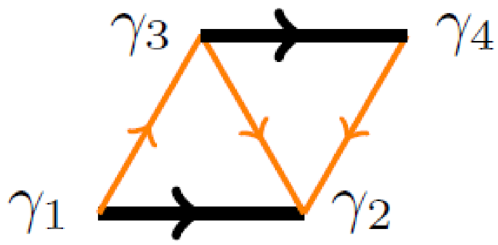
~~$\frac{1}{\sqrt{2}}(|\uparrow\downarrow\rangle - |\downarrow\uparrow\rangle)$~~

Majorana fermions γ_i on sites
Dimer corresponds to paired state of adjacent Majoranas

See also:
Freedman et al 2011
Punk, Allais & Sachdev 2015

Majorana-dimer models

$$|D\rangle|F(D)\rangle =$$



- Dimer configurations $|D\rangle \leftrightarrow$ fermion configurations $|F(D)\rangle$
- Fermion configurations = ground states of parent Hamiltonian

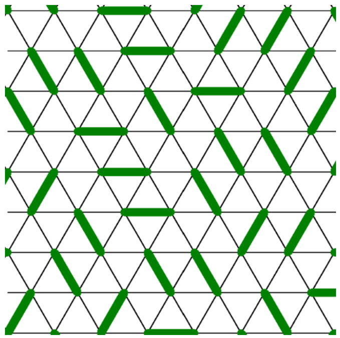
$$H_F(D) = \sum_{(i,j)\in D} i\gamma_i\gamma_j = -i\gamma_1\gamma_2 - i\gamma_3\gamma_4$$

- Goal: analyze $|\psi\rangle = \sum_D |F(D)\rangle|D\rangle$

Freedman et al 2011

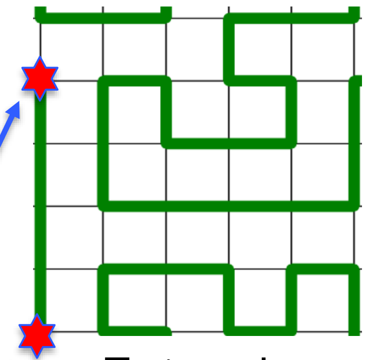
Some intuition from loop gases

Dimer model



Short-range RVB

Loop model



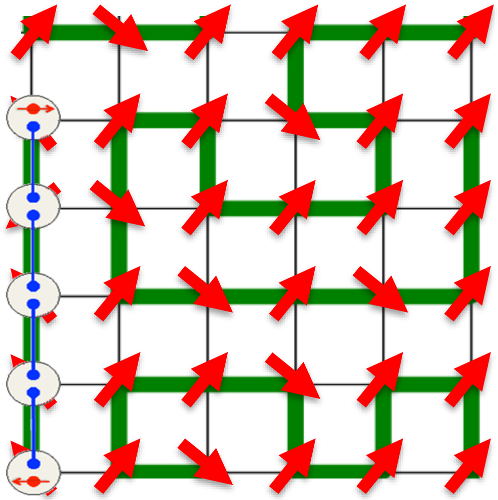
Toric code

One class of excitations: ends of open strings (e excitation of the toric code)

Picture credit: Chris Herdman

Dressed loop model

1d SPT, e.g. AKLT chain
→ e excitation carries spin-*1/2*.

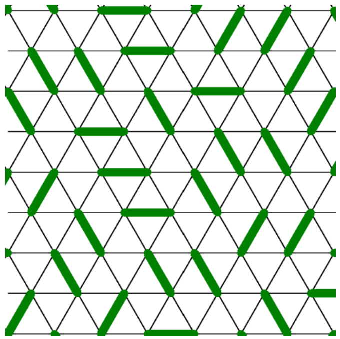


- Enlarged Hilbert space: spins & loops
- Loop = *1d* SPT of the spins
- Quasiparticles carry fractional quantum number: symmetry-enriched topological phase *(Yao, Fu & Qi 2010; Li et al 2014; Huang, Chen & Pollmann 2014)*

