Geodesic paths for quantum many-body systems

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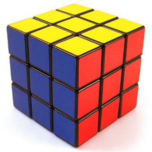
Adiabatic ground-state preparation

$$|\Psi_{\rm gs}(\boldsymbol{\lambda}_0)\rangle \xrightarrow{\hat{\rm H}(\boldsymbol{\lambda}(t))} |\Psi_{\rm gs}(\boldsymbol{\lambda}_f)\rangle$$

$$\boldsymbol{\lambda}(t) = \boldsymbol{\lambda}_0 + (\boldsymbol{\lambda}_f - \boldsymbol{\lambda}_0)\frac{t}{t_f}$$



$$\longrightarrow$$

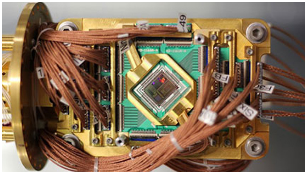
Adiabatic Quantum Computation

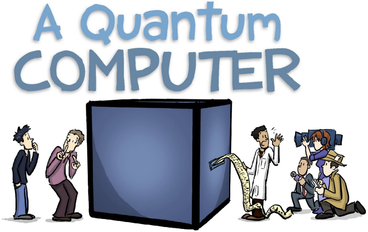
$$\hat{\mathrm{H}} = (1-\lambda)\hat{\mathrm{H}}_0 + \lambda\hat{\mathrm{H}}_f$$

linear interoplation: $\lambda(t) = \frac{1}{t_f} t$

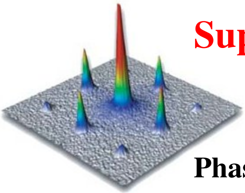
- Ground state of $\hat{\mathrm{H}}_0$ is easily accessible
- Ground state of $\hat{\mathrm{H}}_f$ encodes the solution to a hard computational problem

E. Farhi, et al., Science 20, 472 (2001)





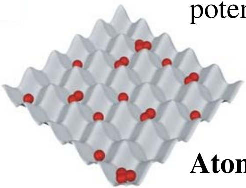
Superfluid - Mott Insulator Transition



Superfluid

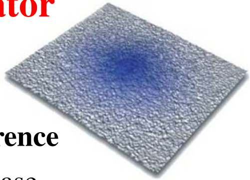
Phase coherence

Macroscopic phase **well defined** in each potential well



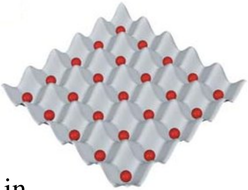
Atom number uncertain in each potential well

Mott Insulator



No phase coherence

Macroscopic phase **uncertain** in each potential well



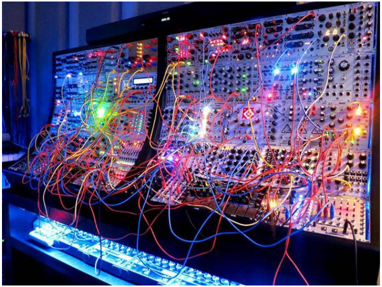
Atom number exactly known in each potential well

M. Greiner, et al., Nature 415, 39-44 (2002)

Outline

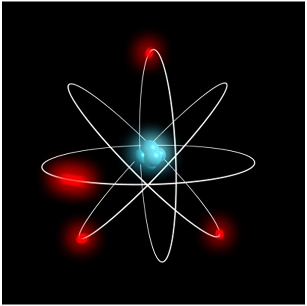
Introduction

Parameters of the Hamiltonian



$$\boldsymbol{\lambda}(t) = (\lambda^1(t), \ldots, \lambda^p(t))^T \in \mathcal{M}$$

