Introduction to the AdS/CFT correspondence

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Strings at Dunes, Natal
July 2016

Plan:

- Lecture 1. Conformal field theories, gauge theories and nonperturbative issues
- Lecture 2. Strings and Anti-de Sitter space
- Lecture 3. The AdS/CFT map and gauge/gravity duality

Lecture 1

Conformal field theories, gauge theories

and nonperturbative issues

1. Conformal field theories: relativistic

.

•Classical scale invariance: $x_\mu \to x'_\mu = \lambda x_\mu$.

•QM scale invariance $\to$ usually conformal invariance. β fct. =0.

•Conformal transformation of flat space: $x'(x)$ such that

$$ds^2 = dx'_\mu dx'^\mu = [\Omega(x)]^{-2} dx_\mu dx^\mu.$$

•Is an invariance of **flat space** under transformations.

•Conformal invariance is not general coord. invariance $\to$ in 2d, $\leftrightarrow \exists$ of generalization that is diff. and Weyl invariant

$$\int d^2z (\partial_\mu \phi)^2 = \int dz d\bar{z} \partial\phi \bar{\partial}\phi \text{ inv.}$$
$$\int d^2z\, m^2 \phi^2 = \int dz d\bar{z}\, m^2 \phi^2$$

•So: conformal transf. = generalization of scale transf. of flat space that change distance between points by local factor.

•Infinitesimal:

$$\begin{aligned} x'_\mu &= x_\mu + v_\mu(x); \quad \Omega(x) = 1 - \sigma_\nu(x) \\ \partial_\mu v_\nu + \partial_\nu v_\mu &= 2\sigma_\nu \delta_{\mu\nu} \Rightarrow \sigma_\nu = \frac{1}{d}\partial \cdot v \end{aligned}$$

•$d = 2$ Euclidean: $ds^2 = dz d\bar{z} \Rightarrow$ Most general solution is holomorphic transformation $z' = f(z)$,

$$ds^2 = dz' d\bar{z}' = f'(z)\bar{f}'(\bar{z}) dz d\bar{z} \equiv [\Omega(z, \bar{z})]^{-2} dz d\bar{z}$$

•$d > 2$: most general solution is

$$\begin{aligned} v_\mu(x) &= a_\mu + \omega_{\mu\nu} x_\nu + \lambda x_\mu + b_\mu x^2 - 2x_\mu b \cdot x \\ \sigma_\nu(x) &= \lambda - 2b \cdot x \end{aligned}$$

•$P_\mu \leftrightarrow a_\mu, J_{\mu\nu} \leftrightarrow \omega_{\mu\nu}$: $ISO(d-1,1)$ (Poincaré). Also $D \leftrightarrow \lambda$ dilatation; $K_\mu \leftrightarrow b_\mu$ special conformal

.

•Form generators of $SO(2,d)$

$$
\begin{aligned}
J_{MN} &= \begin{pmatrix} J_{\mu\nu} & \bar{J}_{\mu,d+1} & \bar{J}_{\mu,d+2} \\ -\bar{J}_{\nu,d+1} & 0 & D \\ -\bar{J}_{\nu,d+2} & -D & 0 \end{pmatrix} \\
\bar{J}_{\mu,d+1} &= \frac{K_\mu - P_\mu}{2}, \quad \bar{J}_{\mu,d+2} = \frac{K_\mu + P_\mu}{2}, \quad \bar{J}_{d+1,d+2} = 0.
\end{aligned}
$$

•Symmetry group of AdS_{d+1} : CFT on $Mink_d$ = gravity in AdS_{d+1}?

•Obs.: Inversion $I : x'_\mu = x_\mu / x^2 \Rightarrow \Omega(x) = x^2$ is conformal also. I & rotation & translation $\Rightarrow$ all finite conformal: e.g. $x^\mu \to \lambda x^\mu$ and special conformal

$$
x^\mu \to \frac{x^\mu + b^\mu x^2}{1 + 2x^\nu b_\nu + b^2 x^2}
$$

d=2 conformal fields and correlators

•Covariant GR tensor: under $z_i \to z'_i(z)$, $\vec{z} = (z_1, z_2)$,

$$T_{i_1 \dots i_n}(z_1, z_2) = T'_{j_1 \dots j_n}(z_1, z_2) \frac{\partial z'^{j_1}}{\partial z^{i_1}} \dots \frac{\partial z'^{j_n}}{\partial z^{i_n}}$$

is generalized to primary field (tensor operator) of CFT, of dimensions $(h, \tilde{h})$

$$\phi^{(h,\tilde{h})}(z, \bar{z}) \equiv T_{z \dots z \bar{z} \dots \bar{z}} = T'_{z \dots z \bar{z} \dots \bar{z}} \left(\frac{dz'}{dz}\right)^h \left(\frac{d\bar{z}'}{d\bar{z}}\right)^{\tilde{h}}$$

•Operator product expansion (OPE) $\to$ in any QFT,

$$\mathcal{O}_i(x_i)\mathcal{O}_j(x_j) = \sum_k C^k{}_{ij}(x_i - x_j)\mathcal{O}_k(x_j)$$

•In CFT, conformal invariance gives for n-point correlators

$$\langle \mathcal{O}_i(x_i)\mathcal{O}_j(x_j) \rangle = \frac{C\delta_{ij}}{|x_i - x_j|^{2\Delta_i}} \Rightarrow \langle \mathcal{O}_i(x_i)\mathcal{O}_j(x_j) \dots \rangle = \sum_k \frac{C^k{}_{ij}}{|x_i - x_j|^{\Delta_i + \Delta_j - \Delta_k}} \langle \mathcal{O}_k \left(\frac{x_i + x_j}{2}\right) \dots \rangle$$

•Know ALL OPEs $\rightarrow$ can solve CFT for correlators.
•Symmetry algebra is infinite dimensional. Energy momentum tensor (not primary!) decomposes as
.

$$T_{zz}(z) = \sum_{m \in \mathbb{Z}} \frac{L_m}{z^{m+2}}, \quad \tilde{T}_{\bar{z}\bar{z}}(\bar{z}) = \sum_{m \in \mathbb{Z}} \frac{\tilde{L}_m}{\bar{z}^{m+2}}$$

Then we have the **Virasoro algebra**

$$[L_m, L_n] = (m-n)L_{m+n} + \frac{c}{12}(m^3 - m)\delta_{m,-n}$$

and idem for $\tilde{L}_m$. Here $L_m^\dagger = L_{-m}$ and c=central charge.
•$\{L_0, L_1, L_{-1}\}$ form a closed subalgebra: $Sl(2,\mathbb{C})$.