Picture credit: Chris Herdman

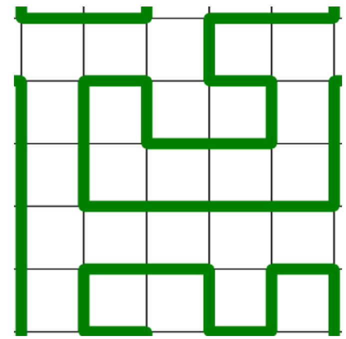
Dual description: loop gas

Dimer model



Short-range RVB

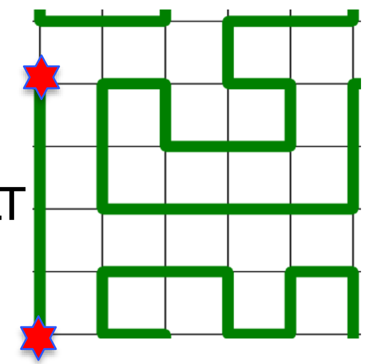
Loop model



Toric code

Dressed loop model

AKLT

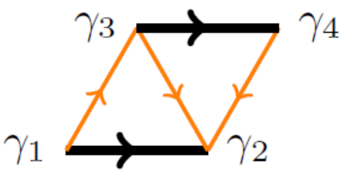


AKLT example:
e excitation has same quantum numbers as spinon in short-range RVB state

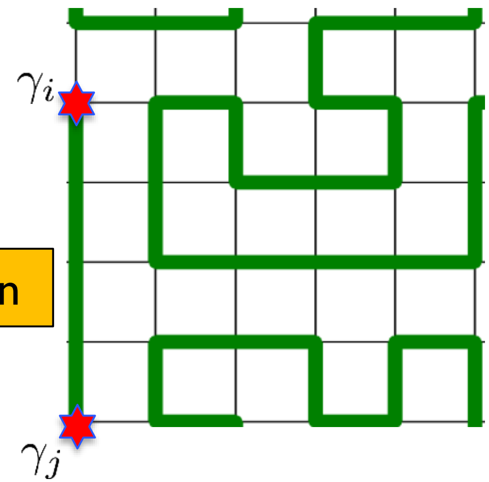
Picture credit: Chris Herdman

Dual description: loop gas

Majorana-Dimer model



Dressed loop model



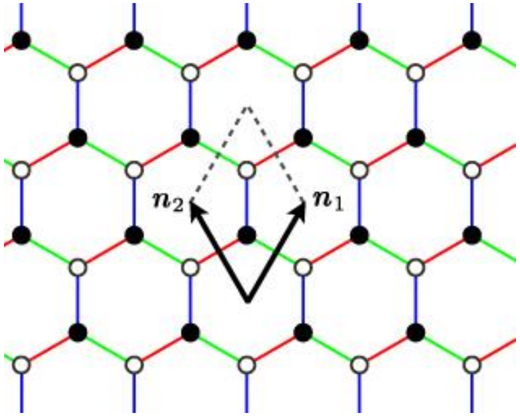
Kitaev chain

Lattice of MZMs + $\mathbb{Z}_2$ gauge fluctuations = Deconfined Ising anyons?

Picture credit: Chris Herdman

Models with Ising anyons

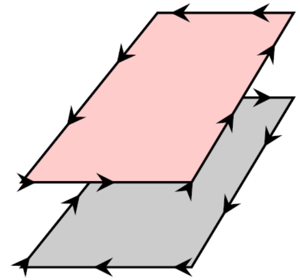
- Kitaev's honeycomb model



Chiral central charge
$c_- = 1/2$

- $\nu = 1$ bosonic Pfaffian QH
  [*Greiter, Wen & Wilczek 1992*]
- Related phase: Moore-Read state

Ising × $(p_x - ip_y)$ phase



- One copy of Ising anyons in the bulk
- Three-fold ground state degeneracy on the torus (periodic BC)
- Fully gapped edge

$$c_-^{\text{Ising}} = 1/2$$

$$c_-^{(p_x - ip_y)} = -1/2$$

$$c_- = c_-^{\text{Ising}} + c_-^{(p_x - ip_y)} = 0$$

Intrinsically fermionic phase of matter!