Hamiltonian



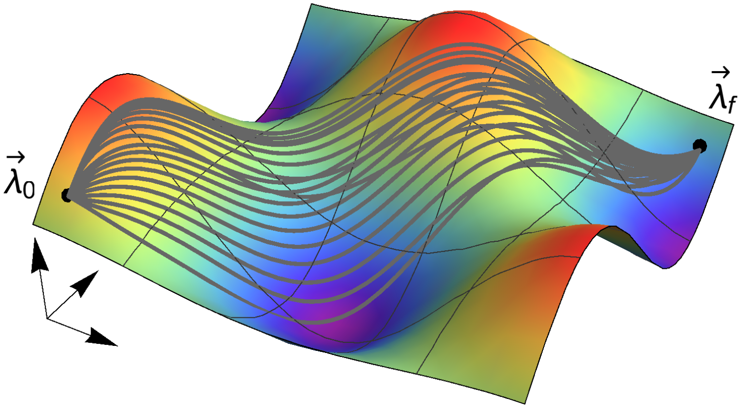
$$H(\boldsymbol{\lambda}(t))$$

$$\hat{\mathrm{H}}(\boldsymbol{\lambda}(t))|\psi_n(\boldsymbol{\lambda}(t))\rangle = E_n(\boldsymbol{\lambda}(t))|\psi_n(\boldsymbol{\lambda}(t))\rangle, \qquad n = 0, 1, 2, \ldots$$

Introduction

$$|\psi_0(\boldsymbol{\lambda}_0)\rangle \quad \xrightarrow{\hat{\mathrm{H}}(\boldsymbol{\lambda}(t))} \quad |\psi_0(\boldsymbol{\lambda}_f)\rangle$$

$$\mathcal{F}[\boldsymbol{\lambda}(t_f)] = |\langle\psi(t_f)|\psi_0(t_f)\rangle|^2$$



Quantum metric tensor - a simplistic approach

Consider two nearby ground states $|\psi_0(\boldsymbol{\lambda})\rangle$ and $|\psi_0(\boldsymbol{\lambda}+d\boldsymbol{\lambda})\rangle$ and define a distance $\Delta(\boldsymbol{\lambda}, d\boldsymbol{\lambda})$ between $\boldsymbol{\lambda}$ and $d\boldsymbol{\lambda}$ in $\mathcal{M}$

$$\boxed{\Delta^2(\boldsymbol{\lambda}, d\boldsymbol{\lambda}) \equiv 1 - |\langle\psi_0(\boldsymbol{\lambda})|\psi_0(\boldsymbol{\lambda}+d\boldsymbol{\lambda})\rangle|^2}$$

$$|\psi_0(\boldsymbol{\lambda}+d\boldsymbol{\lambda})\rangle = |\psi_0(\boldsymbol{\lambda})\rangle + \partial_\mu|\psi_0(\boldsymbol{\lambda})\rangle d\lambda^\mu + \frac{1}{2}\partial_\mu\partial_\nu|\psi_0(\boldsymbol{\lambda})\rangle d\lambda^\mu d\lambda^\nu + \dots$$

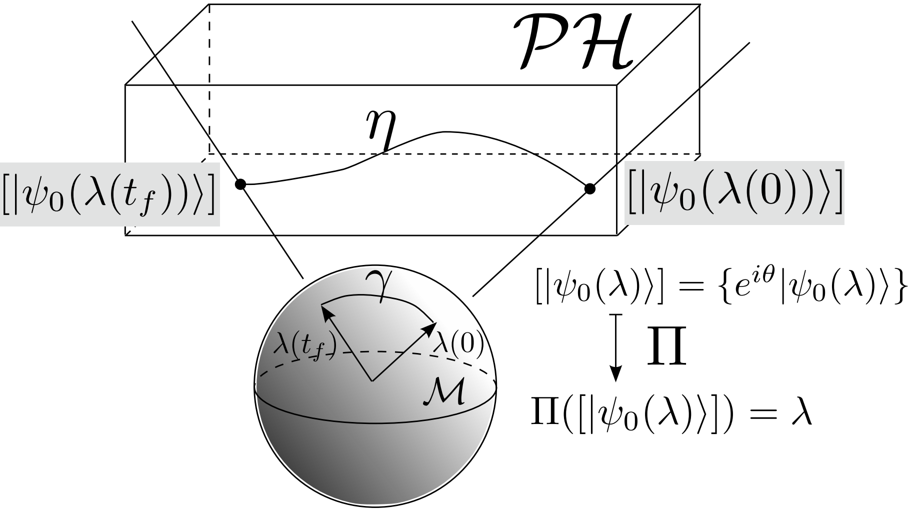
$$\boxed{\partial_\mu \equiv \frac{\partial}{\partial\lambda^\mu}}$$