•Representations: given by "highest weight state" $|h\rangle$. In CFT, $\exists$ operator-state correspondence: $|h\rangle = \lim_{z\to 0} \phi^h(z)|0\rangle$: primary field. L_{-n}: gives "descendants":

$$L_0|h\rangle = h|h\rangle; \quad L_n|h\rangle = 0, \quad L_0(L_{-n}|h\rangle) = (h+n)(L_{-n}|h\rangle)$$

$d > 2$ conformal fields and correlators

•Eigenfcts. of D with eigenv.$-i\Delta$ (like L_0 and "energy" of state)

.
$$\phi(x) \to \phi'(x) = \lambda^{\Delta}\phi(\lambda x).$$

Then $K_\mu \sim L_n$, $P_\mu \sim L_{-n}$: a and $a^\dagger$: create representations

$$\begin{aligned}
[D, P_\mu] &= -iP_\mu \Rightarrow D(P_\mu|\phi) = -i(\Delta + 1)(P_\mu\phi) \\
[D, K_\mu] &= +iK_\mu \Rightarrow D(K_\mu|\phi) = -i(\Delta - 1)(K_\mu\phi)
\end{aligned}$$

•Inversion generates conf. transf. Defined by orthog. matrix

$$\begin{aligned}
R_{\mu\nu}(x) &= \Omega(x)\frac{\partial x'^{\mu}}{\partial x^{\nu}} \Rightarrow \text{for } x'_\mu = x_\mu/x^2, \\
R_{\mu\nu}(x) &\equiv I_{\mu\nu}(x) = \delta_{\mu\nu} - \frac{2x_\mu x_\nu}{x^2}
\end{aligned}$$

•For correlators, 3 point functions of scalars.

$$\langle \mathcal{O}_i(x)\mathcal{O}_j(y)\mathcal{O}_k(z)\rangle = \frac{C_{ijk}}{|x-y|^{\Delta_i+\Delta_j-\Delta_k}|y-z|^{\Delta_j+\Delta_k-\Delta_i}|z-x|^{\Delta_k+\Delta_i-\Delta_j}}$$

amd currents $\to$ analyze properties under inversion

$$\langle J^a_\mu(x)J^b_\nu(y)\rangle = C\frac{\delta^{ab}I_{\mu\nu}(x-y)}{|x-y|^{2(d-1)}}$$

•QCD at high energies $\simeq$ conformal.

.

•Condensed matter systems: Euclidean CFT appears near critical point. Correlation length $\to \infty$.

$$C \propto \left(\frac{T-T_c}{T_c}\right)^{-\alpha}; \quad \chi_m \propto \left(\frac{T-T_c}{T_c}\right)^{-\gamma_m}; \quad \xi \propto \left(\frac{T-T_c}{T_c}\right)^{-\nu} \to \infty$$

•Near T_c : conformal field theory: correlation functions defined by

$$\langle \phi_1^{\mathrm{lat}}(r_1)\phi_2^{\mathrm{lat}}(r_2)...\phi_n^{\mathrm{lat}}(r_n)\rangle = Z^{-1}\sum_{\{s\}}\phi_1^{\mathrm{lat}}(r_1)...\phi_n^{\mathrm{lat}}(r_n)\rangle e^{-\beta H(\{s\})}$$

and take scaling limit $a \to 0$ with ξ fixed, such that

$$\langle \phi_1(r_1)...\phi_n(r_n)\rangle = \lim_{a\to 0}\left(\prod_{i=1}^{n}\frac{1}{a^{\Delta_i}}\right)\langle \phi_1^{\mathrm{lat}}(r_1)\phi_2^{\mathrm{lat}}(r_2)...\phi_n^{\mathrm{lat}}(r_n)\rangle$$

•So: critical point: relativistic.

2. Conformal field theories: non-relativistic

•Condensed matter $\to \exists$ also nonrelativistic scaling near "Lifshitz points": $t \to \lambda^z t, \vec{x} \to \lambda\vec{x}$, $z=$ dynamical critical exponent. e.g. Lifshitz field theory with $z = 2$,

$$\mathcal{L} = \int d^d x dt [(\partial_t \phi)^2 - k^2 (\vec{\nabla}^2 \phi)^2]$$

•Lifshitz algebra generated by Poincaré generators

$$H = -i\partial_t; \quad P_i = -i\partial_i; \quad M_{ij} = -i(x^i \partial_j - x^j \partial_i)$$

and generator of scaling transformations

$$D = -i(zt\partial_t + x^i \partial_i)$$

•Algebra is

$$\begin{aligned}
[D, H] &= izH, \quad [D, P_i] = iP_i, \quad [D, M_{ij}] = 0, \quad [P_i, P_j] = 0, \\
[M_{ij}, P_k] &= i(\delta^i_k P_j - \delta^j_k P_i), \\
[M_{ij}, M_{kl}] &= i[\delta_{ik} M_{jl} - \delta_{jk} M_{il} - \delta_{il} M_{jk} + \delta_{jl} M_{ik}]
\end{aligned}$$

•Larger algebra: **conformal Galilean algebra**, e.g. cold atoms and fermions at unitarity.

•∃ "Galilean boosts" (nonrelativistic version of boosts) and conserved rest mass (particle number) N.

•Represent them by introducing direction ξ.

$$\begin{aligned} D &= -i(zt\partial_t + x^i\partial_i + (2-z)\xi\partial_\xi) \\ K_i &= -i(x^i\partial_\xi - t\partial_i); \quad N = -i\partial_\xi \end{aligned}$$

and the rest, same. Then, algebra = one before, plus

$$\begin{aligned} [D, K_i] &= (1-z)iK_i \quad [D, N] = (2-z)iN, \ \ [D, H] = ziH, \ \ [D, P_i] = iP_i \\ [K_i, P_j] &= i\delta_{ij}N, \ \ [K_i, H] = -iP_i, \ \ [K_i, M_{ij}] = i(\delta_{jk}K_i - \delta_{ik}K_j) \end{aligned}$$

and rest zero. For $z = 2$, ∃ special conformal generator C for **Schrödinger algebra**

$$[C, D] = +2iC, \ \ [C, H] = -iD, \ \ [C, P_i] = -iK_i, \ \ [C, M_{ij}] = [C, K_i] = 0.$$

3. Gauge theory

•∃ gauge field A^a_μ, with field strength

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$$F = dA + gA \wedge A; \quad F = \frac{1}{2} F_{\mu\nu} dx^\mu \wedge dx^\nu, \quad A = A_\mu dx^\mu, \quad A_\mu = A^a_\mu T_a$$

where $[T_a, T_b] = f_{ab}{}^c T_c$ and gauge invariance

$$\delta A^a_\mu = (D_\mu \epsilon)^a = \partial_\mu \epsilon^a + g f^a{}_{bc} A^b_\mu \epsilon^c$$

•The field strength transforms covariantly under a finite transf.

$$\begin{aligned} U(x) &= e^{g\lambda^a(x) T_a}, \quad \epsilon^a \to \lambda^a \\ F'_{\mu\nu} &= U^{-1}(x) F_{\mu\nu} U(x) \end{aligned}$$

