Constraint Structure of the two-dimensional Polynomial BF theory

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Outline

Topological Quantum Field Theories

Topological Quantum Field Theories:

- Witten (Cohomological) Type.
- Schwarz Type.

Schwarz Type: Chern-Simons

- Topologically Massive Gauge Theories: S. Deser, R. Jackiw, S. Templeton, Annals Phys. 140 (1982) 372.
- $(2+1)-$Dimensional Gravity as an Exactly Soluble System: E. Witten, Nucl. Phys. B311 (1988) 46.

Schwarz Type: BF model

- Topological Field Theory, D. Birmingham, M. Blau, M. Rakowski, G. Thompson, Phys. Rep. 209 (1991) 129.

Background Field (BF) Model

Ingredients:

- $n-$dimensional Manifold $\mathscr{M}$.
- Gauge Group G.
- Gauge field (or Connection) $1-$form A.
- $2-$form Field Strength (or Curvature) $F = DA = dA + A \wedge A$.
- $(n-2)-$form B.

Action

$$S_{BF} = \int_{\mathscr{M}} Tr(B \wedge F).$$

Background Field (BF) Model

Equations of Motion:

$$\delta S = \int Tr \left[\delta B \wedge F + (-1)^{n-1} DB \wedge \delta A \right].$$

- $F = 0$: The connection A is flat.
- $DB = 0$.

Gauge Transformation

- Gauge: $\delta A = D\eta$.
- Shift: $\delta B = D\chi$. (Due to Bianchi Identity $DF = 0$)

Background Field (BF) Model

BF in Two Dimensions

$$\mathscr{L}_{2D} = tr\left(B \wedge F\right).$$

BF in Three Dimensions

$$\mathscr{L}_{3D} = tr\left(B \wedge F\right) + \alpha_1\, tr\left(B \wedge B \wedge B\right).$$

BF in Four Dimensions

$$\mathscr{L}_{4D} = tr\left(B \wedge F\right) + \alpha_2\, tr\left(B \wedge B\right) + \alpha_3\, tr\left(F \wedge F\right).$$

Einstein-Hilbert Action

Einstein-Hilbert Action:

$$S_{EH} = \frac{1}{16\pi G}\int d^n x\,\sqrt{g}\,(R - 2\Lambda)\,.$$

Equation of motion:

$$R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R + g_{\mu\nu}\Lambda = 0.$$

Considerations in two dimensions:

- For $\Lambda = 0$ the metric is undetermined.
- For $\Lambda \neq 0$ the metric is zero.
- Two-dim Gravity cannot be described by a Einstein-Hilbert Action!

Jackiw-Teitelboim Gravity

Jackiw-Teitelboim

$$S_{JT} = \int d^2x \sqrt{g} X (R - 2\Lambda) .$$

- R. Jackiw, C. Teitelboim, Quantum Theory of Gravity (1984).
- EoM: $R - 2\Lambda = 0$.

JT from BF Theory

- Gauge Theory of Two-Dimensional Quantum Gravity, K. Isler, C. A. Trugenberger, Phys. Rev. Lett. 63 (1989) 834.
- Gauge theory of Topological Gravity in $1+1$ dimensions, A. H. Chamseddine, D. Wyler, Phys. Lett. B 228 (1989) 75.
- Quadratic gravity from BF theory in two and three dimensions, R. Paszko, R. da Rocha, Gen. Rel. Grav. 47, (2015) 94.

Two-dimensional BF (1)

Ingredients: Group and Generators

- Group: $SO(3)$.
- Generators: $M_{IJ} = -M_{JI}$.
- Algebra: $[M_{IJ}, M_{KL}] = \eta_{IL} M_{JK} - \eta_{IK} M_{JL} + \eta_{JK} M_{IL} - \eta_{JL} M_{IK}$.