$$\begin{aligned}\Delta^2(\boldsymbol{\lambda}, d\boldsymbol{\lambda}) &\equiv 1 - |\langle\psi_0(\boldsymbol{\lambda})|\psi_0(\boldsymbol{\lambda}+d\boldsymbol{\lambda})\rangle|^2 \\ &= g_{\mu\nu} d\lambda^\mu d\lambda^\nu \\ &= \mathrm{Re}\left(\langle\psi_0|\overleftarrow{\partial_\mu}\partial_\nu|\psi_0\rangle - \langle\psi_0|\overleftarrow{\partial_\mu}|\psi_0\rangle\langle\psi_0|\partial_\nu|\psi_0\rangle\right) d\lambda^\mu d\lambda^\nu\end{aligned}$$

Quantum metric tensor

$$\boxed{g_{\mu\nu} \equiv \mathrm{Re}\left(\langle\psi_0|\overleftarrow{\partial_\mu}\partial_\nu|\psi_0\rangle - \langle\psi_0|\overleftarrow{\partial_\mu}|\psi_0\rangle\langle\psi_0|\partial_\nu|\psi_0\rangle\right)}$$

Quantum metric tensor - a rigorous approach



Quantum metric tensor - a rigorous approach

$$\begin{array}{rcl} \langle\cdot|\cdot\rangle_{[|\psi_0\rangle]} \ : \ T_{[|\psi_0\rangle]}\mathcal{PH} \times T_{[|\psi_0\rangle]}\mathcal{PH} & \longrightarrow & \mathbb{C} \\ |u\rangle \quad , \quad |v\rangle & \longmapsto & \langle u|v\rangle_{[|\psi_0\rangle]} \end{array}$$

$|X\rangle \in \mathcal{H}$ projecting to a given vector $|u\rangle \in T_{[|\psi_0\rangle]}\mathcal{PH}$

$$\overline{\chi}(|u\rangle, |v\rangle) \equiv \langle u|v\rangle_{[|\psi_0\rangle]} = \langle X_1|X_2\rangle - \langle X_1|\psi_0\rangle\langle\psi_0|X_2\rangle$$

The non-degenerate hermitian metric $\overline{\chi}$ pulls back to a non-degenerate hermitian metric χ only if pull back is injective

J.P. Provost, G. Valle, (*Commun. Math. Phys.* 76, 289 (1980))

$$\boxed{\chi_{\mu\nu} \equiv \langle\psi_0|\overleftarrow{\partial_\mu}\partial_\nu|\psi_0\rangle - \langle\psi_0|\overleftarrow{\partial_\mu}|\psi_0\rangle\langle\psi_0|\partial_\nu|\psi_0\rangle}$$

where $\partial_\mu \equiv \frac{\partial}{\partial\lambda^\mu}$ and

$$g_{\mu\nu} = \mathrm{Re}(\chi_{\mu\nu}) \qquad F_{\mu\nu} = -2\,\mathrm{Im}(\chi_{\mu\nu})$$

Quantum metric tensor - Fidelity

The quantum metric tensor is closely related to the fidelity $\mathcal{F}$

$$\boxed{\mathcal{F}(\boldsymbol{\lambda}, \boldsymbol{\lambda} + d\boldsymbol{\lambda}) \equiv |\langle \psi_0(\boldsymbol{\lambda}) | \psi_0(\boldsymbol{\lambda} + d\boldsymbol{\lambda}) \rangle|^2}$$