Overview

1. Consistency with fermion parity
2. Local parent Hamiltonian
3. Ground state degeneracy
4. Topological order via modular transformations

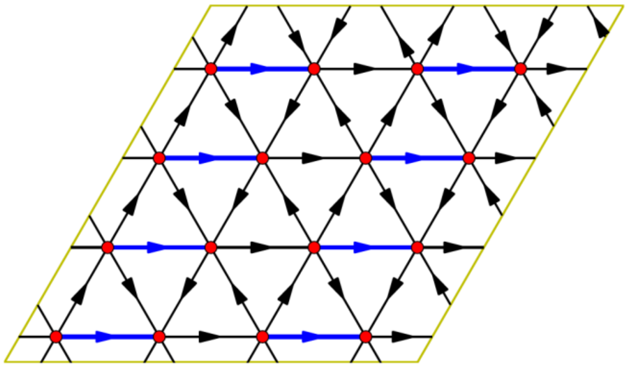
Fermion parity

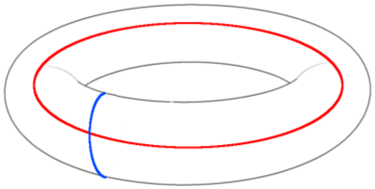
$$|\psi\rangle = \sum_{D \in \mathcal{D}} |F(D)\rangle |D\rangle$$

- Consistency condition for wavefunction: Fermion parity must match!

$$(-1)^{N_f} |D\rangle = P_f(\mathcal{D})$$

- Ensured by Kasteleyn orientation & boundary conditions (math terminology: discrete spin structure)





Dimer Sector	Boundary Cond.			
	PP	PA	AP	AA
$(0,0)$	$+1$	$+1$	$+1$	$+1$
$(1,0)$	-1	$+1$	-1	$+1$
$(0,1)$	-1	-1	$+1$	$+1$
$(1,1)$	-1	$+1$	$+1$	-1

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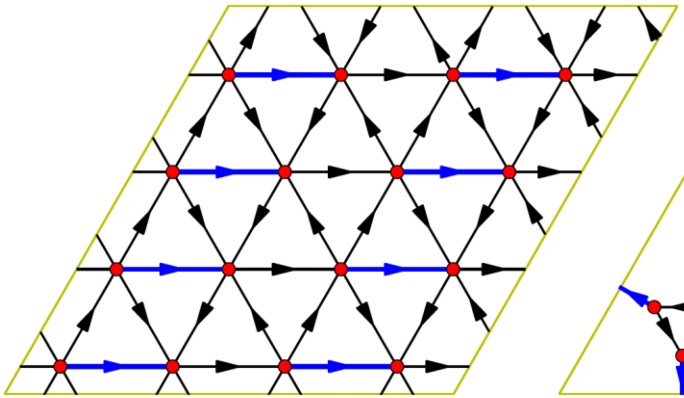
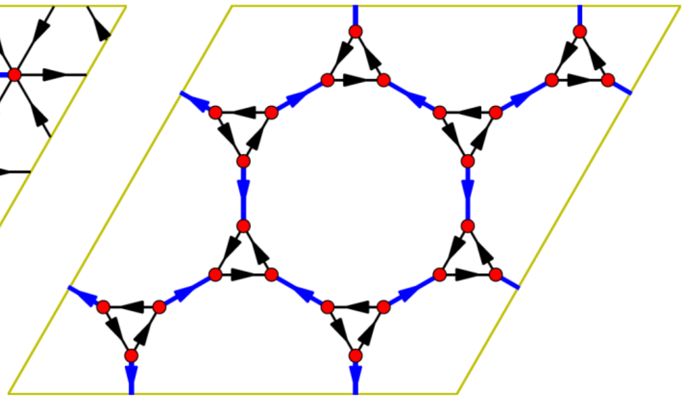
Two lattices