Ingredients: Fields

$$A = \frac{1}{2} A^{IJ}_{\mu} M_{IJ} dx^{\mu}, \quad B = \frac{1}{2} B^{IJ} M_{IJ}.$$

Ingredients: Field Strength

$$F^{IJ}_{\mu\nu} = \partial_{\mu} A^{IJ}_{\nu} - \partial_{\nu} A^{IJ}_{\mu} + A^{I}_{\mu K} A^{KJ}_{\nu} - A^{I}_{\nu K} A^{KJ}_{\mu}.$$

Two-dimensional BF (2)

The Action:

$$S = \frac{1}{4}\int d^2x\, \varepsilon^{\mu\nu} B_{IJ} F^{IJ}_{\mu\nu} = \frac{1}{2}\int d^2x\, \varepsilon^{\mu\nu} B_{IJ}\left(\partial_\mu A^{IJ}_\nu + A^{I}_{\mu K} A^{KJ}_\nu\right).$$

Characteristics of the BF action

- It is first-order: Linear in the velocities $\partial_0 A^{IJ}_\nu$.
- There is no $\partial_0 A^{IJ}_0$ (like electromagnetism).
- The system is constrained.

BF model is constrained

Dirac's Constraint Analysis

- P. A. M. Dirac, *Lectures on Quantum Mechanics* (New York: Yeshiva University) (1964).
- M. Henneaux, C. Teitelboim, *Quantization of Gauge Systems*, Princeton Univ. Press (1992).

BF and Gravity (1)

Generators

- Generators: $M_{IJ} = (M_{ab} = \bar{\varepsilon}_{ab} M,\ M_{a2} = P_a)$.
- Algebra: $[M, P_a] = -\bar{\varepsilon}_a{}^b P_b$, $[P_a, P_b] = -\bar{\varepsilon}_{ab} M$.

Cartan variables $(e_\mu^a, \omega_\mu^{ab})$: Zweibein

$$g_{\mu\nu} = \eta_{ab} e_\mu^a e_\nu^b.$$

Torsion and Curvature

$$\begin{aligned} T^a &\equiv de^a + \omega^a_b \wedge e^b. \\ R^{ab} &\equiv d\omega^{ab} + \omega^a_c \wedge \omega^{cb}. \end{aligned}$$

BF and Gravity (2)

Gauge field:

$$A_\mu = \frac{1}{2}\omega_\mu^{ab} M_{ab} + \frac{1}{l} e_\mu^a P_a.$$

Field Strength:

$$\begin{aligned} F_{\mu\nu}^{ab} &= R_{\mu\nu}^{ab} - \frac{1}{l^2}\left(e_\mu^a e_\nu^b - e_\nu^a e_\mu^b\right), \\ F_{\mu\nu}^{a2} &= \frac{1}{l} T_{\mu\nu}^a. \end{aligned}$$

BF and Gravity (3)

BF action

$$S = \frac{1}{4}\int d^2x\,\varepsilon^{\mu\nu}\left[B_{ab}F^{ab}_{\mu\nu} + 2B_{a2}F^{a2}_{\mu\nu}\right],$$

BF Action

$$S = \frac{1}{2}\int d^2x\left[B\left(\frac{1}{2}\varepsilon^{\mu\nu}\bar{\varepsilon}_{ab}R^{ab}_{\mu\nu} - \frac{2}{l^2}e\right) + \frac{1}{l}\varepsilon^{\mu\nu}B_a T^a_{\mu\nu}\right].$$

Equation of motion for the B_a field:

$$0 = \varepsilon^{\mu\nu}T^a_{\mu\nu}.$$

BF and Gravity (4)

Implication of the equations of motion

- Relation zweibein and spin-connection
  $0 = \varepsilon^{\mu\nu}\left(\partial_\mu e^a_\nu + \vec{\varepsilon}^{\,a}{}_b \omega_\mu e^b_\nu\right)$.
- Curvature form and Riemann tensor $R^{\alpha\beta}_{\mu\nu} = R^{ab}_{\mu\nu} e^\alpha_a e^\beta_b$.
- Furthermore: $\varepsilon^{\mu\nu}\bar{\varepsilon}_{ab}R^{ab}_{\mu\nu} = 2eR$.

BF and JT Gravity

$$S = \frac{1}{2}\int d^2x\sqrt{g}B\left(R - \frac{2}{l^2}\right), \quad \Lambda = \frac{1}{l^2}.$$

More on two-dimensional BF (1)

Modifications

- Group $SO(2,1)$ with metric $\eta^{IJ} = diag(1,-1,1)$ or $\eta^{IJ} = diag(1,-1,-1)$.
- Inonu-Wigner Contraction.

BF SUSY

- Orthosymplectic group $OSP(1,1;1)$.

Noncommutative FJ

- Noncommutative Gravity in two dimensions, S. Cacciatori, et al. Class.Quant.Grav. 19 (2002) 4029.