P. Zanardi, P. Giorda, and M. Cozzini, Phys. Rev. Lett. 99, 100603 (2007)

The fidelity susceptibility $\chi_{\mathcal{F}}$ is defined by the expansion of $\mathcal{F}$

$$\mathcal{F}(\boldsymbol{\lambda}, \boldsymbol{\lambda} + d\boldsymbol{\lambda}) = 1 - \chi_{\mathcal{F}} + \dots$$

and reads

$$\begin{aligned} \chi_{\mathcal{F}} &= \left[\frac{\langle \psi_0 | \overleftarrow{\partial_\mu} \partial_\nu | \psi_0 \rangle + \langle \psi_0 | \overleftarrow{\partial_\nu} \partial_\mu | \psi_0 \rangle}{2} - \langle \psi_0 | \overleftarrow{\partial_\mu} | \psi_0 \rangle \langle \psi_0 | \partial_\nu | \psi_0 \rangle \right] d\lambda^\mu d\lambda^\nu \\ &= g_{\mu\nu} d\lambda^\mu d\lambda^\nu \\ &= \frac{1}{2} \sum_{m \neq 0} \frac{\langle \psi_0 | \partial_\mu \hat{\mathrm{H}} | \psi_m \rangle \langle \psi_m | \partial_\nu \hat{\mathrm{H}} | \psi_0 \rangle + \mu \leftrightarrow \nu}{(E_0 - E_m)^2} d\lambda^\mu d\lambda^\nu \end{aligned}$$

W.-L. You, Y.-W. Li, and S.-J. Gu, Phys. Rev. E 76, 022101 (2007)
V. Mukherjee, A. Polkovnikov, and A. Dutta Phys. Rev. B 83, 075118 (2011)

Quantum metric tensor - Energy fluctuations

$$g(x_f, z_f) = \begin{pmatrix} g_{xx} & g_{xz} \\ g_{zx} & g_{zz} \end{pmatrix}, \quad g_{\mu\nu} = \frac{1}{2} \sum_{m \neq 0} \frac{\langle \psi_0 | \partial_\mu \hat{\mathrm{H}} | \psi_m \rangle \langle \psi_m | \partial_\nu \hat{\mathrm{H}} | \psi_0 \rangle + \mu \leftrightarrow \nu}{(E_0 - E_m)^2}$$

$$z = z_f, \qquad x(t) = v_x t + x_0, \quad 0 \leq t \leq t_f, \quad x(t_f) = x_f$$

$$\delta E^2 \equiv \langle \psi(t_f) | \hat{\mathrm{H}}^2 | \psi(t_f) \rangle - \langle \psi(t_f) | \hat{\mathrm{H}} | \psi(t_f) \rangle^2$$

$$|\psi(t_f)\rangle = |\psi_0\rangle - i v_x \sum_{m \neq 0} \frac{\langle \psi_m | \partial_x \hat{\mathrm{H}} | \psi_0 \rangle}{(E_m - E_0)^2} + \mathcal{O}(v_x^2),$$

$$\delta E^2 \approx v_x^2 \sum_{m \neq 0} \frac{|\langle \psi_m | \partial_x \hat{\mathrm{H}} | \psi_0 \rangle|^2}{(E_m - E_0)^2} = v_x^2 \, g_{xx}$$

M. Kolodrubetz, V. Gritsev, and A. Polkovnikov, Phys. Rev. B **88**, 064304 (2013)

Geodesics

$$ds^2 = g_{\mu\nu} d\lambda^\mu d\lambda^\nu = 1 - |\langle \psi_0(\boldsymbol{\lambda}) | \psi_0(\boldsymbol{\lambda} + d\boldsymbol{\lambda}) \rangle|^2$$

