Four-vertex Model, Random Tilings and Random Walks

N.M. Bogoliubov

Saint Petersburg Department of V.A. Steklov Mathematical Institute RAS, and ITMO University

August 22, 2016
IIP
A four-vertex model on a square grid is defined by the four different arrows (lines) arrangements.

A statistical weight corresponds to each type of the vertices and there are three vertex weights $\omega_a = \omega_2$, $\omega_b = \omega_4$ and $\omega_c = \omega_5 = \omega_6$. The partition function of the model is equal to

$$Z(\omega_a, \omega_b, \omega_c) = \sum \omega_a^{l_a} \omega_b^{l_b} \omega_c^{l_c},$$

where the summation is extended over all allowed configurations of the arrows on a lattice and $l_a, l_b, l_c$ is the number of vertices $a, b, c$ in each configuration.
To apply the Quantum Inverse Scattering Method we use the spin description of the model. With each vertical bond of the grid and horizontal bond one associates the space $\mathbb{C}^2$, with spin up and spin down states forming a natural basis in this space. The spin up state on the vertical bond corresponds to the line pointing up: $\uparrow$, while the spin down state to the line pointing down: $\downarrow$. The total space of the vertical lines is $\mathcal{V} = (\mathbb{C}^2)^\otimes N$ and we shall call it auxiliary space.

The spin up state on the horizontal bond corresponds to the horizontal line pointing to the left, spin down state to the line pointing to the right: $|\leftarrow\rangle_i \equiv \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $|\rightarrow\rangle_i \equiv \begin{pmatrix} 0 \\ 1 \end{pmatrix}$. The total space of the horizontal lines is $\mathcal{H} = (\mathbb{C}^2)^\otimes N$, and we shall call it quantum space. With each vertex of the lattice one associates an operator acting in the full space $\mathcal{V} \otimes \mathcal{H}$. This operator is called $L$-operator and it acts nontrivially only in a single vertical space $\mathbb{C}^2$ and in single horizontal space $\mathbb{C}^2$, while in all other spaces it acts as the unity operator.
The $L$-operator of the four vertex model is equal to

$$L(n|u) = \begin{pmatrix} L_{11}(n|u) & L_{12}(n|u) \\ L_{21}(n|u) & L_{22}(n|u) \end{pmatrix} = \begin{pmatrix} iu\sigma_n^+\sigma_n^- & \sigma_n^- \\ \sigma_n^+ & iu^{-1}\sigma_n^+\sigma_n^- \end{pmatrix},$$

where $u \in \mathbb{C}$, the local spin operators $\sigma^z, \pm$- are the Pauli matrices. The operator with subindex $n$ acts nontrivially only in the $n$-th space:

$$\sigma_n^\# = \mathbb{I} \otimes \cdots \otimes \mathbb{I} \otimes \sigma_n^\# \otimes \mathbb{I} \otimes \cdots \otimes \mathbb{I}.$$

The commutation relations of the spin operators are

$$[\sigma_k^+, \sigma_l^-] = \delta_{k,l} \sigma_k^z, \quad [\sigma_k^z, \sigma_l^\pm] = \pm 2 \delta_{k,l} \sigma_l^\pm.$$
One can represent the matrix elements of the introduced $L$-operator as a dots with the attached arrows.

\[
\begin{array}{c}
\bullet \\
\downarrow \\
(1) \quad (2) \quad (3) \quad (4)
\end{array}
\]

The matrix element $L_{11}(n|u)$ corresponds to a vertex (1), where a dot stands for the operator $iu\sigma^+_n\sigma^-_n$, this operator acts on the local spin state and the only non-zero matrix element of this operator is

\[
n\langle \leftarrow | iu\sigma^+_n\sigma^-_n | \leftarrow \rangle_n
\]

what gives the vertex $b$ with a weight $\omega_b = iu$:
The matrix element $L_{22}(n|u)$ corresponds to a vertex (2), where a dot is the operator $iu^{-1}\sigma_n^+\sigma_n^-$ and the non-zero matrix element $n\langle \leftarrow |iu^{-1}\sigma_n^+\sigma_n^-| \leftarrow \rangle_n$ is the vertex $a$ with a weight $\omega_a = iu^{-1}$:

The matrix elements $L_{12}(n|u)$ and $L_{21}(n|u)$ correspond to the vertices (3) and (4) respectively, and the nonzero matrix elements $n\langle \rightarrow |\sigma_n^-| \leftarrow \rangle_n$ and $n\langle \leftarrow |\sigma_n^+| \rightarrow \rangle_n$ are the vertices $c$ with a weight $\omega_c = 1$: 
The $L$-operator satisfies the intertwining relation

$$R(u, v) (L(n|u) \otimes L(n|v)) = (L(n|v) \otimes L(n|u)) R(u, v),$$

in which $R(u, v)$ is the $4 \times 4$ matrix

$$R(u, v) = \begin{pmatrix}
  f(v, u) & 0 & 0 & 0 \\
  0 & g(v, u) & 1 & 0 \\
  0 & 0 & g(v, u) & 0 \\
  0 & 0 & 0 & f(v, u)
\end{pmatrix},$$

with

$$f(v, u) = \frac{u^2}{u^2 - v^2}, \quad g(v, u) = \frac{uv}{u^2 - v^2}.$$
The $L$-operator of the six vertex model

\[ L_{6v}(n|u) = \begin{pmatrix}
-ue^{\gamma \sigma_n^z} - u^{-1}e^{-\gamma \sigma_n^z} & 2 \sinh(2\gamma) \sigma_n^-
\end{pmatrix} \begin{pmatrix}
2 \sinh(2\gamma) \sigma_n^+
ue^{-\gamma \sigma_n^z} + u^{-1}e^{\gamma \sigma_n^z}
\end{pmatrix}, \]

is associated with the $R$-matrix

\[ \tilde{R}(u, v) = \begin{pmatrix}
\tilde{f}(v, u) & 0 & 0 & 0 \\
0 & \tilde{g}(v, u) & 1 & 0 \\
0 & 1 & \tilde{g}(v, u) & 0 \\
0 & 0 & 0 & \tilde{f}(v, u)
\end{pmatrix}, \]

where

\[ \tilde{f}(v, u) = \frac{u^2 e^{2\gamma} - v^2 e^{-2\gamma}}{u^2 - v^2}, \quad \tilde{g}(v, u) = \frac{uv}{u^2 - v^2} (e^{2\gamma} - e^{-2\gamma}), \]
Consider the following transformation of $L_{6v}(n|u)$:

$$
\tilde{L}(n|u) = e^{h\sigma_n^z}e^{(\omega/2)\sigma^z}L_{6v}(n|u)e^{-(\omega/2)\sigma^z}
$$

$$
= \begin{pmatrix}
-ue^{(h+\gamma)\sigma_n^z} - u^{-1}e^{(h-\gamma)\sigma_n^z} & 2 \sinh(2\gamma) e^{\omega + h\sigma_n^z} \sigma_n^- \\
2 \sinh(2\gamma) e^{-\omega + h\sigma_n^z} \sigma_n^+ & ue^{(h-\gamma)\sigma_n^z} + u^{-1}e^{(h+\gamma)\sigma_n^z}
\end{pmatrix}.
$$

This $L$-operator is intertwined by the following $R$-matrix:

$$
\tilde{R}(u, v) = (1 \otimes e^{-h\sigma^z})\tilde{R}(u, v)(1 \otimes e^{h\sigma^z})
$$

$$
= \begin{pmatrix}
\tilde{f}(v, u) & 0 & 0 & 0 \\
0 & \tilde{g}(v, u) & e^{2h} & 0 \\
0 & e^{-2h} & \tilde{g}(v, u) & 0 \\
0 & 0 & 0 & \tilde{f}(v, u)
\end{pmatrix}.
$$