Triangular lattice

Simple frustration-free Hamiltonian
[ground state is simultaneous
eigenstate of all Hamiltonian terms]

Moessner & Sondhi 2001



Fisher lattice

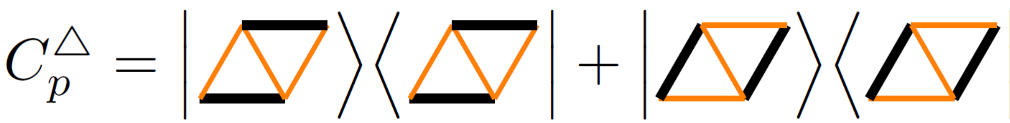
Complicated commuting-
projector Hamiltonian
[in space of valid dimer configurations]

Fisher 1966; Fjaerestad 2008

Structure of the Hamiltonian

$$H_{\mathrm{RK}}^{\triangle} = \sum_p \left(-tB_p^{\triangle} + VC_p^{\triangle} \right)$$

Potential energy term:



Kinetic energy (plaquette-flip) term:

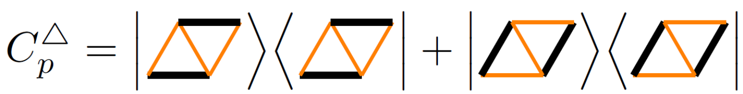
$t = V$: frustration-free Rokhsar-Kivelson point

Rokhsar & Kivelson 1988

Majorana-dimer Hamiltonian

$$H_{\mathrm{RK}}^{\triangle} = -J_e \sum_e \mathbf{A}_e^{\triangle} + \sum_p \left(-t\mathbf{B}_p^{\triangle} + V C_p^{\triangle} \right)$$

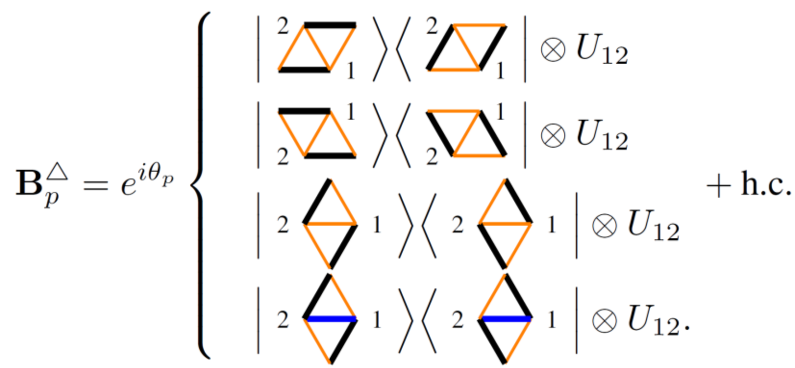
Potential energy term:



Vertex term:

$$\mathbf{A}_e^{\triangle} = \frac{1-\sigma_e^z}{2}\,\frac{1+is_{ij}\gamma_i\gamma_j}{2}$$

Kinetic energy (plaquette-flip) term:



$$U_{12} = (1 + s_{12}\gamma_1\gamma_2)/\sqrt{2}$$

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Overview

1. ✓ Consistency with fermion parity
2. ✓ Local Hamiltonian
3. Ground state degeneracy
4. Topological order via modular transformations

Spectrum

$$h_{DD'} \equiv \langle F(D')|\langle D'|H|F(D)\rangle|D\rangle$$

- If we can choose $|F(D)\rangle$ such that $h_{DD'} = -t\delta_{D',D_p} + V\delta_{DD'}$, then spectrum of Majorana-dimer model equals dimer model.
- Possible for open systems, but more generally?
- Gauge-invariant phases:

$$\Theta_{\{D_k\}} = \operatorname{Arg}\left(h_{D_1D_2}h_{D_2D_3}\ldots h_{D_LD_1}\right)$$