The quantum distance $\mathcal{L}$ between two ground-states $|\psi_0(\boldsymbol{\lambda}_0)\rangle$ and $|\psi_0(\boldsymbol{\lambda}_f)\rangle$ for a path $\boldsymbol{\lambda}(t)$, joining $\boldsymbol{\lambda}_0 = \boldsymbol{\lambda}(0)$ and $\boldsymbol{\lambda}_f = \boldsymbol{\lambda}(t_f)$, is given by

$$\mathcal{L}(\boldsymbol{\lambda}) = \int_i^f ds = \int_{\boldsymbol{\lambda}_0}^{\boldsymbol{\lambda}_f} \sqrt{g_{\mu\nu} d\lambda^\mu d\lambda^\nu} = \int_0^{t_f} \sqrt{g_{\mu\nu} \dot{\lambda}^\mu \dot{\lambda}^\nu} dt.$$

A geodesics is defined by

$$\delta\mathcal{L} = 0$$

and we can find the shortest path $\boldsymbol{\lambda}_{\text{geo}}(t)$ connecting the two ground states at $\boldsymbol{\lambda}_0$ and $\boldsymbol{\lambda}_f$.

Geodesics

$$\mathcal{L} = \int_0^{t_f} \sqrt{g_{\mu\nu} \frac{d\lambda^\mu}{dt} \frac{d\lambda^\nu}{dt}} dt \quad \rightarrow \quad \mathcal{E} = t_f \int_0^{t_f} g_{\mu\nu} \frac{d\lambda^\mu}{dt} \frac{d\lambda^\nu}{dt} dt$$

$$\mathcal{L}^2 \leq \mathcal{E}, \quad \Leftrightarrow \quad g_{\mu\nu} \frac{d\lambda^\mu}{dt} \frac{d\lambda^\nu}{dt} = \text{const.} \approx \delta E^2$$

$$\boxed{\frac{d^2\lambda^\mu}{dt^2} + \Gamma^\mu_{\nu\rho} \frac{d\lambda^\nu}{dt} \frac{d\lambda^\rho}{dt} = 0}$$

where

$$\Gamma^\mu_{\nu\rho} = \frac{1}{2} g^{\mu\xi} \left(\frac{\partial g_{\xi\nu}}{\partial \lambda^\rho} + \frac{\partial g_{\xi\rho}}{\partial \lambda^\nu} - \frac{\partial g_{\nu\rho}}{\partial \lambda^\xi} \right)$$
$$g^{\mu\nu} = (g_{\mu\nu})^{-1}$$
$$\Gamma^\mu_{\nu\rho} = \Gamma^\mu_{\rho\nu}$$

The Landau-Zener model

$$\boxed{\hat{\mathrm{H}}_{\mathrm{LZ}}(t) = x(t)\hat{\sigma}^x + \epsilon(t)\hat{\sigma}^z = \begin{pmatrix} \epsilon(t) & x(t) \\ x(t) & -\epsilon(t) \end{pmatrix}}$$

Eigenstates

$$\boxed{|\psi_{0,1}\rangle = \mp\frac{1}{\sqrt{2}}\frac{\Omega \mp \epsilon}{\sqrt{\Omega(\Omega \mp \epsilon)}}|\uparrow\rangle + \frac{1}{\sqrt{2}}\frac{x}{\sqrt{\Omega(\Omega \mp \epsilon)}}|\downarrow\rangle,}$$

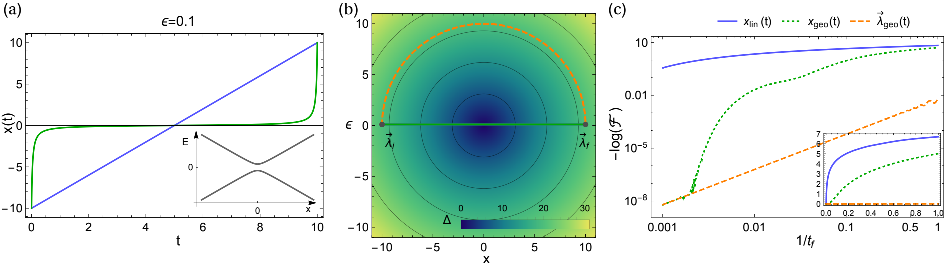
Eigenenergies $E_{0,1} = \mp\Omega \equiv \mp\sqrt{x^2 + \epsilon^2}$