•Couple to fermions and scalars. In Euclidean space,

$$S^E = S^E_A + \int d^4x [\bar{\psi}(\not{D} + m)\psi + (D_\mu \phi)^* D^\mu \phi]$$

where $\not{D} \equiv D_\mu \gamma^\mu$ and

$$D_\mu = \partial_\mu - ieA_\mu \Rightarrow (D_\mu)_{ij} \delta_{ij} \partial_\mu + g(T^a_R)_{ij} A^a_\mu(x).$$

•Green's functions (correlation functions) from partition function = generating functional (in Euclidean space)

$$Z_E[J] = \int \mathcal{D}\phi e^{-S_E[\phi]+i\int d^dx J(x)\phi(x)}$$

.

by

$$\begin{aligned} G_n^{(E)}(x_1,...,x_n) &= \frac{\delta}{\delta J(x_1)}...\frac{\delta}{\delta J(x_n)} Z^{(E)}[J]\Big|_{J=0} \\ &= \int \mathcal{D}\phi e^{-S_E[\phi]}\phi(x_1)...\phi(x_n) \end{aligned}$$

•Can be generalized to existence of composite operators, e.g. *gauge invariant operators in gauge theory* $\mathcal{O}(x)$,

$$Z_{\mathcal{O}}[J] = \int \mathcal{D}\phi e^{-S_E+\int d^dx \mathcal{O}(x)J(x)}$$

giving correlation functions

$$\begin{aligned} \langle \mathcal{O}(x_1)...\mathcal{O}(x_n)\rangle &= \frac{\delta}{\delta J(x_1)}...\frac{\delta}{\delta J(x_n)} Z_{\mathcal{O}}[J]\Big|_{J=0} \\ &= \int \mathcal{D}\phi e^{-S_E[\Phi]}\mathcal{O}(x_1)...\mathcal{O}(x_n) \end{aligned}$$

•Noether theorem: (global) symmetry $\leftrightarrow$ current

$$j^{a,\mu} = \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} (T^a)^i{}_j \phi^j$$

· Classically, $\partial_\mu j^{\mu,a} = 0$. Quantum mechanically, if $\mathcal{D}\phi = \mathcal{D}\phi'$,

$$\langle \partial^\mu j^a_\mu \rangle = 0 = \int \mathcal{D}\phi e^{-S_E[\phi]} \partial^\mu j^a_\mu(x) = 0$$

•If $\mathcal{D}\phi \neq \mathcal{D}\phi' \to \exists$ anomaly.

•Chiral anomaly: $\psi(x) \to e^{i\alpha\gamma_5}\psi(x), \bar{\psi}(x) \to e^{i\alpha\gamma_5}\bar{\psi}(x)$,

$$\begin{aligned} j^5_\mu &= \bar{\psi}\gamma_\mu\gamma_5\psi \Rightarrow \\ \langle \partial^\mu j^5_\mu \rangle &= (2d)\frac{e}{4\pi}\frac{1}{2}\epsilon^{\mu\nu} F^{\text{ext}}_{\mu\nu} \\ &= (4d)\frac{e^2}{16\pi^2}\frac{1}{2}\epsilon^{\mu\nu\rho\sigma} F^{\text{ext}}_{\mu\nu} F^{\text{ext}}_{\rho\sigma} \end{aligned}$$

•Global nonabelian: $\delta\psi^i = \epsilon^a (T^a)^i{}_j \psi^j \Rightarrow$

$$j^a_\mu = \bar{\psi}^i \gamma_\mu (T^a)_{ij} \frac{1+\gamma_5}{2} \psi^j.$$

4. Nonperturbative issues

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•Strong coupling: difficult. In particular, correlators (n-point functions). Also for currents

$$\langle j^{a_1\mu_a}(x_1)...j^{a_n\mu_n}(x_n)\rangle = \frac{\delta^n}{\delta A^{a_1}_{\mu_1}...\delta A^{a_n}_{\mu_n}(x_n)}\int \mathcal{D}(\text{fields})e^{-S_E+\int j^\mu A_\mu}$$

•QCD: conformal at $E \gg \Lambda_{QCD}$. Also, toy model, $\mathcal{N}=4$ SYM (4 susies): $\{A^a_\mu, \psi^{aI}, X^{a[IJ]}$, $I,J=1,...,4$. $SU(4)=SO(6)$ global symmetry. Is exactly conformal ($\beta=0$). From KK dimensional reduction of $\mathcal{N}=1$ SYM,

$$S=\int d^{10}x \text{Tr}\left[-\frac{1}{4}F_{MN}F^{MN}-\frac{1}{2}\bar{\lambda}\Gamma^M D_M\lambda\right]$$

•CFT easy to define in Euclidean space. But Minkowski? Subtle.

•Instantons: nonperturbartive Euclidean solution.

$$S = \frac{1}{4g^2}\int d^4x (F^a_{\mu\nu})^2 = \int d^4x \left[\frac{1}{4g^2} F^a_{\mu\nu} * F^{a\mu\nu} + \frac{1}{8g^2}(F^a_{\mu\nu} - *F^a_{\mu\nu})^2\right]$$

•Only in Euclidean space $F^a_{\mu\nu} = *F^a_{\mu\nu}$ has real solutions, and then $S = S_{\text{inst.}} = 8\pi^2/g^2 n$. Solution

$$A^a_\mu = \frac{2}{g}\frac{\eta^a_{\mu\nu}(x-x_i)_\nu}{g(x-x_i)^2+\rho^2}$$

where $\eta^a_{\mu\nu}$='t Hooft symbol, $\eta^a_{ij} = \epsilon^{aij}$, $\eta^a_{i4} = \delta^a_i$, $\epsilon^a_{4i} = -\delta^a_i$.

•Nonperturbative physics: in *Wilson loops*,

$$W[C] = \text{Tr}\, P \exp\left[i\int A_\mu dx^\mu\right]$$

•For C= reactangle $T \times R$, for $T \to \infty$, we have

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$$\langle W[C] \rangle \propto e^{-TV_{q\bar{q}}(R)}$$

•$V_{q\bar{q}}(R)$= $q\bar{q}$ potential. If $V \sim \sigma R \Rightarrow$ confinement. In CFT, $V_{q\bar{q}}(R) \sim \alpha/R$.

•**Finite temperature:**

$$Z_E[\beta] = \int_{\phi(\vec{x},t_E+\beta)=\phi(\vec{x},t_E)} \mathcal{D}\phi e^{-S_E[\phi]} = \text{Tr}\left(e^{-\beta\widehat{H}}\right)$$

•$\mathcal{N} = 4$ SYM at finite T similar to QCD at finite T. Universality? Finite T QCD: at RHIC, LHC $\to$ *strongly coupled* plasma $\to$ like finite T $\mathcal{N} = 4$ SYM at $g^2_{YM} \to \infty$.