- On torus, non-trivial phases arise (for periodic boundary conditions) *only* in the (0,0) sector

Dimer Sector	PP
$(0,0)$	$+1$
$(1,0)$	-1
$(0,1)$	-1
$(1,1)$	-1

(0,0), +1 → frustrated $h_{DD'}$ → finite-energy state

(1,0), (0,1), (1,1): 3 degenerate ground states

Spectrum

- Condition in Fisher lattice model:

$$\prod \mathcal{B}_p = -P_f$$

- Satisfied in 3 out of 4 dimer sectors for PP. Other sector must have at least one excitation. Similarly for other boundary conditions.

Dimer Sector	Boundary Cond.			
	PP	PA	AP	AA
$(0,0)$	$+1$	$+1$	$+1$	$+1$
$(1,0)$	-1	$+1$	-1	$+1$
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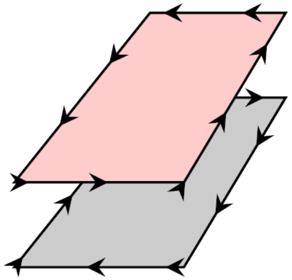
Ising × $(p_x - ip_y)$ phase



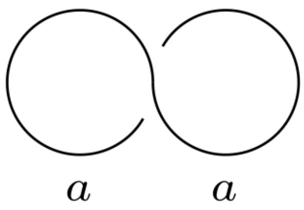
- One copy of Ising anyons in the bulk

✓ Three-fold ground state degeneracy on the torus (periodic BC)

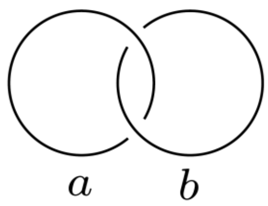
✓ Fully gapped edge

Modular matrices

$T_{aa} \sim$



$S_{ab} \sim$



$\nu = 1/2$ Laughlin state

$$T = e^{i\pi/12}\begin{pmatrix} 1 & 0 \\ 0 & i \end{pmatrix} \qquad S = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$$

$\mathbb{Z}_2$ spin liquid (Toric code):

$$T = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \qquad S = \frac{1}{2}\begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & 1 & -1 \\ 1 & -1 & -1 & 1 \end{pmatrix}$$

- Conjecture: T & S matrices uniquely identify TQFT
- Recent proof for up to 4 particle types: *Rowell et al 2009, Bruillard et al 2013*

Ising phase

Excitations $1, \psi, \sigma$

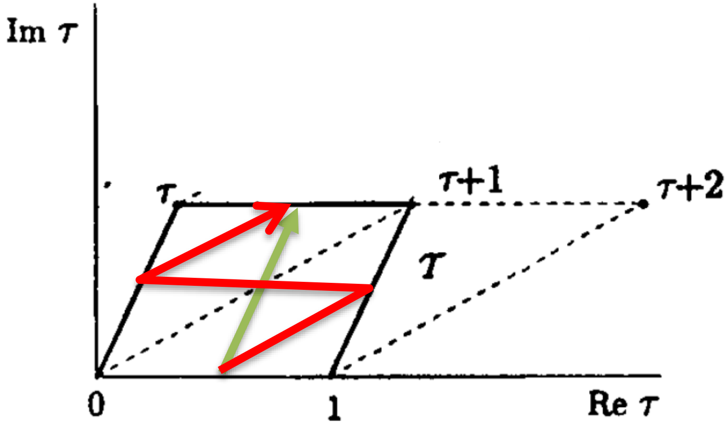
$$\begin{aligned} \psi \otimes \psi &= 1 \\ \sigma \otimes \psi &= \sigma \\ \sigma \otimes \sigma &= 1 + \psi \end{aligned}$$