Putting $h = \omega = \gamma$, we obtain:

$$
\lim_{\gamma \to \infty} e^{-2\gamma \tilde{L}(n|iu)} = L(n|u),
$$

$$
\lim_{\gamma \to \infty} e^{-2\gamma \tilde{R}(u, v)} = R(u, v).
$$
The monodromy matrix is the product of $L$-operators

$$T(u) = L(M|u)L(M-1|u)...L(0|u) = \begin{pmatrix} A(u) & B(u) \\ C(u) & D(u) \end{pmatrix}$$

The commutation relations of the matrix elements of the monodromy matrix are given by the same $R$–matrix

$$R(u, v) (T(u) \otimes T(v)) = (T(v) \otimes T(u)) R(u, v)$$

The most important relations are

$$C(u)B(v) = g(u, v) \{A(u)D(v) - A(v)D(u)\}$$

$$A(u)B(v) = f(u, v)B(v)A(u) + g(v, u)B(u)A(v),$$

$$[B(u), B(v)] = [C(u), C(v)] = 0.$$ 

The transfer matrix $\tau(u)$ is the trace of the monodromy matrix

$$\tau(u) = trT(u) = A(u) + D(u).$$

The intertwining relation means that $[\tau(u), \tau(v)] = 0$ for arbitrary values of parameters $u, v$. 
The matrix elements of the monodromy matrix $T(u)$ are expressed then as sums over all possible configurations of arrows with different boundary conditions on a one-dimensional lattice with $M + 1$ sites. Namely, operator $B(u)$ corresponds to the boundary conditions when arrows on the top and bottom of the lattice are pointing outward (B):

$$B(u) = \sum_{k_M, \ldots, k_1=1}^{2} (L_{4v})_{1k_M} (M|u)(L_{4v})_{k_M k_{M-1}} (M-1|u) \ldots (L_{4v})_{k_1 2} (0|u).$$

Graphical representation of operator $B$ on a lattice with 3 cites:

```
  \begin{center}
    \begin{tikzpicture}
      \draw[->, thick] (0,0) -- (1,0);
      \draw[->, thick] (0,-1) -- (1,-1);
      \draw[->, thick] (0,-2) -- (1,-2);
      \draw[->, thick] (0,-3) -- (1,-3);
      \node at (0.5,0) {$i=2$};
      \node at (0.5,-1) {$i=1$};
      \node at (0.5,-2) {$i=0$};
      \node at (0.5,-3) {$B(u)$};
    \end{tikzpicture}
  \end{center}
```
Operator \( C(u) \) corresponds to the boundary conditions when arrows on the top and bottom of the lattice are pointing inward (C). Operators \( A(u) \) and \( D(u) \) correspond to the boundary conditions when arrows on the top and bottom of the lattice are pointing up (A) and down (D).
One of the main objects in the further consideration will be the state vector

$$|\Psi_N(u_1, u_2, \ldots, u_N)\rangle = \prod_{i=1}^{N} B(u_i) |\leftarrow\rangle,$$

generated by the multiple action of operators $B(u_i)$ on the generating state, the state with all spins up

$$|\leftarrow\rangle = \bigotimes_{i=0}^{M} |\leftarrow\rangle_i = \bigotimes_{i=0}^{M} \left( \begin{array}{c} 1 \\ 0 \end{array} \right)_i.$$

It is easy to verify that

$$C(u) |\leftarrow\rangle = 0, \quad \langle\leftarrow| B(u) = 0,$$

and

$$A(u) |\leftarrow\rangle = \alpha(u) |\leftarrow\rangle; \quad D(u) |\leftarrow\rangle = \delta(u) |\leftarrow\rangle,$$

with the eigenvalues

$$\alpha(u) = (iu)^{M+1}, \quad \delta(u) = \left( \frac{i}{u} \right)^{M+1}.$$
The conjugated state is

\[ \langle \Psi_N(u_1, u_2, \ldots, u_N) | = \langle \leftarrow | \prod_{i=1}^{N} C(u_i). \]