Lecture 2

Strings and Anti-de Sitter space

1. AdS space

•Maximally symmetric spaces in signature $(1,d-1)$: $R_{\mu\nu}-\frac{1}{2}g_{\mu\nu}R = \Lambda g_{\mu\nu}$. Minkowski, dS, AdS. Cosmological const. $\Lambda = 0, > 0, < 0$.
•Sphere (Euclidean signature): embed

.

$$
\begin{aligned}
ds^2 &= +dX_0^2 + \sum_{i=1}^{d-1} dX_i^2 + dX_{d+1}^2 \\
R^2 &= +X_0^2 + \sum_{i=1}^{d-1} X_i^2 + X_{d+1}^2
\end{aligned}
$$

•de Sitter (Minkowski signature): embed

$$
\begin{aligned}
ds^2 &= -dX_0^2 + \sum_{i=1}^{d-1} dX_i^2 + dX_{d+1}^2 \\
R^2 &= -X_0^2 + \sum_{i=1}^{d-1} X_i^2 + X_{d+1}^2
\end{aligned}
$$

•Anti-de Sitter (Minkowski signature): embed

$$
\begin{aligned}
ds^2 &= -dX_0^2 + \sum_{i=1}^{d-1} dX_i^2 - dX_{d+1}^2 \\
-R^2 &= -X_0^2 + \sum_{i=1}^{d-1} X_i^2 - X_{d+1}^2
\end{aligned}
$$

•Poincaré coordinates (Poincaré patch)

$$
\begin{aligned}
ds^2 &= R^2\left[u^2\left(-dt^2+\sum_{i=1}^{d-2}dx_i^2\right)+\frac{du^2}{u^2}\right]\\
&= \frac{R^2}{x_0^2}\left(-dt^2+\sum_{i=1}^{d-2}dx_i^2+dx_0^2\right)
\end{aligned}
$$

•Here $u=1/x_0$. If $x_0/R=e^{-y}$, "warped metric"

$$ds^2=e^{2y}\left(-dt^2+\sum_{i=1}^{d-2}dx_i^2\right)+R^2dy^2$$

•Light ray: $ds^2=0$ at constant $x_i \Rightarrow$

$$t=\int dt = R\int^{\infty}e^{-y}dy<\infty.$$

•$\rightarrow$ light takes **finite** time to reach boundary: can reflect back: This is the only *patch* of a *global* space (universal cover)

•Its boundary at $x_0=\epsilon$: $\mathbb{R}^{1,d-1}$:

$$ds^2=\frac{R^2}{\epsilon^2}\left(-dt^2+\sum_{i=1}^{d-2}dx_i^2\right)$$

•Global coordinates (whole space):

$$ds_d^2 = R^2(-\cosh^2\rho d\tau^2 + d\rho^2 + \sinh^2\rho d\vec{\Omega}_{d-2}^2)$$

.

similar to sphere

$$ds_d^2 = R^2(\cos^2\rho d\tau^2 + d\rho^2 + \sin^2\rho d\vec{\Omega}_{d-2}^2) \equiv d\vec{\Omega}_d^2$$

•Also by $\tan\theta = \sinh\rho \Rightarrow$

$$ds_d^2 = \frac{R^2}{\cos^2\theta}(-d\tau^2 + d\theta^2 + \sin^2\theta d\vec{\Omega}_{d-2}^2)$$

•Boundary of space: $\theta = \pi/2 - \epsilon \Rightarrow$

$$ds^2 = \frac{R^2}{\epsilon^2}(-d\tau^2 + \sin^2\theta d\vec{\Omega}_{d-2}^2)$$

•Analytical continuation to Euclidean signature: $AdS_d \to EAdS_d$. But then, boundary Poincaré vs. global is **radial** time continuation.

$$ds^2 = dt_E^2 + \sum_{i=1}^{d-2} dx_i^2 = d\tilde{\rho} + \tilde{\rho}^2 d\vec{\Omega}_{d-2}^2 = e^{2t_E}(d\tau_E^2 + d\vec{\Omega}_{d-2}^2).$$

Penrose diagram

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•Flat space, under

$$\begin{aligned} u_{\pm} &= t \pm x = \tan \tilde{u}_{\pm} = \tan\left(\frac{\tau+\theta}{2}\right) \Rightarrow \\ ds^2 &= -dt^2 + dr^2 + r^2 d\vec{\Omega}_{d-2}^2 \\ &= \frac{1}{4\cos^2\tilde{u}_+ \cos^2\tilde{u}_-}(-d\tau^2 + d\theta^2 + \sin^2\theta d\vec{\Omega}_{d-2}^2) \end{aligned}$$

where $|\tau \pm \theta| \leq \pi$, $\theta \geq 0 \Rightarrow (\tau, \theta, \vec{\Omega}_{d-2})$ form triangle of revolution.

•AdS space: same, for Poincaré patch (drop $1/x_0^2$).

•Global space: extend to full **cylinder**. Boundary of global space: cylinder $\mathbb{R}_t \times S_{d-2}$. Related to boundary for conformal patch by *conformal* transformation by e^{2t_E}.

2. Holography in AdS space

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•In AdS space, the boundary is a finite time away $\Rightarrow$ natural observables on the *boundary* $\to$ take boundary sources $\phi_0(\vec{x})$, and field

$$\phi(\vec{x}, x_0) = \int d^4\vec{x}' K_B(\vec{x}, x_0; \vec{x}')\phi_0(\vec{x}')$$

where K_B is the bulk-to-boundary propagator, satisfying

$$(\Box_{x,x_0} - m^2)K_B(\vec{x}, x_0; \vec{x}') = \delta^d(\vec{x} - \vec{x}')$$

$$K_{B,\Delta}(\vec{x}, x_0; \vec{x}') = \frac{\Gamma(\Delta)}{\pi^{d/2}\Gamma(\Delta - d/2)}\left[\frac{x_0}{x_0^2 + (\vec{x} - \vec{x}')^2}\right]^\Delta$$

•Massless scalar kinetic on-shell action

$$S_\phi = \frac{1}{2}\int d^4x d\vec{x}' \int d^4\vec{y}' \int d^5x\sqrt{g}\phi_0(\vec{x}')\partial_{\mu_{\vec{x},x_0}} K_B(\vec{x}, x_0; \vec{x}')\partial^\mu_{\vec{x},x_0} K_B(\vec{x}, x_0; \vec{y}')\phi_0(\vec{y}')$$

$$= \frac{C_d d}{2}\int d^d\vec{x}' d^d\vec{y}' \frac{\phi_0(\vec{x}')\phi_0(\vec{y}')}{|\vec{x} - \vec{y}'|^{2d}}$$

•In general, on-shell action for boundary sources defines some holographic quantities (on the boundary)