Find $\quad \boldsymbol{\lambda}_{\mathrm{opt}}(t) = (x_{\mathrm{opt}}(t), \epsilon_{\mathrm{opt}}(t))^T \quad$ maximizing:

$$\boxed{\mathcal{F}(t_f) = |\langle\psi(t_f)|\psi_0(t_f)\rangle|^2}$$

for $|\psi_0(0)\rangle$ at $\boldsymbol{\lambda}_i = (x_i, \epsilon)^T \qquad \longrightarrow \qquad |\psi_0(t_f)\rangle$ at $\boldsymbol{\lambda}_f = (x_f, \epsilon)^T$

The Landau-Zener model



$$(1)\quad x_{\text{lin}}(t) = x_i + (x_f - x_i)t/t_f \qquad \Longrightarrow \qquad \mathcal{F}(t_f) \approx 1 - \exp\left[-\pi \frac{\epsilon^2}{(x_f - x_i)\frac{1}{t_f}}\right]$$

$$(2)\quad g_{xx} = \frac{\epsilon^2}{4(x^2+\epsilon^2)^2} \qquad \Longrightarrow \qquad x_{\text{geo}}(t) = \epsilon \tan\left[\alpha_i + (\alpha_f - \alpha_i)\, t/t_f\right]$$

$$\alpha_{i,f} = \arctan(x_{i,f}/\epsilon)$$

$$(3)\quad \boldsymbol{\lambda}(t) = \Omega(t)\begin{pmatrix}\sin\theta(t)\\ \cos\theta(t)\end{pmatrix} \qquad (g_{\mu\nu}) = \begin{pmatrix}0 & 0\\ 0 & 1/4\end{pmatrix}$$

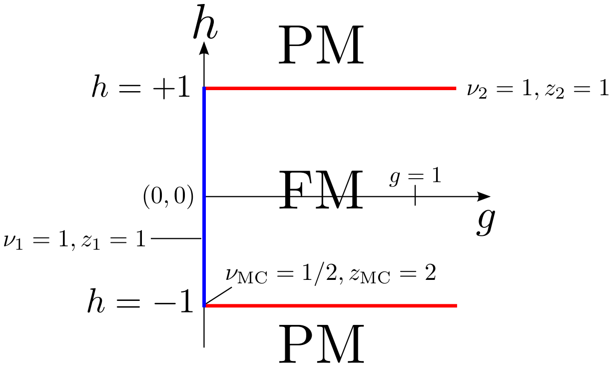
$$\Longrightarrow \qquad \Omega_{\text{geo}}(t) = \Omega_i \qquad \theta_{\text{geo}}(t) = \theta_i + (\theta_f - \theta_i)\, t/t_f$$

$$\theta_{i,f} = \arctan(x_{i,f}/\epsilon_{i,f})$$

XY spin chain in a transverse magnetic field

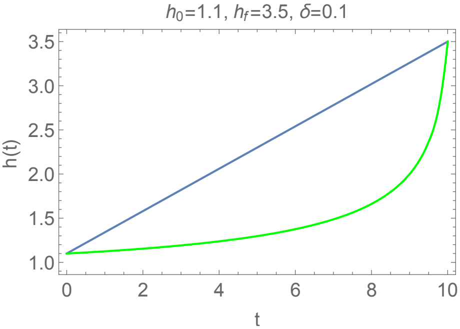
$$\hat{\mathrm{H}}_{XY} = -\sum_{j=1}^{N}\left[\frac{1+g}{2}\hat{\sigma}_j^x\hat{\sigma}_{j+1}^x + \frac{1-g}{2}\hat{\sigma}_j^y\hat{\sigma}_{j+1}^y + h\hat{\sigma}_j^z\right]$$