More on two-dimensional BF (2)

Constraints Analysis

- Canonical Analysis of the Jackiw-Teitelboim Gravity in the Temporal Gauge, C. P. Constantinidis et. al. Class. Quant. Grav. 25 (2008), 125003.
- Quantization of the Jackiw-Teitelboim model, C. P. Constantinidis et. al. Phys. Rev. D 79 (2009) 084007.

Constraints from Hamilton-Jacobi

- Two-dimensional background field gravity: A Hamilton-Jacobi analysis, M. C. Bertin, et. al. J. Math. Phys. 53 (2012), 102901.
- Three-dimensional background field gravity: A Hamilton-Jacobi analysis, N. T. Maia, et. al. Class.Quant.Grav. 32 (2015), 185013.

Palatini formulation

Palatini formulation

$$S_{Pal} = \int d^4x\, \bar{\varepsilon}^{abcd} \left(R_{ab} \wedge e_c \wedge e_d - \frac{\Lambda}{6} e_a \wedge e_b \wedge e_c \wedge e_d \right).$$

MacDowell-Mansouri

$$S_{MM} = \int Tr\left(\widehat{F} \wedge \star \widehat{F}\right) = S_{Pal} + \int Tr\left(R \wedge \star R\right).$$

- Unified Geometric theory o f gravity and supergravity, S.W. MacDowell, F. Mansouri. Phys.Rev.Lett. 38 (1977) 739.

Four-dimensional *BF* (1)

Ingredients for the Riemannian case

- Group $SO(5)$: Spherical space $\Lambda > 0$.
- Group $SO(4,1)$: Hyperbolic space $\Lambda < 0$.
- Group $ISO(4)$: Euclidean space $\Lambda = 0$.

L. Freidel, A. Starodubtsev (2005)

$$S = \int \left[B_{IJ} \wedge F^{IJ} - \frac{\kappa^2}{4} \bar{\varepsilon}^{IJKL4} B_{IJ} \wedge B_{KL} \right].$$

- Quantum gravity in terms of topological observables, e-Print: hep-th/0501191.
- General relativity with a topological phase: an action principle, L. Smolin, A. Starodubtsev, e-Print: hep-th/0311163.

Four-dimensional *BF* (2)

BF Action

$$S = \int \left[B_{ab} \wedge F^{ab} + 2B_{a4} \wedge F^{a4} - \frac{\kappa^2}{4} \varepsilon^{abcd} B_{ab} \wedge B_{cd} \right],$$

Equation of Motion

- $\delta B_{a4} = 0 \to 0 = F^{a4} = \frac{1}{l} T^a$,
- $\delta B_{ab} = 0 \to F^{ab} = \frac{\kappa^2}{2} \bar{\varepsilon}^{abcd} B_{cd} = R^{ab} - \frac{1}{l^2} e^a \wedge e^b$.

MacDowell-Mansouri

$$S = \frac{1}{4\kappa^2} \int \bar{\varepsilon}^{abcd} F_{ab} \wedge F_{cd}.$$

Four-dimensional BF (3)

MacDowell-Mansouri

$$
\begin{aligned}
S &= -\frac{1}{2\kappa^2 l^2}\int \bar{\varepsilon}^{abcd}\left[R_{ab}\wedge e_c\wedge e_d - \frac{1}{2l^2}e_a\wedge e_b\wedge e_c\wedge e_d\right] \\
&+ \frac{1}{4\kappa^2}\int \bar{\varepsilon}^{abcd}\left[R_{ab}\wedge R_{cd}\right].
\end{aligned}
$$

Terms:

- First line: Palatini Action with Cosmological Constant.
- Second line: Euler Class (Topological).
- Conserved charges for Gravity with locally AdS asymptotics, R. Aros et. al. Phys. Rev. Lett. 84 (2000),1964.

Extensions

Adding more topological invariant terms:

- The presence of the term $Tr[B \wedge B]$ adds more topological term: Euler, Pontryagin, Nieh-Yan. We also obtain the Holst term.
- The BF action reduce the number of parameters of this theory.
- The ratio of the cosmological term and the interaction is related to the Immirzi parameter.

Another formulations:

- Supergravity as constrained BF theory, R. Durka, J. Kowalski-Glikman, Phys. Rev. D 83 (2011) 124011.
- Gauged AdS-Maxwell algebra and Gravity, R. Durka et. al. Mod. Phys. Lett. A 26 (2011) 2689.