•Bulk-to-bulk propagator: $(\Box_x - m^2)G(x,y) = -\frac{1}{\sqrt{g_y}}\delta^{d+1}(x-y)$ is ($\nu \equiv m^2R^2 + d^2/4$)

$$G(x,y) = (x_0 y_0)^{d/2} \int \frac{d^d k}{(2\pi)^d} e^{i\vec{k}\cdot(\vec{x}-\vec{y})} I_\nu(k x_0^<) K_\nu(k x_0^>)$$

Constructed out of the two solutions of $(\Box - m^2)\Phi = 0$,

$$\begin{aligned} \Phi \;\; &\propto \;\; e^{i\vec{k}\cdot\vec{x}} x_0^{d/2} K_\nu(k x_0)\phi_0(\vec{k}) \sim x_0^{\Delta_-} \;\; \text{(non − normalizable)} \\ &\propto \;\; e^{i\vec{k}\cdot\vec{x}} x_0^{d/2} I_\nu(k x_0)\phi_0(\vec{k}) \sim x_0^{\Delta_+} \;\; \text{(normalizable)} \end{aligned}$$

•**Lorentzian signature**: (Poincaré) solution

$$\Phi^\pm \propto e^{i\vec{k}\cdot\vec{x}} x_0^{d/2} J_{\pm\nu}(|k| x_0)$$

but now Φ^+, normalizable mode ($\sim x_0^{\Delta_+}$) is finite in center.

3. String theory

•String theory: generalize QFT in worldline particle formalism. Action

$$S_1 = -m\int d\tau\sqrt{-\frac{dx^\mu}{d\tau}\frac{dX^\nu}{d\tau}\eta_{\mu\nu}} \to$$

.

$$\to -mc^2\int dt\sqrt{1-\frac{v^2}{c^2}} \simeq \int dt\left[-mc^2+\frac{mv^2}{2}\right]$$

•Equation of motion $\delta/\delta X^\mu \Rightarrow \frac{d}{d\tau}\left(m\frac{dX^\mu}{d\tau}\right)=0$. Free particle.

•Couple to background fields by adding

$$\int d\tau A_\mu(X^\rho(\tau))\left(q\frac{dX^\mu}{d\tau}\right) \equiv \int d^4x A_\mu(X^\rho(\tau))j^\mu(X^\rho(\tau))$$

•First order particle action in terms of einbein $e(\tau)=\sqrt{-\gamma_{\tau\tau}(\tau)}$,

$$S_P = \frac{1}{2}\int d\tau\left(e^{-1}(\tau)\frac{dX^\mu}{d\tau}\frac{dX^\nu}{d\tau}\eta_{\mu\nu}-em^2\right)$$

•Can put $m=0$ and choose gauge $e(\tau)=1$ for reparametrization invariance, but then $e(\tau)$ equation of motion is constraint

$$\frac{dX^\mu}{d\tau}\frac{dX^\nu}{d\tau}\eta_{\mu\nu} \equiv T = 0.$$

•**Nambu-Goto action** $\rightarrow$ generalization of S_1:

.

$$S_{NG} = -\frac{1}{2\pi\alpha'}\int d\tau d\sigma \sqrt{-\det\left(\partial_a X^\mu \partial_b X^\nu g_{\mu\nu}(X(\xi^a))\right)}$$

•In det, induced metric on *worldsheet* for (σ, τ).

•**Polyakov action** $\rightarrow$ generalization of S_P:

$$S_P = -\frac{1}{4\pi\alpha'}\int d\tau d\sigma \sqrt{-\gamma}\gamma^{ab}\partial_a X^\mu \partial_b X^\nu \eta_{\mu\nu}$$

•It has invariances under:

-spacetime Poincar'e

-worldsheet diff. (with $X'^\mu(\sigma', \tau') = X^\mu(\sigma, \tau)$)

-Weyl invariance $X'^\mu(\sigma, \tau) = X^\mu(\sigma, \tau)$ and $\gamma'_{ab}(\sigma, \tau) = e^{2\omega(\sigma,\tau)}\gamma_{ab}(\sigma, \tau)$.

•Boundary conditions: -**closed strings** (periodic in $\sigma \sim \sigma + 2\pi$) or -**open strings**: Neumann $\partial^\sigma X^\mu(\tau, 0) = \partial^\sigma X^\mu(\tau, l) = 0$ or Dirichlet $\delta X^\mu(\tau, \sigma = 0.l) = 0$.

•Fix a gauge, e.g. conformal gauge $\gamma_{ab} = \eta_{ab}$. Residual invariance is *conformal invariance*.

•Then, equation of motion is $\Box X^\mu(\sigma, \tau) = 0 \Rightarrow X^\mu = X^\mu_R(\tau - \sigma) + X^\mu_L(\tau + \sigma)$ (left- and right- moving modes).
•Constraints: $T_{ab} = 0$ give $T_{+-+} = 0$ and $T_{--} = 0 \rightarrow$ Virasoro constraints (generated by L_n and $\tilde{L}_n$).

•Spectrum: closed string

$$X^\mu_R = \frac{x^\mu}{2} + \frac{\alpha'}{2} p^\mu(\tau - \sigma) + \frac{i\sqrt{2\alpha'}}{2} \sum_{n \neq 0} \frac{1}{n} \alpha^\mu_n e^{-in(\tau - \sigma)}$$

$$X^\mu_L = \frac{x^\mu}{2} + \frac{\alpha'}{2} p^\mu(\tau + \sigma) + \frac{i\sqrt{2\alpha'}}{2} \sum_{n \neq 0} \frac{1}{n} \tilde{\alpha}^\mu_n e^{-in(\tau + \sigma)}$$

•Hamiltonian

.
$$H = \frac{1}{2}\sum_{n\in\mathbb{Z}} \alpha^\mu_{-n}\alpha^\mu_n$$

•Spectrum: act with $\alpha^\mu_{-n}, \tilde{\alpha}^\mu_{-n}$ on $|0\rangle$. But, only α^i_{-n} physical.
•No quantum anomalies (spacetime Lorentz, Weyl, BRST) $\Rightarrow$ dimension (number of scalars X^μ) is $D = 26$ for (bosonic) string.
•Supersymmetry $\to$ introduce fermions $\Rightarrow$ $D = 10$ for superstring.
•String theory has ∞ number of modes $\leftrightarrow$ worldline particles $\leftrightarrow$ fields. (from α^μ_{-n}, $n \in \mathbb{N}$)

•Background fields: from (massless) modes of (super)string. Closed, bosonic $(g_{\mu\nu}, B_{\mu\nu}, \phi) \Rightarrow$

$$\begin{aligned} S &= -\frac{1}{4\pi\alpha'}\int d^2\sigma[\sqrt{-\gamma}\gamma^{ab}\partial_a X^\mu \partial_b X^\nu g_{\mu\nu}(X^\rho) \\ &\quad +\alpha'\epsilon^{ab}\partial_a X^\mu d_b X^\nu B_{\mu\nu}(X^\rho) - \alpha'\sqrt{-\gamma}\mathcal{R}^{(2)}\Phi(X^\rho)] \end{aligned}$$

•It also contains nonperturbative objects: D-branes.