Let us consider the scalar product of the state vectors

\[ W(u_1, \ldots, u_N; v_1, \ldots, v_N) = \langle \leftarrow | C(v_1) \cdots C(v_N) B(u_1) \cdots B(u_N) | \leftarrow \rangle, \]

where \( \{u\} \) and \( \{v\} \) are the sets of independent parameters. For arbitrary \( N \) and \( M \) this scalar product is evaluated by means of the commutation relations and may be represented in determinantal form

\[
W(u_1, \ldots, u_N; v_1, \ldots, v_N) = (-1)^{MN} \left\{ \prod_{j>k} \frac{v_j v_k}{v_k^2 - v_j^2} \prod_{l>m} \frac{u_l u_m}{u_l^2 - u_m^2} \right\} \det H,
\]

where the matrix elements of matrix \( H \) are

\[
H_{jk} = \left\{ \left( \frac{v_j}{u_k} \right)^{M-N+2} - \left( \frac{v_j}{u_k} \right)^{-M+N-2} \right\} \times \frac{1}{\frac{u_k}{v_j} - \left( \frac{u_k}{v_j} \right)^{-1}}.
\]
We can represent the scalar product of the state vectors

\[
W(u_1, \ldots, u_N; v_1, \ldots, v_N) = \langle \left\langle C(v_1) \ldots C(v_N) B(u_1) \ldots B(u_N) \right| \left| \right \rangle
\]

as the two-dimensional square lattice with \(2N \times (M + 1)\) sites. First \(N\) vertical rows of the lattice are associated with the operators \(C(v_j)\) and the last \(N\) vertical rows with operators \(B(u_j)\).
The horizontal rows of the lattice are associated with the local spin spaces, $i$-th raw with the $i$-th space respectively. The scalar product is equal to the sum over all allowed configurations of vertices on a square lattice with the arrows on first $N$ vertical rows pointing inwards, on the last $N$ ones pointing outwards; on the right and on the left boundaries all arrows are pointing to the left. The path is running from one of the $N$ down left vertices to the top $N$ right ones and always moves east or north. The paths cannot touch each other and arbitrary number of consequent steps are allowed in vertical direction while only one step at a time is allowed in the horizontal one.
The scalar product is proportional to the partition function of the inhomogeneous four-vertex model

\[ W(u_1, \ldots, u_N; v_1, \ldots, v_N) = (-1)^{MN} Z(u_1, \ldots, u_N; v_1, \ldots, v_N) \]

\[ = (-1)^{MN} \sum \prod_{k=-1}^{-N} v_k^{b-a} \prod_{j=1}^{N} u_j^{b-a}, \]

where the summation is extended over all allowed configurations of the arrows on a lattice and \( l_k^a, l_k^b, l_k^c \) is the number of vertices \( a, b, c \) in the \( k \)-th lattice column. The partition function of the homogenous four-vertex model with the fixed boundary conditions is equal to

\[ Z(\omega_a, \omega_b, \omega_c) = (\omega_a \omega_b)^{N(M-N+1)} \omega_c^{2N^2} (-1)^{MN} W(1; 1). \]
Each configuration may be associated with an $N \times N$ array $(\pi_{i,j})$. The $m$–th path may be thought of as the $m$–th column in the array with a matrix elements $\pi_{j,m}$ equal to the number of the cells in the subsequent columns $j$ of the lattice (starting from the right) under the path. The array

$$\pi = \begin{pmatrix} 6 & 4 & 3 \\ 5 & 3 & 1 \\ 2 & 1 & 0 \end{pmatrix}$$

corresponds to the introduced configuration.
A plane partition $\pi$ is an array $(\pi_{i,j})$ of non-negative integers that are non-increasing as functions of both $i$ and $j$ ($i, j = 1, 2, \ldots$). The integers $\pi_{i,j}$ are called the parts of the plane partition, and $|\pi| = \sum \pi_{i,j}$ is its volume. Each plane partition has a three dimensional diagram which can be interpreted as stacks of cubes (three-dimensional Young diagrams). The height of a stack with coordinates $(i, j)$ is $\pi_{i,j}$. If we have $i \leq L, j \leq N$ and $\pi_{i,j} \leq M$ for all cubes of the plane partition, it is said that the plane partition is contained in a box with side lengths $(L, N, M)$.
The strict plane partition is the partition \( \pi_{spp} \) that is decaying along each column and each raw \( (\pi_{i,j} > \pi_{i+1,j} \text{ and } \pi_{i,j} > \pi_{i,j+1}) \).