Polynomial BF

Back to two dimensions!

$$S = \int_{\mathscr{M}} tr \left[-\frac{1}{2} B \wedge F + \kappa^2 (PB)(PB)(QA) \wedge A \right].$$

- Quadratic gravity from BF theory in two and three dimensions, R. Paszko, R. Rocha, Gen. Rel. Grav. 47 (2015) 8, 94.

Projetors

$$P_{IJ,KL} \equiv \frac{1}{2} \bar{\varepsilon}_{IJ2} \bar{\varepsilon}_{KL2}, \quad Q_{IJ,KL} \equiv \frac{1}{2} \bar{\varepsilon}_{IJM} \bar{\varepsilon}_{KLN} \bar{\varepsilon}_{MN2}.$$

Action of the projectors

- $(PM)_{ab} = M_{ab}$, $(PM)_{a2} = 0$.
- $(QM)_{a2} = \bar{\varepsilon}_a{}^b M_{b2}$ $(QM)_{ab} = 0$.

Polynomial BF (2)

Polynomial BF Action

$$S = \int d^2x \left[\frac{1}{2} B \left(\frac{1}{2} \varepsilon^{\mu\nu} \bar{\varepsilon}_{ab} R^{ab}_{\mu\nu} - \frac{2}{l^2} e \right) + \frac{1}{2l} \varepsilon^{\mu\nu} B_a T^a_{\mu\nu} + \frac{2\kappa^2}{l^2} B^2 e \right].$$

- Torsion free condition: $\varepsilon^{\mu\nu} T^a_{\mu\nu} = 0$.
- $B = -\frac{l^2}{8\kappa^2} \left(R - \frac{2}{l^2} \right)$.

Quadratic Gravity from Polynomial BF: $2\Lambda = 1/l^2$

$$S = \frac{1}{8\kappa^2} \int d^2x \sqrt{g} (R - 2\Lambda) - \frac{1}{64\Lambda\kappa^2} \int d^2x \sqrt{g} R^2.$$

Polynomial BF (3)

Some consideretions

- R^2 is the lowest order higher derivative term introduced to get rid of the triviality of the Einstein-Hilbert action in two-dimensions.
- R^2 is important in UV renormalization.

Canonical Formalism

Foliation of space-time

$$S = \int d^2x \left[B\partial_0\omega + \frac{1}{l} B_a \partial_0 e_1^a + \omega_0 \mathscr{G} + \frac{1}{l} e_0^a \mathscr{G}_a \right].$$

- Where: $\mathscr{G} \equiv D_1 B$ and $\mathscr{G}_a \equiv D_1 B_a + \frac{2\kappa^2}{l} \bar{\varepsilon}_{ab} e_1^b B^2$.
- We do not have $\partial_0 \omega_0$ and $\partial_0 e_0^a$.
- Canonical Momenta $\Pi^0 = 0$, $\pi_a^0 = 0$.
- Primary Constraints: $\phi \equiv \Pi^0 \approx 0$, $\phi_a \equiv \pi_a^0 \approx 0$.

- $\{\omega(x), B(y)\} = \delta(x-y)$, $\{e_1^a(x), B_b\} = l\delta_b^a \delta(x-y)$.
- Canonical Hamiltonian: $\mathscr{H}_0 = -\omega_0 \mathscr{G} - \frac{1}{l} e_0^a \mathscr{G}_a$.

Secondary Constraints

Consistency Condition: Primary Constraints must be conserved in time

- Primary Hamiltonian: $\mathscr{H}_P \equiv \mathscr{H}_0 + v\phi + v^a\phi_a$.
- Consistency: $\dot{\phi} = \{\phi, \mathscr{H}_P\} = 0$.