4. AdS as a limit of D-branes

.

•D-branes = enpoints of strings with $D-(p+1)$ Dirichlet boundary conditions $\delta X^{\mu}(\tau,\sigma=0,l)=0$. Wall with $p+1$ dimensions.

•A graviton $\delta g_{\mu\nu}$ (or $\delta\phi$, etc.) can hit wall and excite modes that live on it $\rightarrow$ gives action

$$S_P = -T_p \int d^{p+1}\xi e^{-\phi}\sqrt{-\det\left(\frac{\partial X^{\mu}}{\partial \xi^a}\frac{\partial X^{\nu}}{\partial \xi^b}(g_{\mu\nu}+\alpha' B_{\mu\nu})+2\pi\alpha' F_{ab}\right)}+S_{WZ}$$

•But: Dp-branes = (same masses and charges as) p-brane solutions of supergravity (= low energy of string theory)

$$\begin{aligned} ds^2_{\text{string}} &= H_p^{-1/2}(-dt^2+d\vec{x}_p^2)+H_p^{1/2}(dr^2+r^2 d\Omega^2_{8-p}) \\ e^{-2\phi} &= H_p^{\frac{p-3}{2}} \\ A_{01\ldots p} &= -\frac{1}{2}(H_p^{-1}-1) \end{aligned}$$

•Here the harmonic function is

.

$$H_p = 1 + \frac{2C_pQ_p}{r^{7-p}} \equiv 1 + \frac{R^4}{r^4}$$

and colorGreen$R^4 = 4\pi g_s N\alpha'^2$.

•Note: "string frame". "Einstein frame" $ds_E^2 = e^{-\phi/2}ds_s^2$.

•D3-branes ($p = 3$) for $r \to 0$: $H \simeq R^4/r^4 \Rightarrow$

$$\begin{aligned} ds^2 &= \frac{r^2}{R^2}(-dt^2 + d\vec{x}_3^2) + \frac{R^2}{r^2}dr^2 + R^2 d\Omega_5^2 \\ &= R^2\frac{-dt^2 + d\vec{x}_3^2 + dx_0^2}{x_0^2} + R^2 d\Omega_5^2 \end{aligned}$$

•$\Rightarrow$ $AdS_5 \times S^5$ space! Therefore AdS space appears from N ($N \to \infty$) D-branes, in the near-horizon limit $r \to 0$.

5. String theory in AdS space

•Supergravity (low energy) limit:
•On $AdS_5 \times S^5 \rightarrow$ KK reduction on $S^5 \rightarrow$ *gauged* supergravity in AdS_5 background.

•$\exists$ cosmological constant, $SO(6)$= symmetry group of S^5 is *gauged* (local), with coupling $g \neq 0$.

•We have also solitonic objects, e.g. D-instantons in $AdS_5 \times S^5$, charged under $a = a_\infty + e^{-\phi} - \frac{1}{g_s}$, with

$$e^\phi = g_s + \frac{24pi}{N^2} \frac{x_0^4 \tilde{x}_0^2}{[\tilde{x}_0^2 + |\vec{x} - \vec{x}_a|^2]^4} + \ldots$$

•Also Dp-brane that can wrap cycles in geometry $\rightarrow$ e.g. D5 on S^5 for $AdS_5 \times S^5$.

•We also have long (classical) strings that can end on D-branes

•If the D-brane is on boundary at infinity, and string ends on a contour C, the string stretches into AdS by gravity, and is stopped by tension:

$$\begin{aligned} ds^2 &= \alpha' \frac{r^2}{R^2}(-dt^2 + d\vec{x}^2) + ... \\ &\sim (1 + 2V_{\text{Newton}})(-dt^2 + ...) \end{aligned}$$

•Quantum strings in AdS space $\rightarrow$ hard to quantize: highly nonlinear worldsheet action.

•e.g. in embedding space for $S^3 \subset S^5$,

$$S = \frac{R^2}{4\pi\alpha'} \int d\tau \int_0^{2\pi} d\sigma [-(\partial_a X^0)^2 + \sum_{i=1}^{4} (d_a X^i)^2]$$

where $\sum_i X^i X^i = 1$. $\rightarrow$ is actually nonlinear.

•Exception: Penrose limit (near *null* geodesic in $AdS_5 \times S^5$)

.

$$ds^2 = -2dx^+dx^- - \mu^2(\vec{r}^2 + \vec{y}^2)(dx^+)^2 + d\vec{y}^2 + d\vec{r}^2$$

•Polyakov string action

$$S = -\frac{1}{2\pi\alpha'}\int_0^l d\sigma \int d\tau \frac{1}{2}\sqrt{-\gamma}\gamma^{ab}[-2\partial_a X^+ \partial_\beta X^- - \mu^2 X_i^2 \partial_a X^+ \partial_b X^+ + \partial_a X^i \partial_b X^i]$$

•In light-cone gauge, $X^+(\sigma, \tau) = \tau$,

$$S = -\frac{1}{2\pi\alpha'}\int d\tau \int_0^l d\sigma \left[\frac{1}{2}\eta^{ab}\partial_a X^i \partial_b X^i + \frac{\mu}{2}X_i^2\right]$$

Lecture 3

The AdS/CFT map and gauge/gravity duality

1. AdS/CFT and state-operator map

.

•We have seen that $AdS_5 \times S^5$ appears in the near-horizon of N D3-branes, and AdS space is likely holographic.

•More precise: two open strings on D3 collide: closed string peels off into bulk as Hawking radiation $\Rightarrow$ relation between bulk gravity theory and D3-brane field theory in *decoupling limit* of D-branes.

•So: $SU(N)$ at large N $\mathcal{N} = 4$ SYM = gravity theory at $r \to 0$, for $\alpha' \to 0$ (for no string worldsheet corrections) and $g_s \to 0$ (for no quantum string corrections): $AdS_5 \times S^5$.

•More precisely: $\lambda = g_{YM}^2 N = R^4/\alpha'^2$ large and fixed.

•In fact, duality is believed to hold for all g_s and N.

•Map: couplings $4\pi g_s = g^2_{YM}$, $\lambda = g^2_{YM} N = R^4/\alpha'^2$

.

•$SO(6)$ R-symmetry $\leftrightarrow$ $SO(6)$ gauge symmetry in AdS_5 $\leftrightarrow$ isometry of S^5 on which we KK reduce.

•Conformal symmetry $SO(4,2)$ $\leftrightarrow$ isometry of AdS_5.