Each strict plane partition in \((N, N, M + 2N - 2)\) may be transferred into plane partition in \((N, N, M)\) by subtracting the \( N \times N \) matrix

\[
\pi_{\text{min}} = \begin{pmatrix}
2N - 2 & 2N - 3 & \cdots & N - 1 \\
2N - 3 & 2N - 4 & \cdots & N - 2 \\
\vdots & \vdots & \ddots & \vdots \\
N - 1 & N - 2 & \cdots & 0
\end{pmatrix}.
\]

The volumes of \(|\pi|\) and \(|\pi_{spp}|\) are related by the formula

\[
|\pi| = |\pi_{spp}| - N^2(N - 1).
\]
A plane partition in the box \((N, N, M)\) is equivalent to tiling a hexagon by elementary lozenges with angles \(\pi/3\) and \(2\pi/3\).

Admissible lozenge tilings give a projection of three-dimensional Young diagrams with gradient lines.
The generating function for plane partitions

\[ S(N, N, M|q) = \sum_{\pi} q^{\mid \pi \mid} , \]

where \( q^{\mid \pi \mid} \) is the statistical weight, and the sum is taken over all plane partitions in \((N, N, M)\).

The parametrization \( u_j = q^{-(j-1)/2} \), \( v_j = q^{j/2} \) of the partition function

\[ Z(u_1, \ldots, u_N; v_1, \ldots, v_N) = \sum \prod_{k=-1}^{-N} v_k^{l_k^b - l_k^a} \prod_{j=1}^{N} u_j^{l_j^b - l_j^a} \]

gives

\[ Z(q) = \sum q^{\sum_{k=-1}^{-N} \frac{k}{2} (l_k^a - l_k^b) + \sum_{j=1}^{N} \frac{j-1}{2} (l_j^a - l_j^b)} . \]

The volume of the strict plane partition in a box \( N, N, M \) is equal to

\[ \mid \pi_{spp} \mid = \frac{N^2 M}{2} + \sum_{k=-1}^{-N} \frac{k}{2} (l_k^a - l_k^b) + \sum_{j=1}^{N} \frac{j-1}{2} (l_j^a - l_j^b) , \]
The partition function of the four-vertex model with the fixed boundary conditions in $q$-parametrisation is

$$Z(q) = q^{-\frac{N^2 M}{2}} \sum_{spp} q^{\pi_{spp}} = q^{-\frac{N^2 M}{2}} S_{spp}(N, N, M|q),$$

where the summation is performed over all strict plane partitions in a box $(N, N, M)$.

The partition function of the homogeneous four-vertex model with the fixed boundary conditions is

$$Z(\omega_a, \omega_b, \omega_c) = (\omega_a \omega_b)^{N(M-N+1)} \omega_c^{2N^2} Z(1)$$

$$= (\omega_a \omega_b)^{N(M-N+1)} \omega_c^{2N^2} S_{spp}(N, N, M|1),$$

where $S_{spp}(N, N, M|1)$ is the number of strict plane partitions in a box $(N, N, M)$. 
Substituting the introduced $q$-parametrization into the determinantal representation of the scalar product we obtain

$$W_q(N, M) = (-1)^{NM+N(N-1)/2} \left\{ \prod_{j > k} \left( q^{j-k} - q^{-j-k} \right)^2 \right\} \det \mathcal{H},$$

where

$$\mathcal{H}_{jk} = \frac{s^{k+j-1/2} - s^{-k+j-1/2}}{q^{k+j-1/2} - q^{-k+j-1/2}},$$

with $s = q^{M-N+2}$. This determinant of the matrix may be calculated and is equal to