Secondary Constraints

$$\dot{\phi} = 0 \rightarrow \mathscr{G} \quad \dot{\phi}^a = 0 \rightarrow \mathscr{G}_a.$$

Algebra of Constraints

$$\begin{aligned} \{\mathscr{G}(x), \mathscr{G}_a(y)\} &= -\bar{\varepsilon}_a{}^b \mathscr{G}_b(x). \\ \{\mathscr{G}_a(x), \mathscr{G}_b(y)\} &= -\bar{\varepsilon}_{ab}\left(1 - 4\kappa^2 B\right)\mathscr{G}(x). \end{aligned}$$

Generators of gauge transformations

Smeared constraints:

$$\mathscr{G}(\zeta) \equiv \int dx\, \zeta(x)\mathscr{G}(x), \qquad \mathscr{G}_a(\zeta^a) \equiv \int dx\, \zeta^a(x)\mathscr{G}_a(x),$$

$$\widetilde{\mathscr{G}}(\lambda) \equiv \int dx\, \lambda(x)\phi(x), \qquad \widetilde{\mathscr{G}}_a(\lambda^a) \equiv \int dx\, \lambda^a(x)\phi_a(x).$$

Generators

$$\begin{aligned}
\left\{e_0^a, \widetilde{\mathscr{G}}_b\left(\lambda^b\right)\right\} &= \lambda^a, \left\{\omega_0, \widetilde{\mathscr{G}}(\lambda)\right\} = \lambda. \\
\left\{e_1^a, \mathscr{G}(\zeta) + \mathscr{G}_b\left(\zeta^b\right)\right\} &= lD_1\zeta^a. \\
\left\{\omega, \mathscr{G}(\zeta) + \mathscr{G}_b\left(\zeta^b\right)\right\} &= -D_1\zeta + \frac{4\kappa^2}{l}\bar{\varepsilon}_{ab}\zeta^a e_1^b B.
\end{aligned}$$

Degrees of Freedom

Counting degrees of freedom (d.o.f) in Dirac's canonical formalism:

$$d.o.f = N - 2M - S.$$

- N is the dimension of the phase-space.
- M is the number of first-class constraints.
- S is the number of second-class constraints.

For Polynomial BF action we have

- $N = 12$: Due to $\left(e^a_\mu, \omega_\mu\right)$ and their canonical momenta.
- $M = 6$. Since we have $(\phi, \phi_a, \mathscr{G}, \mathscr{G}_a)$
- $S = 0$. There is no second-class constraints

BFV Quantization

Batalin-Fradkin-Vilkovisky Formalism:

- E.S. Fradkin, G.A. Vilkovisky, Phys. Lett. B 55 (1975) 244.
- A. Batalin, G.A. Vilkovisky, Phys. Lett. B 69 (1977) 309.

Conditions for BFV

- The collection of primary and secondary constraints $G_A = (\phi, \phi_a, \mathscr{G}, \mathscr{G}_a)$.
- Algebra $\{G_A, G_B\} = U_{AB}{}^C G_C$ and $\{H_0, G_A\} = V_A^B G_B$.
- It is possible: $U_{AB}{}^C = U_{AB}{}^C(q, p)$.

BFV quantization

For the Polynomial BF action we have

$$U^4_{34} = -\bar{\varepsilon}_a{}^b,\ U^3_{44} = -\bar{\varepsilon}_{ab}\left(1 - 4\kappa^2 B\right).$$

$$\begin{aligned} V^3_1 &= -1,\ V^4_2 = -\frac{1}{l}\delta^a_b,\ V^4_3 = -\frac{1}{l}\bar{\varepsilon}_a{}^b e^a_0. \\ V^3_4 &= -\frac{1}{l}\bar{\varepsilon}_{ab} e^b_0 \left(1 - 4\kappa^2 B\right),\ V^4_4 = \bar{\varepsilon}_a{}^b \omega_0. \end{aligned}$$

Introduction of ghost

- Vector of ghost $\eta^A = (P, P^a, c, c^a)$ and $\overline{\mathscr{P}}_A = \left(\bar{c}, \bar{c}_a, \overline{P}, \overline{P}_a\right)$.
- Brackets $\left\{\eta^A, \overline{\mathscr{P}}_B\right\} = -\delta^A_B \delta\left(x - y\right)$.