•Supergravity modes couple to (are sources for) boundary gauge invariant operators. e.g. scalar ϕ with KK expansion (on S^5)

$$\phi(x,y) = \sum_n \sum_{I_n} \phi^{I_n}_{(n)}(x) Y^{I_n}_{(n)}(y)$$

then $\phi^{I_n}_{(n)}$ couples with operator $\mathcal{O}^{I_n}_{(n)}$ of dimension

$$\Delta = \frac{d}{2} + \sqrt{\frac{d^2}{4} + m^2 R^2}$$

•Gravity dual: string theory in the background ($U \equiv r/\alpha'$)

.

$$
\begin{aligned}
ds^2 &= \alpha' \left[\frac{U^2}{\sqrt{4\pi g_s N}}(-dt^2 + d\vec{x}_3^2) + \sqrt{4\pi g_s N}\left(\frac{dU^2}{U^2} + d\Omega_5^2\right)\right] \\
F_5 &= 16\pi g_s \alpha'^2 N(1+*)\epsilon_{(5)}
\end{aligned}
$$

•Solve $(\Box - m^2)\phi = 0$ in background $\Rightarrow$ we obtain $\phi \sim x_0^{\Delta_\pm}\phi_0$, where ϕ_0= boundary source, and

$$
\Delta_\pm = \frac{d}{2} \pm \sqrt{\frac{d^2}{4} + m^2 R^2}
$$

•Then $x_0^{\Delta_-}\phi_0 = x_0^{d-\Delta}\phi_0$ is non-normalizable mode, and $x_0^{\Delta_+}\Phi_0$ is normalizable mode.

•In non-normalizable mode, ϕ_0 = source for $\mathcal{O}$ = composite, gauge invariant operator $\to$ need partition function $Z_{\mathcal{O}}[\phi_0]$,

$$Z_{\mathcal{O}}[\phi_0] = \int \mathcal{D}[SYM] e^{-S+\int \mathcal{O}\cdot\phi_0}$$

•Witten prescription for duality: partition function is same

$$Z_{\mathcal{O}}[\phi_0]_{\text{CFT}} = Z_{\phi}[\phi_0]_{\text{string}}$$

•Moreover, in $\alpha' \to 0$ limit, $g_s \to 0 \Rightarrow$ classical, on-shell, supergravity limit

$$Z_{\phi}[\phi_0] = e^{-S_{\text{sugra}}[\phi[\phi_0]]}$$

•S_{sugra} is on-shell, for classical $\phi[\phi_0]$ depending on boundary value ϕ_0, i.e., as we saw

$$\phi(\vec{x}, x_0) = \int d^4\vec{x}' K_B(\vec{x}, x_0; \vec{x}')\phi_0(\vec{x}')$$

Correlators

$$\langle \mathcal{O}(x_1)...\mathcal{O}(x_n)\rangle = \frac{\delta^n}{\delta\phi_0(x_1)...\delta\phi_0(x_n)} Z_{\mathcal{O}}(\phi_0)\Big|_{\phi_0=0}$$

.

$$\langle \mathcal{O}(x_1)\mathcal{O}(x_2)\rangle = -\frac{\delta^2 S_{\text{sugra}}[\phi[\phi_0]]}{\delta\phi_0(x_1)\delta\phi_0(x_2)}\Big|_{\phi_0=0}$$

•Given the form of the quadratic part of S_{sugra} in lecture 2, we obtain

$$\langle \mathcal{O}(x_1)\mathcal{O}(x_2)\rangle = -\frac{C_d d}{|\vec{x}_1 - \vec{x}_2|^{2d}}$$

as required by CFT.

•In general, "Witten diagrams" from $S_{\text{sugra}}[\phi[\phi_0]]$: tree (classical) Feynman diagrams in x space with endpoints on boundary.

•Lorentzian case: $\exists$ good normalizable and non-normalizable modes $\Rightarrow$ normalizable modes $\leftrightarrow$ VEVs (states). For $\phi \sim \alpha_i x_0^{d-\Delta} + \beta_i x_0^{\Delta}$,

$$H = H_{\text{CFT}} + \alpha_i \mathcal{O}_i; \quad \langle \beta_i | \mathcal{O}_i | \beta_i \rangle = \beta_i + (\alpha_i \text{ piece})$$

Nonperturbative states

.

•e.g. instanton maps to D-instantons in $AdS_5 \times S^5$, by

$$\begin{aligned} \frac{\delta S}{\delta\phi_0(\vec{x})} &= -\frac{\delta}{\delta\phi_0(\vec{x})}\frac{1}{4\pi\kappa_5^2}\int d^5x\sqrt{-g}g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi \\ &= -\frac{48}{4\pi g_s}\frac{\tilde{z}^4}{[\tilde{z}^2+|\vec{x}-\vec{x}_a|^2]^4} \\ &= \frac{1}{2g_{YM}^2}\langle \mathrm{Tr}\,[F_{\mu\nu}^2(\vec{x})]\rangle \end{aligned}$$

matches exactly.

•Also, D5-brane wrapping $S^5 \leftrightarrow$ baryon vertex operator in $SU(N)$ SYM, for connecting N external quarks.

Wilson loops

•Nonperturbative gauge theory $\to$ encoded in Wilson loops
•$\langle W[C]\rangle$ encodes $V_{q\bar{q}}(r)$.
.

•In a susy theory, susy Wilson loop

$$W[C] = \frac{1}{N}\mathrm{Tr}\, P\exp\left[\oint (A_\mu \dot{x}^\mu + \theta^I X^I(x^\mu)\sqrt{\dot{x}^2})d\tau\right]$$

where $X^\mu(\tau)$ parametrizes the loop, θ^I unit vector on S^5. Then

$$\langle W[C]\rangle = e^{-S_{\mathrm{string}}[C]-l\phi}$$

Here $l\phi$= renormalization $\to$ exract divergence: straight string forming parallelipiped.

•Non-susy loop can also be defined.
•Calculation in AdS_5 gives

$$V_{q\bar{q}}(L) = -\frac{4\pi^2}{\Gamma(1/4)^4}\frac{\sqrt{2g_{YM}^2 N}}{L}$$

PP wave correspondence

.

- Penrose limit of $AdS_5 \times S^5 \rightarrow$ worldsheet string is free (massive).

- In SYM, corresponds to large charge J (for $U(1) \subset SO(6)_R$). $Z = \Phi^5 + i\Phi^6$ is charged under it, $\Phi^1, ..., \Phi^4$ aren't. Vacuum of string:

$$|0, p^+\rangle = \frac{1}{\sqrt{J} N^{J/2}} \text{Tr}\,[Z^J]$$

- String states $\rightarrow$ insertions of $\Phi^1, ..., \Phi^4$, etc. inside the trace. e.g.

$$a^\dagger_{n,4} a^\dagger_{-n,3} |0, p^+\rangle = \frac{1}{\sqrt{J}} \sum_{l=1}^{J} \frac{1}{N^{J/2}} \text{Tr}\,[\Phi^3 Z^l \phi^4 Z^{J-l}] e^{\frac{e\pi i n l}{J}}$$

- $\exists$ Hamiltonian acting on these $\rightarrow$ Hamiltonian of discretized string on the pp wave: only way to obtain quantum string.