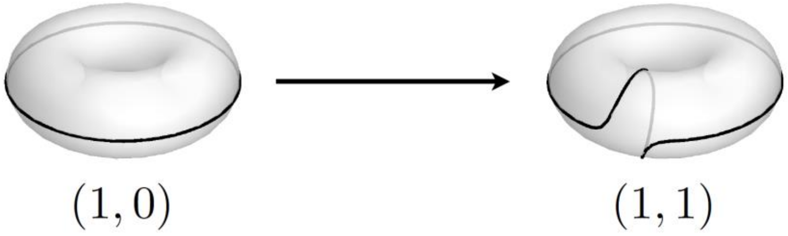
$$S = \frac{1}{2} \begin{pmatrix} 1 & 1 & \sqrt{2} \\ 1 & 1 & -\sqrt{2} \\ \sqrt{2} & -\sqrt{2} & 0 \end{pmatrix}$$

$$T = e^{-\frac{\pi i}{24}} \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & e^{\frac{\pi i}{8}} \end{pmatrix}$$

Modular matrices

- Modular transformations = SL(2,$\mathbb{Z}$) transformations of ground states
- S, T: generators of modular group



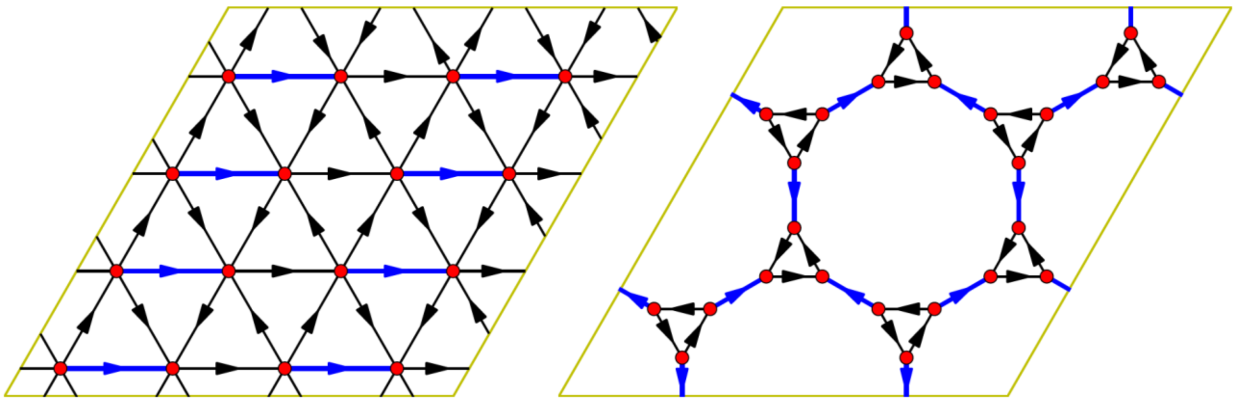


Dehn twist → T

See, e.g., Di Francesco et al

Computing modular matrices

- Key property: $(ST)^{-1} = R_{2\pi/3}$



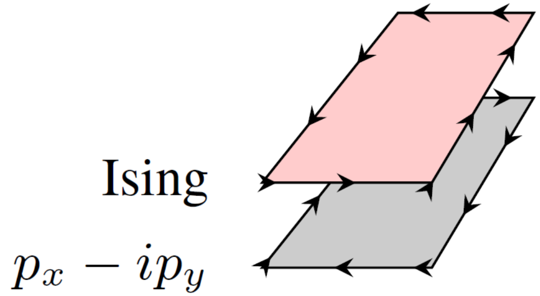
- Symmetries of Fisher lattice & vanishing correlation length make computation feasible!