BRST Charge and Invariant Hamiltonian

BRST Charge

$$\Omega = \eta^A G_A + \frac{1}{2}\overline{\mathscr{P}}_C U^C_{AB}\eta^A\eta^B.$$

$$\Omega = P\phi + P^a\phi_a + c\mathscr{G} + c^a\mathscr{G}_a - \frac{1}{2}\bar{\varepsilon}_{ab}\left(1 - 4\kappa^2 B\right)\overline{P}c^a c^b + \bar{\varepsilon}^a_b\overline{P}_a c c^b.$$

BRST Invariant Hamiltonian: $\{\mathscr{H}_B, \Omega\} = 0$

$$\begin{aligned} \mathscr{H}_B &= \mathscr{H}_0 + \eta^A V^B_A \overline{\mathscr{P}}_B. \\ \mathscr{H}_U &= \mathscr{H}_B + \{\Psi, \Omega\}. \end{aligned}$$

Gauge Fermion

Gauge Fermion $\Psi = \overline{\mathscr{P}}_A \chi^A$

- Gauge functions $\chi^A = (\chi, \chi^a, 0, 0)$. Then $\Psi = \overline{c}\chi + \overline{c}_a \chi^a$.
- Temporal Gauge $\chi = \frac{1}{\gamma}\omega_0$, $\chi^a = \frac{1}{\gamma}\left(e_0^a - \delta_1^a\right)$.

$$\{\Psi, \Omega\} = -\frac{1}{\gamma}\omega_0 \Pi^0 - \frac{1}{\gamma}\left(e_0^a - \delta_1^a\right)\pi_a^0 - \frac{1}{\gamma}P\overline{c} - \frac{1}{\gamma}P^a \overline{c}_a.$$

Transition Amplitude:

$$Z = \int D[Fields] D[Ghosts] \times \exp i \int dt \left[\dot{q}_i p^i + \dot{\eta}^A \overline{\mathscr{P}}_A - \mathscr{H}_U\right].$$

Path Integral Transition Amplitude

Transition amplitude:

$$Z = \int De_1^a D\omega DB_a DB \exp i \int dt \left[\dot{e}_1^a B_a + \dot{\omega} B + \frac{1}{l} \mathscr{G}_1 \right] \times Z_{gh},$$

Ghost part

$$\begin{aligned} Z_{gh} &= \int D\bar{c} Dc D\bar{c}^a Dc^a \exp i \int dt \left[-\bar{c}\dot{c} - l\bar{c}_a \dot{c}^a \right] \\ &\quad \exp i \int dt \left[-\frac{1}{l} \bar{\varepsilon}_{a1} \left(1 - 4\kappa^2 B \right) \bar{c} c^a + \bar{\varepsilon}_1^a \bar{c}_a c \right]. \end{aligned}$$

Quadratic Gravity From Dilaton Theory

Two dimensional dilaton theories

$$S(g,X)=\int d^2x\sqrt{g}\left[\frac{R}{2}X-\frac{1}{2}U(X)(\partial X)^2-V(X)\right].$$

- *Dilaton gravity in two dimensions*, D. Grumiller, W. Kummer, D. V. Vassilevich, Phys. Rep. 369,(2002) 327.
- Jackiw-Teitelboim Gravity: $U(X)=0$ and $V(X)=\Lambda X$.
- KV Model: $U(X)=cte$ and $V(X)=\frac{\beta}{2}X^2-\Lambda$. ($R^2$ gravity with dynamical torsion).
- Almheiri-Polchinski model: Quadratic potential and RX^2 (2014).

Modeling the potential

Let us consider $U(X) = 0$. Then, for the dilaton field we have

$$R = 2\frac{dV}{dX}.$$

We can choose $V(X) = 2\Lambda X\left(1 - 2\kappa^2 X\right)$

$$S = \frac{1}{8\kappa^2}\int d^2x\,\sqrt{g}\,(R - 2\Lambda) - \frac{1}{64\Lambda\kappa^2}\int d^2x\,\sqrt{g}\,R^2.$$

Dilaton with Cartan variables

Introducing Cartan variables:

$$S = \int \left[X d\omega + X_a \left(de^a + \omega^a_b \wedge e^b \right) - \frac{1}{2} \bar{\varepsilon}_{ab} e^a \wedge e^b V(X) \right].$$

Pros and Cons

- Dilaton theories englobe a large class of two-dimensional gravity models.
- BF theory is build as a gauge theory.
- Dilaton Gravity can be written as a Poisson-Sigma model.
- The first-order Dilaton Gravity is valid on two dimensions.
- BF can be build in three and four dimensions.

Perspectives

In two dimensions

- Are valid the same extensions of Jackiw-Teitelboim gravity?
- Are valid the same extensions of MacDowell-Mansouri gravity?
- It is possible to introduce a dynamical torsions?
- Loop quantum gravity as in Jackiw-Teitelboim?

In three dimensions

- Is it useful to introduce a cosmological term in the three dimensional Polynomial BF action?.