$$\det \mathcal{H} = (-1)^{N(N-1)/2} \left\{ \prod_{j > k} \left( q^{j-k} - q^{-j-k} \right)^2 \right\} \prod_{1 \leq j, k \leq N} \frac{s^{1/2} q^{j-k} - s^{-1/2} q^{-j-k}}{q^{k+j-1/2} - q^{-k+j-1/2}}.$$
The partition function of the four-vertex model with the fixed boundary conditions in the $q$-parametrisation is equal to

$$Z(q) = q^{-\frac{N^2 M}{2}} S_{spp}(N, N, M|q).$$

The generating function of the strict plane partition is

$$S_{spp}(N, N, M|q) = q^{N^2(N-1)} \prod_{1 \leq j, k \leq N} \frac{1 - q^{M-N+2+j-k}}{1 - q^{k+j-1}}$$

$$= q^{N^2(N-1)} \prod_{1 \leq j, k \leq N} \frac{1 - q^{M+3-j-k}}{1 - q^{k+j-1}}.$$

For the partition function of the homogeneous four-vertex model with the fixed boundary conditions we have respectively

$$Z(\omega_a, \omega_b, \omega_c) = (\omega_a \omega_b)^{N(M-N+1)} \omega_c^{2N^2} \prod_{1 \leq j, k \leq N} \frac{M - j - k + 3}{j + k - 1}.$$
The generating functions of strict plane partitions and plane partitions are connected:

\[ S_{\text{spp}}(N, N, M|q) = q^{N^2(N-1)} S_{\text{pp}}(N, N, M - 2N + 2|q). \]
Let us consider now the periodic boundary conditions. It means that on a square lattice with \((M + 1)\) raws and \(L\) columns the boundary arrows are pointing in the same direction. In this figure the typical configuration of arrows and lattice paths is represented:

The periodic boundary conditions mean that to calculate the partition function of the model one must take all different configurations of operators \(A\) and \(D\) and then sum up over all allowed configurations of boundary arrows of rows.
The function $\tau_L(u) = (A(u) + D(u))^L$ is a generating function of different combinations of $A$ and $D$ operators. The transfer matrix commutes with the total spin operator $S^z = \frac{1}{2} \sum_{i=0}^{M} \sigma_i^z$: $[S^z, \tau(u)] = 0$. It means that the number $l_c = 2n$ of vertices ($c$) per column (the number of arrows pointed to the right in the column) is conserved. Taking the trace in the quantum space we shall obtain that

$$Tr\tau^L(u) = \sum (iu)^b (iu^{-1})^a,$$

where the sum is taken over all possible configurations of the vertices $a; b; c$. The partition function is equal to

$$Z_L(\mu) = e^{-i \frac{\pi}{2} (M+1-2n)L} Tr\tau^L(e^{\mu}),$$

where $u = e^{\mu}, (\omega_a = e^{-\mu}, \omega_b = e^{\mu})$. 

N.M. Bogoliubov
\[ Z_L(\mu) = e^{-i \frac{\pi}{2} (M+1-2n)L} \text{Tr} \tau^L(e^\mu). \]

To calculate this expression one have to solve the eigenvalue problem for the transfer matrix \( \tau(u) \), which means finding a vector \( |\Psi_n(v_1, ..., v_n)\rangle \) such that

\[ \tau(u)|\Psi_n(v_1, ..., v_n)\rangle = \Theta_N(u; v_1, ..., v_n)|\Psi_n(v_1, ..., v_n)\rangle. \]

This problem may be solved by algebraic Bethe ansatz method. The Bethe equations are

\[ (v_k^2)^{M+1-n} = (-1)^{n-1} \prod_{j=1}^{n} v_j^{-2}, \quad k = 1, ..., N \]
By putting $u_k^2 = e^{ip_k}$ we can express solutions of the Bethe equations in the form

$$p_k = \frac{2\pi I_k - P}{M + 1 - n}, \quad -\pi \leq p_k \leq \pi,$$

where $I_k$ are integers or half-integers depending on $n$ being odd or even, $P = \sum_{j=1}^{n} p_j$, and $P = \frac{2\pi K}{M+1}$, where $K$ is an integer, $-\frac{M+1}{2} \leq K \leq \frac{M+1}{2}$. The eigenvalue is equal to