Fermionic Ising phase

$$S = \frac{1}{2}\begin{pmatrix} 1 & 1 & \sqrt{2} \\ 1 & 1 & -\sqrt{2} \\ \sqrt{2} & -\sqrt{2} & 0 \end{pmatrix} \qquad T = e^{-\frac{\pi i}{24}}\begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & e^{\frac{\pi i}{8}} \end{pmatrix}$$

- Chiral central charge $c_{-} = 1/2$:
  - Incompatible with commuting projector Hamiltonian
- Bosonic topological phase obeys $R_{2\pi} = R_{2\pi/3}^3 = (ST^{-1})^3 = 1$
- Resolution: fermionic system!

$$R_{2\pi/3}^3 = (ST^{-1})^3 = P_f$$

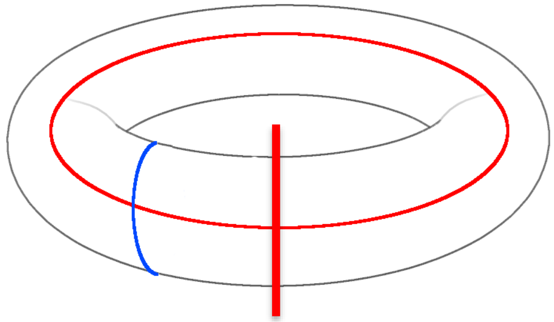


$$S = S_{\text{Ising}} \otimes S_{(p_x - ip_y)} = S_{\text{Ising}}\ e^{\frac{\pi i}{4}}$$

$$T = T_{\text{Ising}} \otimes T_{(p_x - ip_y)} = T_{\text{Ising}}\ e^{-\frac{\pi i}{12}}$$

You & Cheng, 2015
also: Halperin et al, 2012

Minimally entangled states



Entanglement cut

$$S = \alpha L - 2 \ln \frac{D}{d_a}$$

Basis of states that minimize entanglement = states with well-defined topological flux through torus.

Diagonalize T matrix.

Computation of modular matrices

1. Find ground states in winding number basis: $|n_1, n_2\rangle$. Fix phases such that

$$R_{\frac{2\pi}{3}} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & -1 \\ 1 & 0 & 0 \end{pmatrix}$$

2. Find basis of minimally entangled states & compute rotations

$$|1\rangle = \frac{1}{\sqrt{2}}\left(|1,0\rangle - e^{\frac{3i\pi}{8}}|1,1\rangle\right)$$

$$|2\rangle = \frac{1}{\sqrt{2}}\left(|1,0\rangle + e^{\frac{3i\pi}{8}}|1,1\rangle\right)$$

$$S = 3\ln 2$$

$$|3\rangle = |0,1\rangle \qquad S = 4\ln 2$$

$$R_{\frac{2\pi}{3}} = e^{\frac{3\pi i}{8}} \begin{pmatrix} \frac{1}{2} & -\frac{1}{2} & \frac{e^{\frac{\pi i}{4}}}{\sqrt{2}} \\ \frac{1}{2} & -\frac{1}{2} & -\frac{e^{\frac{\pi i}{4}}}{\sqrt{2}} \\ \frac{e^{-\frac{3\pi i}{8}}}{\sqrt{2}} & \frac{e^{-\frac{3\pi i}{8}}}{\sqrt{2}} & 0 \end{pmatrix}$$

3. Resolve S and T from

$$R_{\frac{2\pi}{3}} = PD\left(ST^{-1}\right)D^{\dagger}P^{\dagger}.$$

Generalizations

1. Replace toric code with double semion.
2. Have n Majorana at each site (and couple in the same way).

Families of gauged topological superconductors
Kitaev 2006

Odd n

$$S = S_{(p_x - ip_y)^n} \cdot \frac{1}{2} \begin{pmatrix} 1 & 1 & \sqrt{2} \\ 1 & 1 & -\sqrt{2} \\ \sqrt{2} & -\sqrt{2} & 0 \end{pmatrix}$$

$$T = T_{(p_x - ip_y)^n} \cdot e^{-\frac{\pi i n}{24}} \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & e^{\frac{\pi i n}{8}} \end{pmatrix}$$