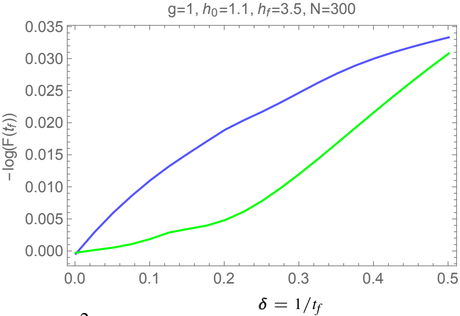
$$\hat{\mathrm{H}}_{XY} = \sum_k \begin{pmatrix} h-\cos(k) & g\sin(k) \\ g\sin(k) & -[h-\cos(k)] \end{pmatrix}$$



XY spin chain in a transverse magnetic field

$$\mathcal{F} = |\langle\psi(t_f)|\psi_0(h_f)\rangle|^2$$

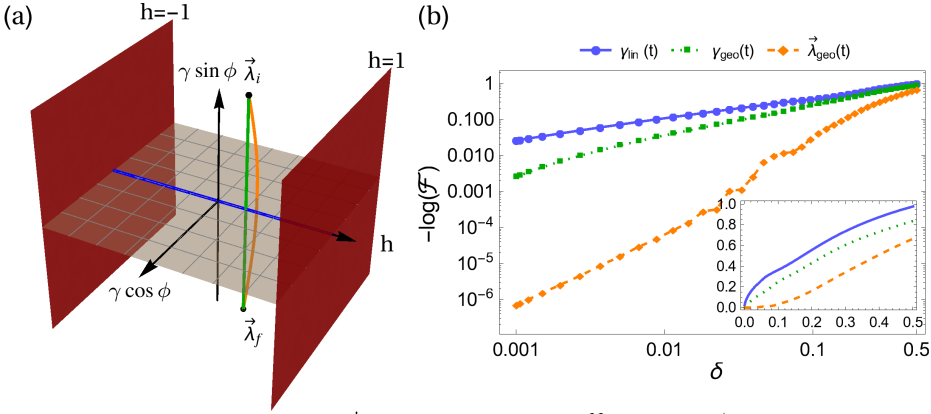




$$g_{hh} = \frac{1}{16}\frac{h\,g^2}{(h^2-1)(h^2-1+g^2)^{3/2}}$$

M. Kolodrubetz, V. Gritsev, and A. Polkovnikov, Phys. Rev. B **88**, 064304 (2013)
P. Zanardi, P. Giorda, and M. Cozzini, Phys. Rev. Lett. **99**, 100603 (2007)

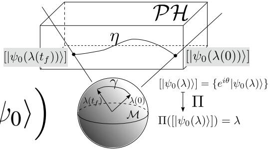
XY spin chain in a transverse magnetic field



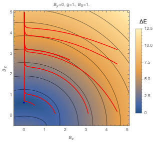
$$\tilde{H}_{XY}(h,\gamma,\phi) = R_z(\phi) H_{XY} R_z^\dagger(\phi), \qquad R_z(\phi) = \prod_{l=1}^N \exp(-i\tfrac{\phi}{2}\sigma_l^z)$$

$$g_{\gamma\gamma} = \frac{1}{16\gamma(1+\gamma)^2}, \quad g_{\phi\phi} = \frac{\gamma}{8(\gamma+1)}, \quad g_{\gamma\phi} = 0$$

Summary



- Quantum metric tensor:
  $$g_{\mu\nu} = \mathrm{Re}\left(\langle\psi_0|\overleftarrow{\partial_\mu}\partial_\nu|\psi_0\rangle - \langle\psi_0|\overleftarrow{\partial_\mu}|\psi_0\rangle\langle\psi_0|\partial_\nu|\psi_0\rangle\right)$$



- Geodesics:
  $$\frac{d^2\lambda^\mu}{dt^2} + \Gamma^\mu_{\nu\rho}\frac{d\lambda^\nu}{dt}\frac{d\lambda^\rho}{dt} = 0 \qquad g_{\mu\nu}\frac{d\lambda^\mu}{dt}\frac{d\lambda^\nu}{dt} = \text{const.} \approx \delta E^2$$
- Landau-Zener model and XY spin chain
- Used in experiments with superconducting quantum circuits (P. Roushan, et al.)