2. Gauge/gravity duality

.

•$AdS_5 \times S^5$ best understood example. But $\exists$ other cases.

•Conformal: -N M2-branes and N M5-branes in 11d M-theory (strong coupling string theory) $\Rightarrow$ $AdS_4 \times S^7$ and $AdS_7 \times S^4$.
-orbifolds/orientifolds of $AdS_5 \times S^5 \rightarrow$ less susy.
- ABJM model in 2+1 dimensions: N M2-branes on $\mathbb{C}^4/\mathbb{Z}_k$, IR limit $\leftrightarrow$ $AdS_4 \times S^7/\mathbb{Z}_k\Big|_{k\rightarrow\infty} = AdS_4 \times \mathbb{CP}^3$.

•Nonconformal, less (or no) susy: "gravity dual backgrounds" $\rightarrow$ no AdS factor.

•Extra dimension $r \leftrightarrow$ energy scale U. $\exists$ β function $\rightarrow$ running with $U \leftrightarrow r \Rightarrow$ nontrivial metric as a function of r.

•To holographically simulate QCD-like gauge theory, need:

-large N gauge theory (e.g. $SU(N)$): small string corrections
-(flat) boundary for field theory , and sections of constant $r \leftrightarrow U$.
-compact space $X_n \leftrightarrow$ global symmetry of field theory
-RG flow in r between low energy (low r) and high energy (large r).

•Map:

-Gauge invariant SYM operators ("glueballs") $\leftrightarrow$ sugra fields in gravity dual
-Gauge invariant operators with "quarks" ("mesons") $\leftrightarrow$ SYM fields on brane in gravity dual
-Wavefunctions in field theory, e.g. $e^{ik\cdot x}$ $\leftrightarrow$ wavefunctions in gravity dual (times wavefunctions on compact space), e.g.

$$\phi(x, U, X_m) = e^{ik\cdot x}\psi(U, X_m)$$

Phenomenological gauge/gravity duality

.

•QCD "bottom-up" models or condensed matter models: no decoupling limit of brane theory.

•Instead, gravity dual with some *field theory* on it (not string theory) with right properties. (could be truncation of a low energy sugra limit of string theory)

•Could be nonrelativistic, e.g. Lifshitz system (add $-iu\partial_u$ to dilatation D)

$$ds^2_{d+1} = R^2\left(-\frac{dt^2}{u^{2z}} + \frac{d\vec{x}^2}{u^2} + \frac{du^2}{u^2}\right)$$

or gravity dual of Schrödinger algebra (add $-iu\partial_u$ to D)

$$ds^2_{d+2} = R^2\left(-\frac{dt^2}{u^{2z}} + \frac{d\vec{x}^2}{u^2} + \frac{du^2}{u^2} + \frac{2dt\, d\xi}{u^2}\right)$$

3. Finite temperature

.

•For sQGP and condensedf matter applications, need *finite temperature*.

•Hawking radiation from Wick rotation of black hole

$$ds^2 = +\left(1 - \frac{2MG_N}{r}\right) d\tau^2 + \frac{dr^2}{1 - \frac{2MG_N}{r}} + r^2 d\Omega_2^2$$

has conical singularity ($ds^2 \simeq A(d\rho^2 + \rho^2 d\tau^2)$) near $r = 2MG_N$, unless $\theta = \tau/(4MG_N)$ has periodicity $2\pi \Rightarrow$

$$T_{BH} = \frac{1}{8\pi MG_N} \Rightarrow C = -\frac{\partial M}{\partial T} < 0$$

So it is not thermodynamically stable system.

•But a black hole in AdS space is. $\exists C > 0$ branch.

•For a black hole in AdS space,

.

$$ds^2 = -\left(\frac{r^2}{R^2} + 1 - \frac{w_n M}{r^{n-2}}\right) dt^2 + \frac{dr^2}{\frac{r^2}{R^2} + 1 - \frac{w_n M}{r^{n-2}}} + r^2 d\Omega^2_{n-2}$$

one finds (r_+ is largest solution to $\frac{r^2}{R^2} + 1 - \frac{w_n M}{r^{n-2}} = 0$)

$$T = \frac{nr_+ + (n-2)R^2}{4\pi R^2 r_+}$$

•$T(M)$ has a minimum, followed by a uniformly increasing branch $\rightarrow$ stable.

•Need to take also $R \cdot T \rightarrow \infty \Rightarrow M \rightarrow \infty$.

•More generally, put black hole in gravity dual $\Rightarrow$ Put dual field theory at finite temperature.

4. AdS/CMT and transport

•Strongly coupled CMT field theories $\to$ not easy to describe $\to$ phenomenological models.
•Need finite temperature $\Rightarrow$ black holes.

•For transport properties, need spectral functions $\to$ retarded Green's functions $G^R_{\mathcal{O}_A\mathcal{O}_B}$.

•Linear response theory:

$$\delta\langle\mathcal{O}_A\rangle(\omega,k) = G^R_{\mathcal{O}_A\mathcal{O}_B}(\omega,k)\delta\phi_{B(0)}(\omega,k)$$

•Im $G^R_{\mathcal{O}_A\mathcal{O}_B}(\omega,k)$ is spectral functions for χ, since

$$\chi \equiv \lim_{\omega\to 0+i0} G^R_{\mathcal{O}_A\mathcal{O}_B}(\omega,x) = \int_{-\infty}^{+\infty}\frac{d\omega'}{\pi}\frac{\text{Im}G^R_{\mathcal{O}_A\mathcal{O}_B}(\omega',x}{\omega'}$$

•To calculate $G^R_{\mathcal{O}_A\mathcal{O}_B}$ holographically, prescription by Son and Starinets.

•For asymptotically AdS gravity dual with black hole $\leftrightarrow$ event horizon H, if

$$S = \int \frac{d^d k}{(2\pi)^d} \phi_{(0)}(-k)\mathcal{F}(k,z)\phi_{(0)}(k)\Big|_{z=z_B}^{z=z_H}$$

then, G^R is

$$G^R(k) = -\,2\mathcal{F}(k,z)|_{z_B} = \frac{2\Delta_A - d}{R}\frac{\delta\phi^{(2\Delta-d)}_{A,\text{norm}}}{\delta\phi_{B,\text{norm.}}}$$

•Results: Kubo formulas:

$$\sigma(\omega,\vec{k}) = \frac{iG^R_{J_xJ_x}(\omega,\vec{k})}{\omega}; \quad \eta(\omega,\vec{k}) = \frac{iG^R_{T_{xy}T_{xy}}(\omega,\vec{k})}{\omega}$$

•For gravity duals with black holes, generically one finds $\eta/s = 1/(4\pi)$ ($s =$ entropy density).