Even n

n	Phase	Twists
0	Toric Code	$1, -1, 1, 1$
2	$\mathrm{U}(1)_4$	$1, -1, e^{i\pi/4}, e^{i\pi/4}$
4	$\mathrm{U}(1)_2 \times \mathrm{U}(1)_2$	$1, -1, e^{i\pi/2}, e^{i\pi/2}$
6	$\mathrm{SO}(6)_1$	$1, -1, e^{3i\pi/4}, e^{3i\pi/4}$
8	$\mathrm{SO}(8)_1$	$1, -1, -1, -1$

See also: Gu, Wang & Wen 2014, Gaiotto & Kapustin 2015

Overview

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✓ 2. Local Hamiltonian

✓ 3. Ground state degeneracy

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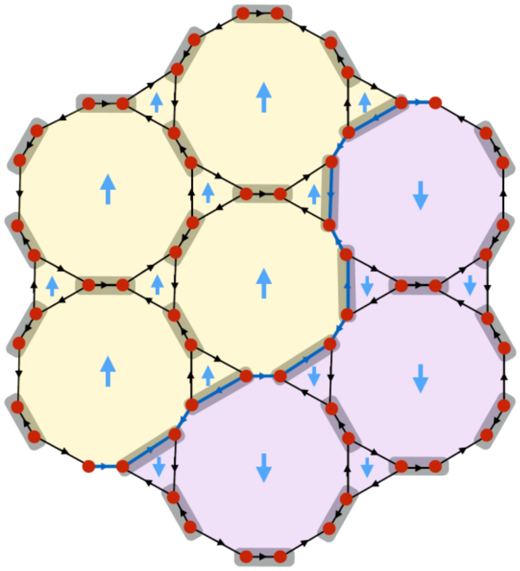
Fermion SPTs

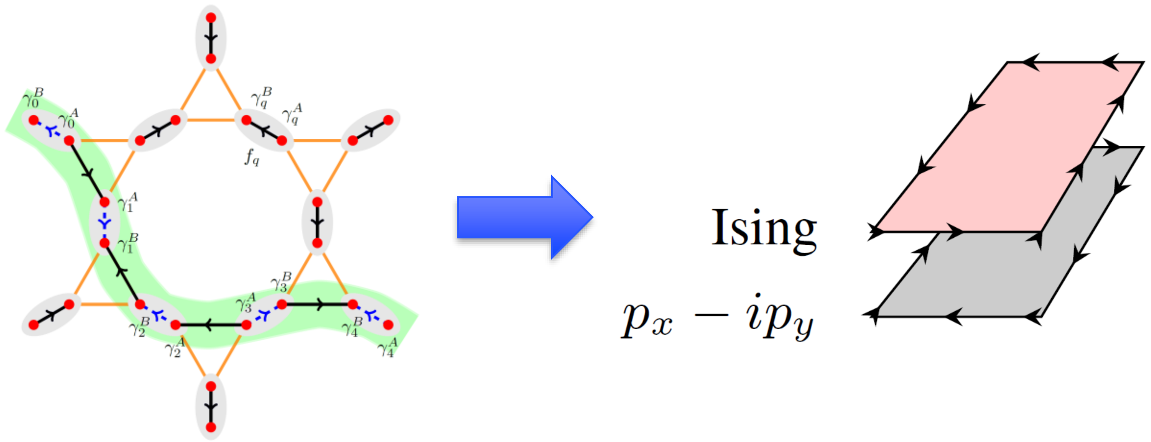
Discrete spin structures and commuting projector models for 2d fermionic symmetry protected topological phases

Nicolas Tarantino[1] and Lukasz Fidkowski[1]

PRB 94, 115115 (2016)



Conclusions



- Majorana-dimer models: playground for new fermionic phases!
- Explicit construction of intrinsically fermionic topological phase

- Open questions:
  - Realizations? Chiral topological superconductor *without* gapless edge
  - Tensor network representations beyond bosonic doubles