$$\Theta_n(u; \{v\}) = e^{\frac{i\pi}{2}(M+1-2n)} \left\{ u^{M+1}Y^2 + (-1)^nu^{-(M+1-2n)} \right\} \prod_{j=1}^{n} \frac{1}{u_j^2 - v_j^2},$$

where $Y^2 = \prod_{j=1}^{n} u_j^2$. The partition function of the four vertex model with the periodic boundary conditions is then given by the equation

$$Z_L(\mu) = e^{-i\frac{\pi}{2}(M+1-2n)L} \sum_{\{v\}} \Theta_n^L(e^\mu; \{v\}).$$

The simplest case corresponds to $n = 0$. The partition function is $Z_L(\mu) = \{2 \cosh(M + 1)\mu\}^L$, and $Z_L(0) = 2^L$. The other limiting case is when $2n = M + 1$. In this case one can find that $Z_L(\mu) = 2$. 

N.M. Bogoliubov
The four vertex model with the periodic boundary conditions is equivalent to a random tiling model. This tiling model consists of squares and parallelograms

The vertex weights $\omega_a = e^{-\mu}$, $\omega_b = e^{\mu}$ are assigned to a parallelograms (r), (l) respectively and $\omega_c = 1$ to a square (s). These tiles can be put together randomly on a torus as long as they do not overlap with each other and there are no gaps between them. The following interpretation of vertices

transfers the four-vertex model into a random tiling model.
The example of the grid and correspondent tiling:

When $2n = M + 1$ ($l_a = l_b = 0$) the torus is tiled by squares, when $n = 0$ ($l_c = 0$) we can tile a torus - with the left and with the right oriented parallelograms.

When a chemical potential $\mu = 0$ the partition function is equal to a number of possible tilings on a torus with a fixed number of squares $n$ in a column. For $n = 0$ there are $2^L$ possible variants of tilings, while for $2n = M + 1$ there is only two ways to tile a torus.
The transfer matrix $\tau(u)$ of the four-vertex model commutes with the Hamiltonian

$$[\tau(u), H_{SA}] = 0,$$

where

$$H_{SA} = -\frac{1}{2} \sum_{k=0}^{M} \mathcal{P} (\sigma_{k+1}^- \sigma_k^+ + \sigma_{k+1}^+ \sigma_k^-) \mathcal{P}, \quad \mathcal{P} \equiv \prod_{k=0}^{M} (I - \hat{q}_{k+1} \hat{q}_k),$$

and $\hat{q}_k \equiv \frac{1}{2} (1 - \sigma_k^z)$. The nearest neighbours with spins “down” are not allowed. This Hamiltonian commutes with the Hamiltonian of the $XXX$ model

$$H_{XXX} = -\frac{1}{2} \sum_{k=0}^{M} (\sigma_{k+1}^- \sigma_k^+ + \sigma_{k+1}^+ \sigma_k^- + \frac{\Delta}{2} (\sigma_k^z \sigma_{k+1}^z - I) + (\sigma_k^z - I)), $$

in the strong anisotropy limit $\Delta \to -\infty$

$$\lim_{\Delta \to -\infty} \frac{1}{\Delta} H_{XXX} = H_{IZ} \equiv -\frac{1}{4} \sum_{k=0}^{M} (\sigma_{k+1}^z \sigma_k^z - I).$$
The $N$-particle state-vectors of the model, the states with $N$ spins “down”, are convenient to express by means of the Schur functions:

$$ |\Psi_N(u)\rangle = \sum_{\lambda \subseteq \{(M-2(N-1))N\}} S_{\lambda}(u^2) \left( \prod_{k=1}^{N} \sigma_{\mu_k}^{-} \right) |\uparrow\rangle. $$

The summation is over all partitions $\lambda$ satisfying $M + 2(1 - N) \geq \lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_N \geq 0$. The sites with spin “down” states are labeled by the coordinates $\mu_i$, $1 \leq i \leq N$. These coordinates constitute a strictly decreasing partition $M \geq \mu_1 > \mu_2 > \ldots > \mu_N \geq 0$; $\mu_i > \mu_{i+1} + 1$. The relation $\lambda_j = \mu_j - 2(N - j)$, where $1 \leq j \leq N$, connects the parts of $\lambda$ to those of $\mu$.

The Schur functions $S_{\lambda}$ are defined by the Jacobi-Trudi relation:

$$ S_{\lambda}(x_N) \equiv S_{\lambda}(x_1, x_2, \ldots, x_N) \equiv \frac{\text{det}(x_j^{\lambda_k + N - k})_{1 \leq j, k \leq N}}{\mathcal{V}(x_N)}, $$

in which $\mathcal{V}(x_N)$ is the Vandermonde determinant

$$ \mathcal{V}(x_N) \equiv \text{det}(x_j^{N-k})_{1 \leq j, k \leq N} = \prod_{1 \leq m < l \leq N} (x_l - x_m). $$
The conjugated state-vectors are given by

$$\langle \Psi(v_N) | = \sum_{\lambda \subseteq \{(M-2(N-1))^N\}} \langle \uparrow | \left( \prod_{k=1}^{N} \sigma_{\mu_k}^+ \right) S_\lambda(v^{-2}_N).$$

The state vectors are the eigenvectors of the Hamiltonian if parameters $v_1, \ldots, v_N$ satisfy Bethe equations

$$(v_k^2)^{M+1-n} = (-1)^{n-1} \prod_{j=1}^{n} v_j^{-2}, \quad k = 1, \ldots, N.$$ 

The scalar product of the state vectors is calculated by the Binet-Cauchy formula for the Schur functions

$$P_M(y, x) = \sum_{\lambda \subseteq \{\lambda^N\}} S_\lambda(x) S_\lambda(y) = \left( \prod_{l=1}^{N} y_l^n x_l^n \right) \frac{\det(T_{kj})_{1 \leq k, j \leq N}}{V_N(x) V_N(y)},$$

where the summation is over all partitions $\lambda$ satisfying: $M \geq \lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_N \geq 0$. The entries

$$T_{kj} = \frac{1 - (x_k y_j)^{M+N}}{1 - x_k y_j}.$$
In the $q$-parameterized form the scalar product is equal to

$$\sum_{\lambda \subseteq \{(M-2(N-1))^N\}} S_{\lambda}(q)S_{\lambda}(q) = q^{N^2(N-1)}S_{pp}(N, N, M - 2N + 2|q) = S_{spp}(N, N, M|q).$$
There is a natural way of representing a Schur function as a nests of self-avoiding lattice paths with prescribed start and end points:

\[ S_\lambda(x_1, x_2, \ldots, x_N) = \sum_{C} \prod_{j=1}^{N} x_j^{l_j}. \]

A nest \( C \) consists of paths going from points \( C_i = (i, 2(N - i)) \) to points \((N, \mu_i = \lambda_i + 2(N - i))\). It makes \( \lambda_i \) steps to the north. The power \( l_j \) of \( x_j \) in the weight of any particular nest of paths is the number of steps to north taken along the vertical line \( x_j \).

Figure: A semistandard tableau of shape \( \lambda = (6, 3, 0) \).
We can represent the Schur function as a nest of conjugated self-avoiding lattice paths

\[ S_{\lambda}(y_1, y_2, \ldots, y_N) = \sum_{B} \prod_{j=1}^{N} y_j^{(M-l_j)}, \]

where summation is over all admissible nests \( B \) of \( N \) self-avoiding lattice paths.
The scalar product may be graphically expressed as a nest of $N$ self-avoiding lattice paths starting at the equidistant points $C_i$ and terminating at the equidistant points $B_i$ ($i = 1, \ldots, N$). This configuration is known as watermelon.

Figure: Watermelon configuration and correspondent plane partition.
N.M. Bogoliubov, *Four-vertex model and random tilings*, Theoretical and Mathematical Physics, **155** (2008), 523.
